An undergraduate experiment on the propagation of thermal waves

A. Bodas, V. Gandía,[a)] and E. López-Baeza[b)]
Universitat de València, Facultat de Física, Departament de Termodinàmica, Calle Dr Moliner 50, Burjassot, 46100 València, Spain

(Received 30 May 1997; accepted 26 November 1997)

When a long thin bar of a material is subject to periodic heating, a temperature wave propagates along the sample, and its propagation properties can be used to determine the thermal diffusivity of the material. In this paper, we present a simple experiment that can be used by undergraduate students to better understand thermal waves and their propagation and to measure thermal diffusivity. In spite of its simplicity, the experiment provides acceptable results for a copper sample.

I. INTRODUCTION

During their first undergraduate years, physics students become acquainted with the wave equation mainly through vibrating strings and electromagnetic field propagation. It is highly likely that only a few of these students are familiar with the phenomenon of heat propagation in solids. One of the reasons for this may be that they have spent less time on the in-depth study of heat propagation phenomena as compared with other specific phenomena in mechanics or electromagnetism during their first years of study. This paper gives students a concrete example of a wave equation with properties slightly different from waves resulting from the more complicated Schrödinger or Maxwell equations. Our aim is to give an account of a simple experiment that we have been offering to our undergraduate students for some years now. Besides providing a conceptual approach to thermal waves and their propagation, the experiment also allows students to perform some measurements on the thermal diffusivity of solids. In fact, the measurements proved to be quite accurate in spite of the simplicity of the experimental equipment involved.

Thermal waves are a simple example to initiate students into the study of wave phenomena that are of fundamental importance in physics. They may facilitate the subsequent study of more complex wave phenomena related to the Maxwell and Schrödinger equations. It is worth noting that although these equations belong to fields in physics other than heat conduction, they still have common points such as the conservation laws associated with the different fluxes. These fluxes, namely, the heat flux, the current density, and the probability current lead to their corresponding continuity equations. This similarity of the heat equation to other field equations provides the heat flux with an intrinsic importance, comparable to that of the current density in electromagnetism or the probability current in quantum physics.

The importance of the heat conduction equation may be seen when we realize that it is implicitly present in many everyday objects. For example, the coiled design of fireplace pokers and soldering iron holders provides increased heat dissipation in order to reduce conduction toward the handler. Also, the development of new clothing materials may respond to the necessity of having good thermal insulation at the same time as we require a lightweight resistant fabric, and even with waterproof qualities. Other examples may refer more specifically to the propagation of thermal waves, as in the case of those waves which are not damped in liquid Helium-II due to its very high thermal conductivity, which approaches infinity for small heat currents. Another interesting example is related to the oscillations of soil temperature. Thus the daily temperature variations penetrate the soil more rapidly, but less deeply than the annual variations due to seasonal changes. This may be explained in terms of the dependence of the wave propagation velocity and damping coefficient on the frequency of oscillation.

The reason for the interest in the heat conduction equation is that it is one of the fundamental linear field equations of physics. It is not as important as Schrödinger's or Maxwell's equations, but is somewhat simpler. This simplicity, combined with the many properties in common with the more essential field equations (i.e., the superposition principle), makes it ideal for learning physics in an experimental setting where most variables are measurable.

Our experiment consists of periodically heating one of the ends of an isolated long thin metal rod and, once the dynamical thermal equilibrium is achieved, determining the material thermal diffusivity just by monitoring temperature variations at two points along the bar. There are in fact two different approaches that exploit two different characteristic wave properties, namely, the amplitude decay along the bar as we travel away from the heating end, and the thermal-wave phase velocity.

The paper is divided into the following sections: Through the theoretical analysis, Sec. II relates thermal-wave properties, as experimentally measured, to thermal properties of matter. Section III describes the experimental equipment and the measuring procedure applied. Some results for a copper sample are given in Sec. IV and, finally in Sec. V, some conclusions are drawn.

II. THEORETICAL ANALYSIS

The fundamental heat conduction equation that gives the temperature distribution in a homogeneous long thin bar is Fourier's equation, which, for the one-dimensional case and for an isolated bar, can be written as[1]

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\kappa} \frac{\partial \theta}{\partial t}, \tag{1}$$

where $\theta(x,t)$ represents the temperature oscillation with respect to the mean temperature in (x,t) and κ is the material thermal diffusivity. It is easy to see that this equation is not valid for a nonhomogeneous solid. The ''heat current'' is $I = a(x)\ \partial\theta/\partial x$, where $a(x)$ is related to the heat conductivity. Conservation of energy implies that the divergence of I (i.e., $\partial I/\partial x$) is proportional to $\partial\theta/\partial t$. For a homogeneous rod, $\partial[a\,\partial\theta/\partial x]/\partial x = a(\partial^2\theta/\partial x^2)$, which leads to Eq. (1).

