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RESEARCH PAPER

Yasser Aboelkassem · Georgios H. Vatistas
Nabil Esmail

Viscous dissipation of Rankine vortex profile in zero meridional flow

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Abstract An analytical solution is given for a time-decay Rankine vortex profile due to viscous effects. The vortex filament is assumed to be isolated, strong, concentrated and having zero-meridional flow (i.e. radial and axial velocities are equal to zero). Zero-meridional renders the governing equations for an unsteady, incompressible and axisymmetric vortex in a simple form. Based on the tangential momentum equation, the spatial-temporal distributions of the swirl velocity are given in terms of Fourier-Bessel series by using separation of variables technique. A general formula is derived by total differentiation of the swirl velocity with respect to time, depicting the viscous dissipation for Oseen and Taylor-like vortex profiles. This analysis is validated by comparison with previous experimental data.

Keywords Rankine vortex · Dissipation · Fourier series · Monopolar · Zero-Meridional

1 Introduction

Concentrated intense vortices appear in many of real fluid motions, their life cycle consisting of vortex evolution and decay phases. The dissipation of vortices is a subject that is frequently visited by aerodynamicists, hydrodynamicists, and fluid mechanicians. More than a century ago interest in this topic was first expressed in connection with Lord Kelvin's vortex atomic model, Kelvin [1].

The diffusion of the vorticity and decay of concentrated vortices are important in many of engineering applications. For instance, the hazard associated with vortices shed by a large, heavily loaded aircraft to an incoming smaller plane is well known. A safe separation distance assures that the generated vortices have decayed to a level that is safe for the following aircraft.

Steady state vortices are well predicted in the past, the most prominent of the steady vortex formulations are referenced to Rankine and Helmholtz [2]. The model assumes that both radial and axial-velocity components are zero. The latter produces a Caucy type differential equation for the azimuthally momentum with two possible solutions for the tangential velocity. The first suggests a linear distribution (forced motion) inside the core, while the second gives a hyperbolic variation outside. Burgers [3] model provides a smooth transition of the velocity from forced to free-vortex modes. Another theoretical model was proposed by Scully [4]. The advantages of this model are: both radial and axial velocities are finite and satisfy the general vortex boundary conditions. The disadvantage of this model is: it underestimates the maximum swirl velocity, see Leishman & Bhagwat [5]. Vatistas et al. [6] proposed an alternative formulation that gives rise to a family of smooth velocity distributions having the Rankine-Helmholtz free and forced modes as asymptotes.

The decay of eddies possesses highly complicated properties, nevertheless a simplified time-dependent solutions might capture some of their fundamental physical characteristics involved during the energy dissipating phase. Two exact asymptotic solutions to the Navier-Stokes equations that describe the decay of line vortices have been given previously. The first solution dealt with the diffusion of an initially potential (free) vortex in the absence of a meridional flow, this work was accomplished by Oseen [7], Hamel [8], and Lamb [9]. In this case, the initial vorticity is zero everywhere except at the origin where it is singular, both tangential velocity and static pressure are also singular at the origin. As time evolves, the velocity begins to diminish, the vorticity diffuses outwards, and the core dilates. The second solution was brought about by Taylor [10], a decaying vortex with remarkable properties was derived by noting the similarity of vorticity diffusion to that of heat conduction equation, where the solution is well known, and the resulting class of vortices was later known as monopolar type of vortices.

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Y. Aboelkassem (✉) · G. H. Vatistas · N. Esmail
Department of Mechanical and Industrial Engineering,
Concordia University, Montreal, Quebec H3G 1M8, Canada
E-mail: yaboelka@alcor.concordia.ca
Tel.: 514-848-2424 ext 3158
Fax: 514-848-3175

Neufville [11] presented a solution in terms of Laguerre-polynomials based on similarity transformation of time and space for the diffusion term only in the tangential momentum. Kirde [12] also reported a solution in terms of the hyper-geometric series to further elaborate in the same phenomenon. Both series solutions are restricted to the potential vortex as their initial condition. A theoretical study with regard to the time decay of different types of vortex profiles has developed by Aboelkassem [13], while Bennett [14] has experimentally studied the decay of Rankine's vortex, where a typical forced-free vortex was generated by spinning a single thin walled cylinder inside a larger water tank for long period of time, the instantaneous decay of the tangential velocity are recorded.