Heating in our experiment was actually achieved by inserting a soldering iron in the end of the sample, whose temperature oscillated as a periodic step function. The temperature oscillation at the origin of the bar could then be expressed as

$$\theta(0,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4\theta_0}{n\pi} \sin\frac{2n\pi t}{\tau}, \tag{2}$$

where the time origin has been chosen so that $\theta(0,t)$ is odd.

At the opposite end, we assume that there is no temperature oscillation, $\theta(\infty,t)=0$. It is clear that the bar is not semi-infinite, and this condition actually means that the thermal wave should have been completely damped out at the far extreme so that there is no wave reflection. In our case, we were able to check that the wave almost completely vanished at about 40 cm, the bar being 50 cm long.

Since we are interested in the long-term temperature distribution once the system has forgotten the initial conditions, we try the Fourier series[2]

$$\theta(x,t) = \sum_{n=1}^{\infty} A_n(x)\sin(\omega_n t - k_n x) \tag{3}$$

as a solution, where A_n, ω_n, and k_n are, respectively, the amplitude, frequency, and wave number of the nth harmonic.

By introducing this function in the Fourier equation and applying the boundary conditions, we obtain the solution

$$\theta(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \theta_n e^{-\epsilon_n x} \sin(\omega_n t - \epsilon_n x), \tag{4}$$

where

$$\theta_n = \frac{4\theta_0}{n\pi}, \tag{5}$$

$$\omega_n = \frac{2n\pi}{\tau}, \tag{6}$$

and

$$\epsilon_n = \sqrt{\frac{\omega_n}{2\kappa}} \tag{7}$$

is the nth harmonic damping coefficient.

We can see that each term has its own associated damping coefficient, and that the higher harmonics damp out more quickly because the damping coefficient increases with frequency. As a consequence, we may approximate the temperature distribution by just the first harmonic, if sufficiently far from the origin, and therefore we can redefine our coordinate origin to the point of the sample where the presence of the second harmonic is already negligible, thus approximating $\theta(x,t)$ from this point by means of

$$\theta(x,t) \cong A_0 e^{-\epsilon x} \cos \omega(t - x/v), \quad (8)$$

where we have chosen the time origin so that θ is maximum at $x=0$. In our case, this new origin is given by the first measuring point, 8 cm from the sample origin.

Temperature oscillation propagates to the neighboring points with the following characteristics:

(1) the oscillation amplitude decreases with the distance from the origin O, damping out exponentially, and
(2) the maximum temperature value appears at a point in time after it appears at O.

This progressive propagation of the temperature oscillation is what we call a thermal wave.

We see that the amplitude decays exponentially with distance and that the thermal wave is periodic both in time, with a period τ, and in space, with a wavelength $\lambda = v\tau$.

Taking into account Eq. (8) and the first harmonic of the complete solution, Eq. (4), we can then obtain the expressions

$$\kappa_\epsilon = \frac{\pi}{\tau\epsilon^2} \quad (9)$$

and

$$\kappa_v = \frac{v^2\tau}{4\pi} \quad (10)$$

that relate the thermal properties of the material, in our case thermal diffusivity, κ, to the two wave properties velocity, v, and damping coefficient, ϵ, that can be experimentally determined. In principle, we have distinguished between κ_ϵ and κ_v to account for the eventual differences we might get in thermal diffusivity, coming from the application of different experimental parameters. This means that although thermal diffusivity of a particular material should be the same whether calculated from ϵ or from v, however, the influence of thermal losses in the experimental determination of the two wave properties could be different.[3] Then, by calculating ϵ/v from Eqs. (9) and (10) and assuming $\kappa_\epsilon = \kappa_v$, we may obtain an expression for thermal diffusivity that combines both wave properties simultaneously. Thus

$$\kappa = \frac{v}{2\epsilon}. \quad (11)$$

Equation (1) has been established assuming the ideal case of not considering thermal losses in the experiment. Since this is not the actual case, the inclusion of a loss term in Eq. (1) would provide other expressions for ϵ and v, depending explicitly on the losses. However, it can be shown[3] that the ratio ϵ/v remains constant because thermal losses cancel out mathematically when combining both wave properties, and Eq. (11) still holds. The determination of thermal diffusivity by accounting for the effects of heat losses has been studied in more detail,[4,5] but is not considered here.