In this paper, the vortex flow solution that belongs to a particular class, in which the velocity field has the general form $\boldsymbol{q} = (0, V_\theta, 0)$, is given. This category represents a subset of viscous vortices that are decaying in a zero-meridional field. The present analysis is a natural extension to Batchelor [15]. The solution to the problem is achieved by using Fourier-Bessel series to solve the conservation of tangential momentum. The series solution is firstly applied to predict the decay of initially potential vortex, where a similarity solution is known, refer to Oseen [7] that gave us the chance to gauge the effectiveness of the method. The solution is then used to study the decay of the Rankine's vortex profile as well as to generate a family of monopolar-like vortices by total differentiation of the swirl velocity expression with respect to time.

2 Formulations and mathematical analysis

Two concentric cylinders are used to represent theoretically a class of columnar vortex filament under consideration here, i.e. Rankine's-vortex profile. A schematic plane normal to the axis of rotation is shown in Fig. 1. Let the vortex core

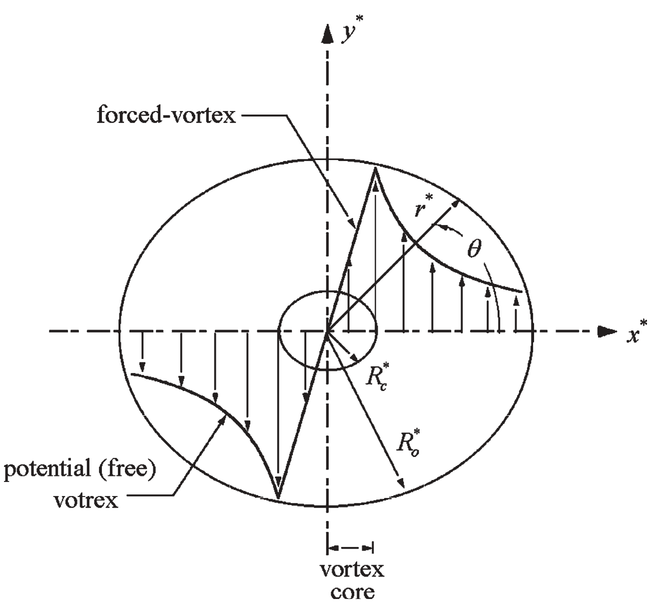

Fig. 1 Schematic of the Rankine-vortex velocity profile

(defined as the radius where the tangential velocity attains its maximum) initially located at the inner cylinder radius R_c^*. Since we are interested in modeling the dissipation of line vortex within unconfined domain, therefore, the outer radius R_o^* should extend theoretically to infinity. In other words, β parameter attains a sufficiently small value where β is defined as the ratio of the core radius to the outer radius. Since there is no energy supplied to the vortex, therefore it supposes to dissipate under the action of viscosity consequently and the hazards will diffuse outward. The general boundary conditions require that, at the center of rotation the tangential velocity and the gradient of the pressure and vorticity are also zeros. Far away from the axis of rotations, theoretically infinity, all the flow properties completely dissipate and vanish.

2.1 Equations and solution

Consider the motion of pure swirling, incompressible, and axially invariant line vortex. According to these assumptions, the continuity is identically satisfied while the radial and tangential momenta in addition to the vorticity transport are given respectively as follows:

$$\frac{V_\theta^2}{r} = \frac{\partial \Pi}{\partial r}, \tag{1}$$

$$\frac{\partial V_\theta}{\partial \tau} = \frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r^2}, \tag{2}$$

$$\Omega_z = \frac{1}{r}\frac{\partial (r V_\theta)}{\partial r}, \tag{3}$$

where,

$$r = \frac{r^*}{R_c^*},\quad \tau = \frac{\nu t}{R_c^{*2}},\quad V_\theta = \frac{V_\theta^*}{V_{\theta c}},\quad \Pi = \frac{p - p_\infty}{\rho V_{\theta c}^2},\quad V_{\theta c} = \frac{\Gamma_\infty}{2\pi R_c^*}.$$

Γ_∞ is the vortex circulation, p is the pressure, p_∞ is the pressure far from the vortex center ($r \to \infty$), ν is the kinematic viscosity and t is the actual time. The subscript c denotes value of the parameter at the core.

The required initial and boundary conditions are:

i. $V_\theta(\tau = 0, r) = F(r)$,
ii. $V_\theta(\tau \geq 0, r = 0) = 0$,
iii. $V_\theta\left(\tau \geq 0, r \to \frac{1}{\beta}\right) = \beta$.

An analytical solution to the above tangential momentum equation is feasible under the separation of variable technique. However, it is difficult to obtain explicit formula for the tangential velocity because the second boundary condition is non-homogenous. Here, we propose a proper velocity transformation in order to make the boundary conditions homogenous and achieve an expression for the Eigen values and consequently the swirl velocity distribution. Let the tangential velocity be given as:

$$\tilde{V}_\theta(r,\tau) = \beta^2 r - V_\theta(r,\tau), \tag{4}$$

where $\beta = R_c^*/R_o^*$.

One can notice that, Eq.(4) satisfies the original momentum equation. In this case, the initial and boundary conditions are:

i. $\tilde{V}_\theta(\tau = 0, r) = \tilde{F}(r)$ where, $\tilde{F}(r) = \beta^2 r - F(r)$,
ii. $\tilde{V}_\theta(\tau \geq 0, r = 0) = 0$,
iii. $\tilde{V}_\theta\left(\tau \geq 0, r \to \frac{1}{\beta}\right) = 0$.

Now, the standard separation of variables technique is used, and the swirl velocity is given as:

$$\tilde{V}_\theta(r,\tau) = [A\mathrm{J}_1(mr) + B\mathrm{Y}_1(mr)]\mathrm{e}^{-m^2\tau}, \tag{5}$$

where, J_1 is the Bessel function of first kind first order, Y_1 is the Bessel function of second kind first order and A,B are arbitrary constants. Owing to the fact that $\lim_{r\to\infty} \mathrm{Y}_1(mr) = \infty$ and the swirl velocity must be zero at the origin, the constant B must be zero, *i.e.* the first boundary condition is now satisfied. Application to the second boundary conditions gives $A\mathrm{J}_1(m/\beta) = 0$, since A can not be equal to zero, therefore $\mathrm{J}_1(m/\beta) = 0$. Let $\lambda_n = m/\beta$ be the Eigen values of J_1, which are given by following expression:

$$\lambda_n = \pi\left\{n + 0.25 - \frac{0.151982}{4n+1} + \frac{0.0151399}{(4n+1)^3} - \frac{0.245275}{(4n+1)^5}\right\} \qquad (n = 1, 2, 3, 4, \ldots). \tag{6}$$

Therefore, the general solution for the velocity is then,

$$\tilde{V}_\theta(r,\tau) = \sum_{n=1}^{\infty} A_n \mathrm{J}_1(\lambda_n \beta r)\mathrm{e}^{-\lambda_n^2\beta^2\tau}, \tag{7}$$

or

$$V_\theta(r,\tau) = \beta^2 r - \sum_{n=1}^{\infty} A_n \mathrm{J}_1(\lambda_n \beta r)\mathrm{e}^{-\lambda_n^2\beta^2\tau}. \tag{8}$$

The constant A_n is evaluated by employing the initial condition; different initial profiles of the velocity will produce distinct velocity time distributions. By using the orthogonal boundary condition of the Bessel functions, the constant A_n is given by

$$A_n = \frac{\int_0^{1/\beta} \tilde{F}(r)\mathrm{J}_1(\lambda_n\beta r) r\,\mathrm{d}r}{\int_0^{1/\beta} \mathrm{J}_1^2(\lambda_n r) r\,\mathrm{d}r} = \frac{2\beta^2}{\mathrm{J}_0^2(\lambda_n)}\int_0^{1/\beta} \tilde{F}(r)\mathrm{J}_1(\lambda_n\beta r) r\,\mathrm{d}r, \tag{9}$$

where J_0 is the Bessel function of first kind zero order, $\tilde{F}(r) = \beta^2 r - F(r)$, and $F(r)$ is an initial vortex's velocity profile such as the potential or Rankine vortex profiles. The other flow properties such as the static pressure distribution and the axial vorticity can be evaluated by using Eqs.(1) and (3). One may note that the pressure requires the standard numerical integrations.

2.2 General decay model

In this section, we extended the solution to include the decay of other class of vortices, monopolar (Taylor)-like profiles. A general decay expression is derived based on the remark made by Neufville [11], that all the temporal differentiation of swirl velocity is also a feasible vortical flow,

$$\begin{aligned}
V_1 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{V}_\theta(r,\tau) = \sum_{n=1}^{\infty} A_n \mathrm{J}_1(\lambda_n\beta r) \times \mathrm{e}^{-\lambda_n^2\beta^2\tau}(-\lambda_n^2\beta^2),\\
V_2 &= \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\tilde{V}_\theta(r,\tau) = \sum_{n=1}^{\infty} A_n \mathrm{J}_1(\lambda_n\beta r) \times \mathrm{e}^{-\lambda_n^2\beta^2\tau}(-\lambda_n^2\beta^2)^2,\\
&\vdots\\
V_k &= \frac{\mathrm{d}^k}{\mathrm{d}\tau^k}\tilde{V}_\theta(r,\tau) = \sum_{n=1}^{\infty} A_n \mathrm{J}_1(\lambda_n\beta r) \times \mathrm{e}^{-\lambda_n^2\beta^2\tau}(-\lambda_n^2\beta^2)^k.
\end{aligned} \tag{10}$$