Both the damping coefficient and the wave velocity can be obtained from the temperature oscillations at two points along the bar. The damping coefficient can be determined from the amplitude decay and the wave velocity from the wave shift at these two points, as we shall show later on. This can be done if the wave is perfectly harmonic at these two points because both quantities depend on the frequency.

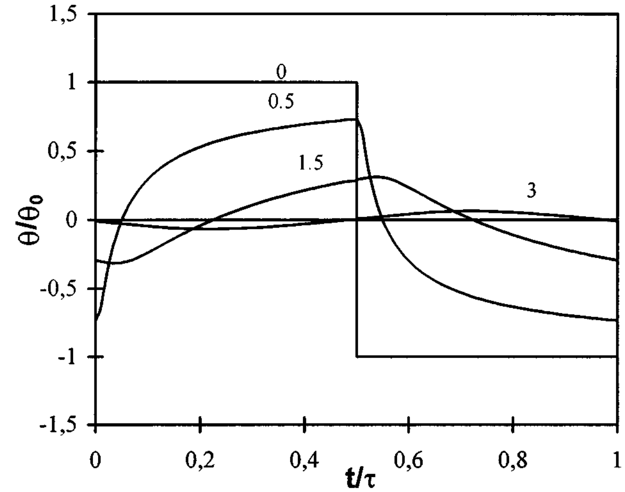

Fig. 1. Theoretical progressive harmonization of the wave at different distances from the sample origin.

In order to see the progressive damping and shifting of the wave as it advances along the bar more clearly, Fig. 1 shows $\theta(x,t)/\theta_0$ vs t/τ for four different values of $x\sqrt{n\pi/\kappa}$, namely, 0 (the very beginning of the bar), 0.5, 1.5, and 3. In the graph we may observe the progressive harmonization of the wave, thus raising the question as to when the wave is sufficiently harmonic to be able to approximate it by a cosine function as in Eq. (8). We can make some estimates of the minimum distance from where the wave can be represented only by the fundamental harmonic by calculating the ratio of successive harmonics to the fundamental one and establishing *a priori* the weight allowed to the second harmonic at the first measuring point. This ratio will be given by

$$P_n(x) = \frac{A_n}{A_1} = \frac{1}{n}\, e^{-\sqrt{\pi/\tau\kappa}(\sqrt{n}-1)x}, \quad n = 1,3,5,\ldots . \quad (12)$$

For example, by introducing an approximate value of the thermal diffusivity of copper, we obtain $\sqrt{\pi/(\tau\kappa)} = 0.132\ \mathrm{m}^{-1}$, so that at the first measuring point, we would have $P_5(8\ \mathrm{cm}) \approx 0.05$, meaning that the third-harmonic amplitude represents about 5% of the fundamental one. It is worth noting that the actual weight of these harmonics at the measuring points is notably less because, in our case, the boundary condition at the initial end was not exactly a step function, strictly speaking. The heater was not completely inserted into the sample but part of it remained outside, thus making the thermal wave undergo the beginning of its harmonization before actually reaching the sample, in the heater itself. In this way, the degree of wave harmonization is actually greater than the one represented by the series solution, Eq. (4), which is the exact limiting case that has permitted us to carefully choose the first measuring point to carry out the experiment.

III. EXPERIMENTAL METHOD

In this section we describe the experimental equipment used in the laboratory and the method followed to conduct the experiment.

A. Equipment

Figure 2 shows the experimental setup. The sample was a cylindrical copper bar, 50 cm long, and 15 mm diameter with

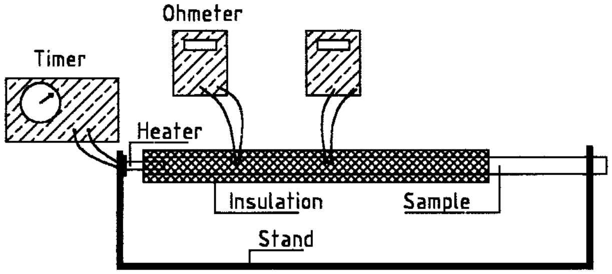

Fig. 2. Experimental setup.

temperature measuring holes of 2.5 mm diameter and 5 mm depth, placed at 4 cm intervals. The bar was isolated by introducing it snugly within a cylindrical tube of 5-mm-thick transparent methacrylate. These two elements were mounted on a plastic stand.

The sample was heated by an 11-W soldering iron partly embedded in the end of the sample and connected to a timer.