From Eqs.(8) and (10), a general expression for the decay problem is found as follow,

$$V_\theta = \beta^2 r - \begin{cases} \sum\limits_{n=1}^{\infty} A_n \mathrm{J}_1(\lambda_n\beta r)\mathrm{e}^{-\lambda_n^2\beta^2\tau} & k = 0, \quad \text{Oseen's-like},\\ \sum\limits_{n=1}^{\infty} A_n \mathrm{J}_1(\lambda_n\beta r)\mathrm{e}^{-\lambda_n^2\beta^2\tau}(-\lambda_n^2\beta^2)^k & k = 1, 2, \ldots, \quad \text{Tylor's-like}. \end{cases} \tag{11}$$

The above formula combined the decay phase for both Oseen-like and Taylor-like velocity profiles. In addition, since the tangential momentum is a linear partial differential equation, therefore the superposition principle for Eq.(11) is applicable. One can sum up all the above feasible solutions to provide an expression that holds for linear combinations of a single Oseen and single/multiple Taylor-like vortices, the formula is give as:

$$V = \beta^2 r - \sum_{n=1}^{\infty} A_n \mathrm{J}_1(\lambda_n\beta r)\mathrm{e}^{-\lambda_n\beta^2\tau} \times \underbrace{[1 + (-\lambda_n^2\beta^2) + (-\lambda_n^2\beta^2)^2 + \cdots (-\lambda_n^2\beta^2)^k]}_{\text{Geometric series}}$$

or

$$V = \beta^2 r - \sum_{n=1}^{\infty} A_n \mathrm{J}_1(\lambda_n \beta r) \mathrm{e}^{-\lambda_n \beta^2 \tau} \times \frac{1-(-\lambda_n^2\beta^2)^{k+1}}{1-(-\lambda_n^2\beta^2)} \quad (k = 0, 1, 2, \ldots). \tag{12}$$

The proposed methodology is kept as general as to accommodate any initial swirl velocity distribution provided that it satisfies the original steady state governing equations.

3 Results and discussion

In order to test the accuracy of the present solution; firstly, we deal with the decay of the potential (free) vortex model, where the similarity solution is known. Upon verifying the results, we go further to study in details the decay of initially Rankine's vortex and its derived monopolar-like subset of vortices, refer to Eq.(11).

3.1 Decay of potential vortex

Here, we considered the decay of the potential vortex. Its mathematical description constitutes one of the exact solutions of the Navier-Stokes equations tackled by Oseen [7]. For this type of problem, the following known similarity solution is available from the literature:

$$V_\theta(r,\tau) = \frac{1}{r}(1 - \mathrm{e}^{(-r^2/4\tau)}). \tag{13}$$

The equivalent form in terms of Fourier-Bessel functions is derived by using the potential vortex as an initial velocity distribution, which yields

$$F(r,\tau=0) = \frac{1}{r} \quad \text{or} \quad \tilde{F}(r,\tau=0) = \beta^2 r - \frac{1}{r}. \tag{14}$$

Using Eq.(9), an expression for determining the coefficients of A_n with the help of the integral table is given as follows:

$$A_n = \frac{-2\beta}{\lambda_n \mathrm{J}_0^2(\lambda_n)}.$$

The velocity distribution as functions of the space and time is then,

$$V_\theta(r,\tau) = \beta^2 r + 2\beta \sum_{n=1}^{\infty} \frac{\mathrm{J}_1(\lambda_n \beta r)}{\lambda_n \mathrm{J}_0^2(\lambda_n)} \mathrm{e}^{-\lambda_n^2 \beta^2 \tau}. \tag{15}$$

The purpose of comparing the series results with the existing solution from the literature at this point is firstly to serve as a guide in deciding the value of β that will render the calculations of the unconfined vortex hypothesis reliable and secondly to gauge the effectiveness of the present method before using it to study the decay of Rankine profile problem where there is no exact solution existing in the literature. Based on the boundary condition that as $\beta \to 0$, the flow approaches the condition of decaying vortex in a fluid domain extending to infinity. The predicted swirl velocity from the series solution was compared with the exact solution at the same decay time and for three values of $\beta = 1/25$, 1/50, 1/100, as shown in Fig. 2.

We found that, when $R_o^* = 100R_c^*$ or $\beta = 0.01$, the two methods of solution produced no differences up to and including the fifth decimal place, therefore we decided to use the latter value of β in our calculations. All the previous and subsequent calculations were carried out by taking $n = 1000$ terms from the Bessel function.

3.2 Decay of Rankine's vortex

Rankine's vortex has been used extensively in various studies. For instance, to predict the tangential velocity of trailing vortices, to estimate the noise level produced by vortexes and vibrations, to model the natural phenomena such as hurricanes and tornados, and others. Here, we applied the proposed method to study the time-decay of Rankine's vortex profile. Consider the standard velocity distribution for Rankine's vortex,

$$F(r,\tau=0) = \begin{cases} r, & 0 \le r \le 1, \\ \dfrac{1}{r}, & 1 < r \le \dfrac{1}{\beta}, \end{cases}$$

or

$$\tilde{F}(r,\tau=0) = \begin{cases} \beta^2 r - r, & 0 \le r \le 1, \\ \beta^2 r - \dfrac{1}{r}, & 1 < r \le \dfrac{1}{\beta}. \end{cases} \tag{16}$$

The corresponding A_n coefficients in Eq.(9) and an expression for the tangential velocity are given respectively in Eqs.(17) and (18).

$$A_n = \frac{-4\mathrm{J}_1(\lambda_n \beta)}{\lambda_n^2 \mathrm{J}_0^2(\lambda_n)}, \tag{17}$$

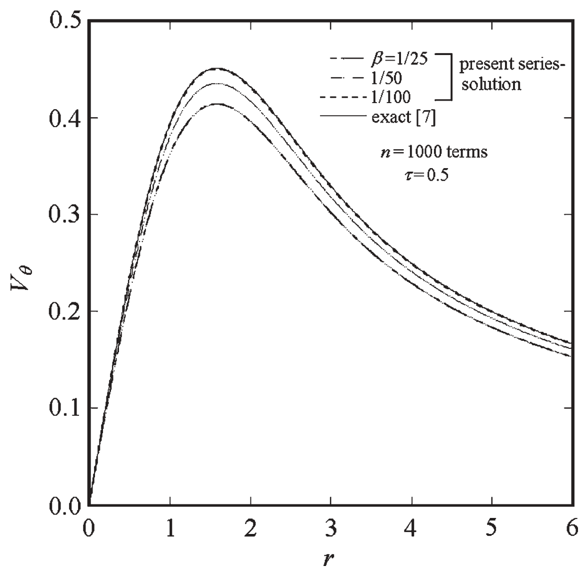

Fig. 2 Solution validation at different β

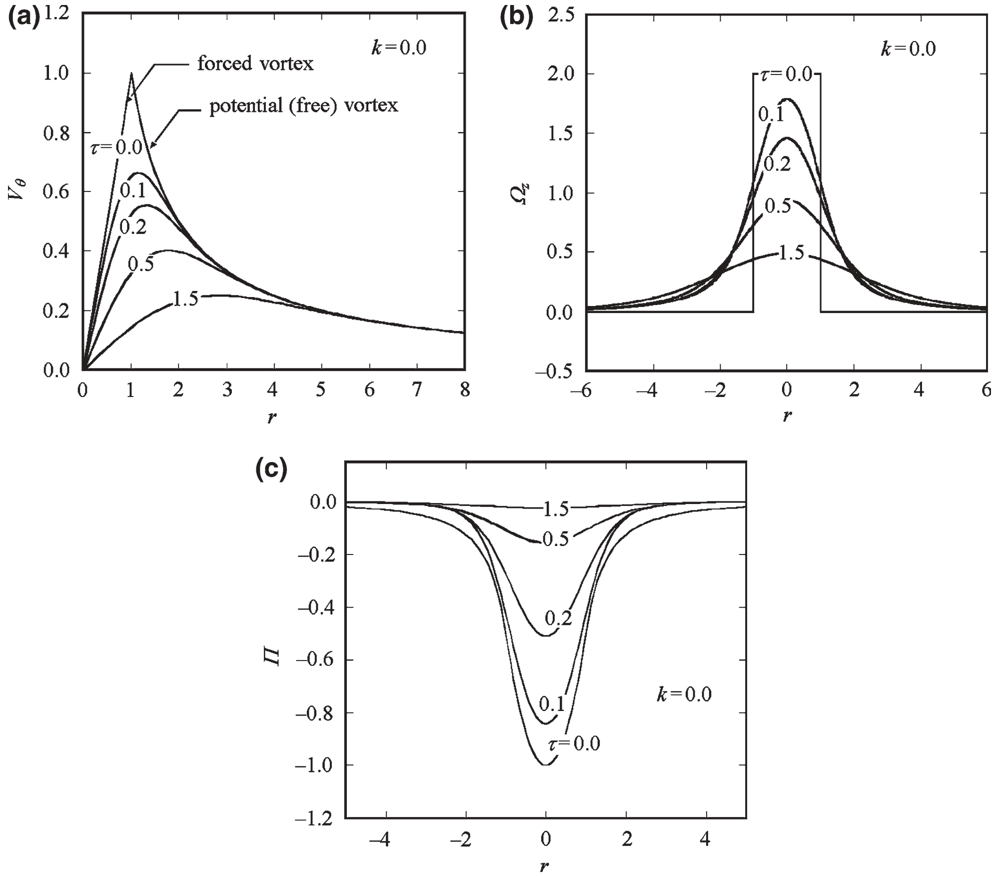

Fig. 3 Time-decay of (a) velocity, (b) vorticity and (c) pressure

$$V_\theta(r,\tau) = \beta^2 r + 4\sum_{n=1}^{\infty} \frac{J_1(\lambda_n\beta)}{\lambda_n^2 J_0^2(\lambda_n)} \times J_1(\lambda_n\beta r)\exp(-\lambda_n^2\beta^2\tau). \qquad (18)$$

The time decay distributions of the induced tangential velocity, axial vorticity and static pressure for $k = 0$ are shown in Figs. 3(a)-(c).

It is clear that, as the time progresses, the swirl velocity diminishes and the maximum velocity is seen to decrease monotonically and tend to zero asymptotically as $\tau \to \infty$. As the vortex core enlargement is accompanied by a rapid vorticity decrease, as shown in Fig. 3(b), the static pressure property diminishes by an outwards viscous diffusion in the radial direction. This response of the system is in accord with the entropy principle, which requires every disturbance to diminish by, in this case, dispersion. The static pressure develops in such a way as to balance the generated centrifugal force field. As time goes on the tangential velocity of the eddy decreases, the sides of the inverted bell-like pressure profile become less steep and the central depression gradually increases, as shown in Fig. 3(c). The present solution is compared with the experimental data given by Bennett [14] at three different times, which describe the early stage of decay. The results are found in fair agreement as shown in Fig. 4(a). The maximum values of the tangential velocity core radius are predicted adequately and compared fairly well with the experimental maximum values as shown in Fig. 4(b).

In Fig. 4(b), the divergence of $V_{\max}$ and $r_{\max}$ between the potential and Rankin vortices is given. Results show large difference during the initial stages of the decaying process, however, for $\tau >\sim 2.5$, the maximum velocity and core radius curves merge and tend asymptotically to those of the potential vortex, thus indicating that beyond this time threshold there are no appreciable differences between the two vortex models. Taylor-like vortex profiles are produced by using Eq.(11) for different k values. For instance, at $k = 1$ the time decay distributions for both tangential velocity and axial vorticity are given. The predicted profiles are found to be identical to the standard Taylor-vortex, as shown in Fig. 5. A family of monopolar vortices might exist in the same main flow. For instance, at $k = 2$ it seems that the original vortex has broken up into two vortices with different intensity and opposite direction of rotation. In this case, the induced tangential velocity changes sign after reaches the stagnation point right at the interface between those two vortices, while the vorticity changes its sign twice, as shown in Fig. 6. Comparisons between the tangential velocity and axial vorticity at the same time scale for different values of k are given in Fig. 7. The magnitude of induced swirl velocity resulting

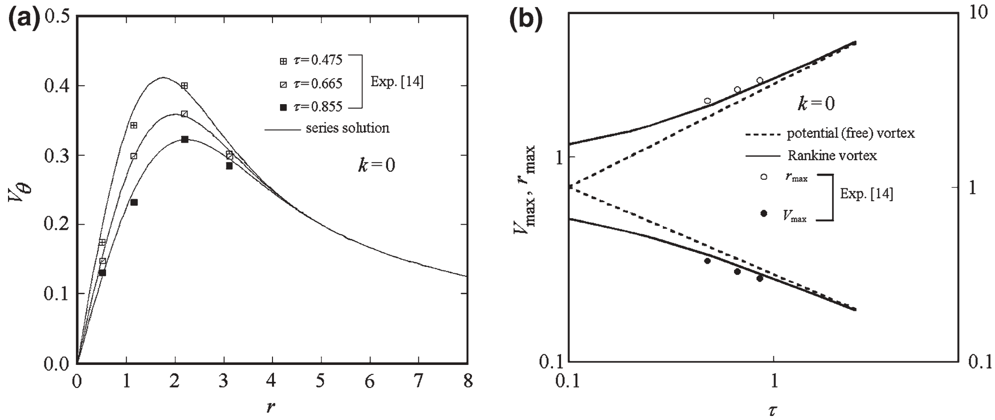

Fig. 4 (a) Theory versus experiment and (b) maximum values of the velocity and core radius versus experimental data

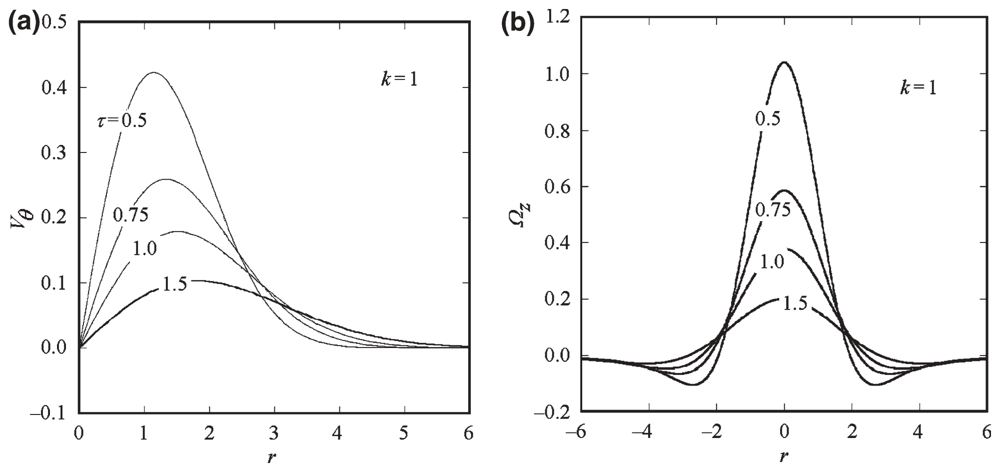

Fig. 5 Time decay of (a) velocity, (b) vorticity for $k = 1$

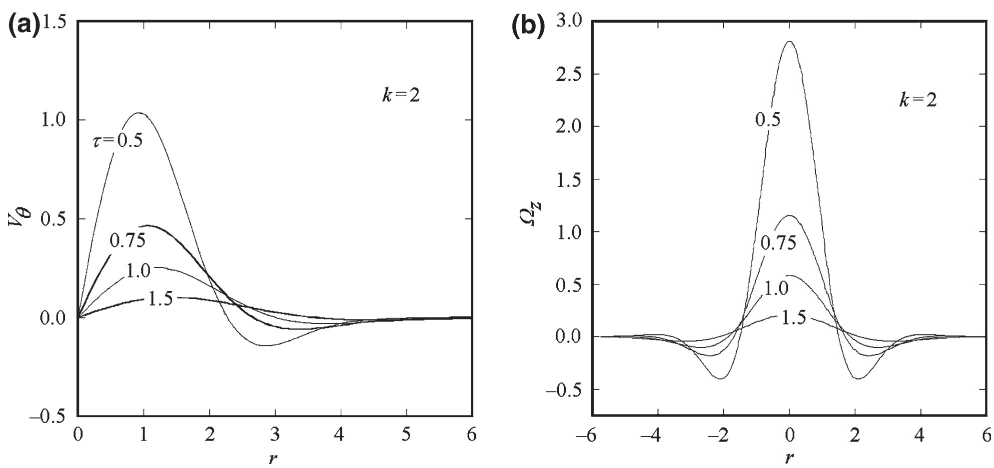

Fig. 6 Time decay of (a) velocity, (b) vorticity for $k = 2$

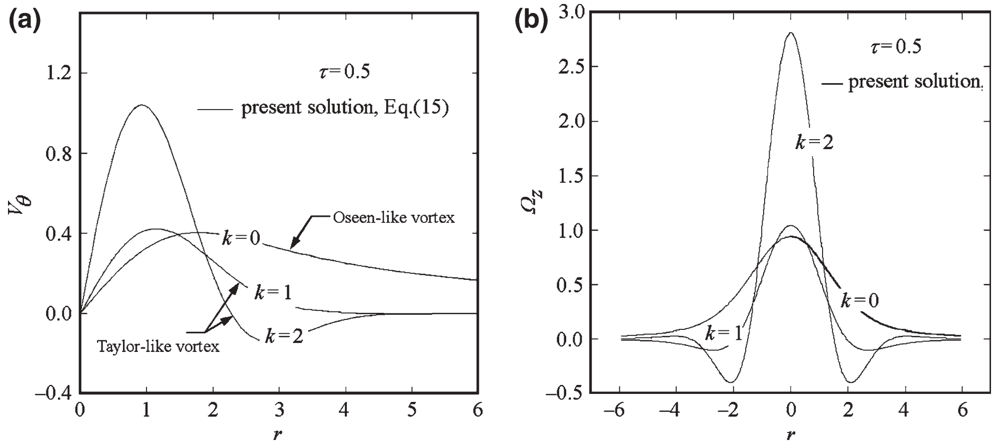

Fig. 7 Time decay of (a) velocity, (b) vorticity for $k = 1$, 2, and 3 at time $= 0.5$

from two eddies $k = 2$ is larger than the velocity induced from a single eddy of Taylor-like vortex $k = 1$ or Oseen-like vortex $k = 0$, as shown in Fig. 7.

The vortices that appear for larger value of k seem to be unstable and break up into separate vortices with different intensities. These emerged vortices rotates in opposite directions consequently, the induced swirl velocity changes periodically its sign. In general, the number of eddies created are equal to the number of k used in the series solution.

4 Conclusions

In this paper, the time decay of Rankin vortex profile is given in terms of Fourier-Bessel series. The present solution is restricted to the vortices that exhibits zero-meridional flow and of concentrated intensity. This method is kept general for any initial velocity distribution that satisfies the steady-momenta equations, such as the classical asymptotic solution given by Oseen [7]. The present results are found in fair agreement with experimental data. Expressions for monopolar-like vortices are derived from the original momentum equation by total differentiation with respect to time. Vortices for $k > 1$ are unstable and break up into small vortices.

References

1. Thomson, W. (Lord Kelvin): Vibrations of a vortex column. Phil., Mag. 10, 155 (1880), also in Gray A, Mathews G.B.: A Treatise on Bessel Functions and Their Applications to Physics, MacMillan and Co., London, 1922
2. Rankine, W.J.M.: Manual of Applied Mechanics. C. Griffen Co., London, England, 1858
3. Burgers, J.M.: A mathematical model illustrating the theory of turbulence. Adv. App. Mech. **1**, 171–199 (1948)
4. Scully, M.P.: Computation of helicopter rotor wake geometry and its influence on rotor harmonic air loads, Aeroelastic and Structures Research Laboratory, Massachusetts Institute of Technology, ASRL TR 178-1, Cambridge, MA, March, 1975
5. Gordon, L.J., Bhagwat M.J.: Generalized viscous vortex model for application to free-vortex wake and aero acoustics calculations. In: Proc. of the 58th Annual Forum and Technology Display of the Helicopter Society International, Montreal, Canada, June, 2002
6. Vatistas, G.H., Kozel, V., Minh, W.: A simpler model for concentrated vortices. Exp. Fluids **11**, 73–76 (1991)
7. Oseen C.W.: Uber Wirbelbewegune in einer reiben-den flussigkeit. Ark. J. Mat. Astrom. Fys. **7**, 14–21 (1912)
8. Hamel, G.: Spiralformige bewegung zaher flussigkeiten. Jahresber. Dt. Mathematiker-Vereinigung **25**, 34–60 (1916)
9. Lamb, H.: Hydrodynamics. Cambridge University Press. Cambridge, UK, 1932
10. Taylor, G.I.: On the Dissipation of Eddies Reprinted in the Sientific Papers of Sir. Geoffrey Ingram Taylor, 1918
11. Neufville, A. De.: The dying vortex. In: Proc. Fifth Midwestern Conf. on Fluid Mechanics, University of Michigan 1957, pp. 365–375
12. Kirde, K.: Untersuehungen uber die zeitliche weiterentwicklung eines wirbels mit vorgegebener anfangsverteilung. Ingenieur Archive. **Xxxi**, 385–404 (1962)
13. Aboelkassem, Y.: On the decay of strong isolated columnar vortices. Master of Applied Science, Department of Mechanical and Industrial Engineering, Concordia University, Montreal Canada, 2003
14. Bennett T. J.: Vortex coalescence and decay. [Ph.D. dissertation], Department of Civil and Environmental Engineering, Washington State University, USA, 1988
15. Batchelor G. K.: An Introduction to Fluid Dynamics. Cambridge University Press, 1970