Since the temperature changes periodically, it is necessary to take as many measurements as possible within a given period of time, and this was done by using a thermistor of 4.7 kΩ at 25 °C that has very small thermal inertia. It was a bead thermistor of 2 mm diameter embedded in glass with a thermal characteristic given by the expression

$$R = Ae^{B/T}, \tag{13}$$

which had previously been calibrated between 30 and 40 °C, giving the values $A \cong 1.9 \times 10^{-2}$ Ω and $B \cong 3600$ K.

The accuracy achieved in measuring temperature with this thermistor can be estimated from its temperature coefficient α, given by

$$\alpha = \frac{1}{R}\frac{dR}{dT} \cong -\frac{B}{T^2} = -0.04 \ \ \mathrm{K}^{-1}. \tag{14}$$

Since the temperature excursions produced on the bar are small, we can approximate the resistance variations linearly with respect to temperature variations so that

$$\frac{\Delta R}{\Delta T} \cong R\alpha, \tag{15}$$

and therefore the error in measuring temperatures is of the order of $\Delta T \cong -0.011 \Delta R$.

The measurement of thermistor resistance is performed by means of a digital multimeter used as an ohmmeter.

Periodic sample heating is achieved by using a combining switch that activates the heating system at time intervals according to two timers, one to control the heating time period and the other to control the period during which the heating system is switched off. The connecting system is fed by a 9-V battery and changes their contacts when it is activated by the corresponding timer.

B. Experimental procedure

The metal bar is heated in a pulsing periodic way at 80-s intervals, using the timing system mentioned above and controlling temperature until a dynamical equilibrium state is achieved without temperature drifts. This state is reached when temperatures at the measuring points oscillate around their respective mean values. From this moment, we measured thermistor resistances at two points situated at 8 and 16 cm, respectively, from the bar origin. These measurements

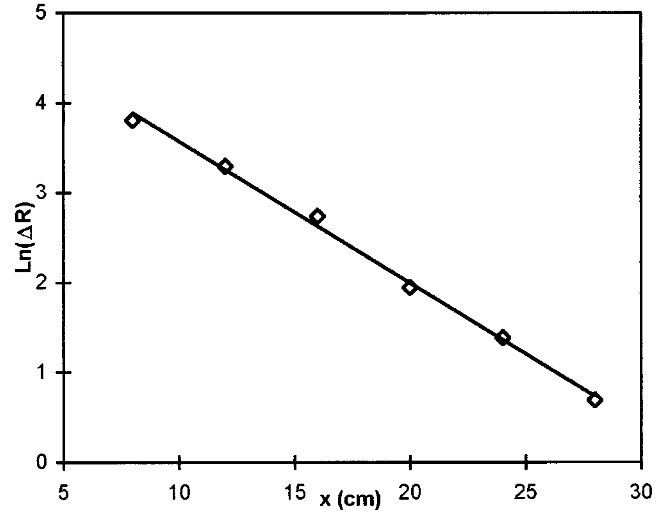

Fig. 3. Variation of the natural logarithm of the amplitude with length.

were taken every 10 s during several periods with the aid of a digital stopwatch. With these data, we may obtain the amplitudes of the thermal wave at both measuring points. If we consider that the temperature oscillations are proportional to the thermistor resistances in that temperature interval, we can determine the thermal-wave damping coefficient from the amplitude decrement between both points according to

$$\epsilon = \frac{1}{\Delta x} \ln \frac{A_1}{A_2}, \tag{16}$$

where Δx is the distance between the measuring points, 8 cm in our case, and A_1 and A_2 are the respective amplitudes. Equation (16) has been obtained from the theoretical analysis that shows that the amplitude decays exponentially. To confirm this, we have measured temperature oscillations along the bar at four different positions situated at 8, 12, 16, 20, 24, and 28 cm from the sample origin, and Fig. 3 shows the variation of the natural logarithm of the amplitude with length.

The wave velocity may be obtained from the graphical analysis of $R - \bar{R}$ versus time, determining the time lag Δt between both measuring points. This is done by measuring the time difference between two equal-phase points from both graphs such as two maxima or two minima or two intercepts with the time axis. The wave velocity will then be $v = \Delta x / \Delta t$.

IV. DATA AND RESULTS FOR A COPPER SAMPLE

In this section we present an example of the experiment described applied to a copper sample, using data and results actually obtained by undergraduate students.

Figure 4 shows the data obtained at the two measuring points situated at 8 and 16 cm, respectively, once the dynamical state of equilibrium was achieved and we were sure that there were no temperature drifts. These data correspond to thermistor resistances directly measured on the sample, and it is easy to check that the temperature oscillations are small by means of Eq. (13). Although we can determine the wave properties related to thermal diffusivity through Eqs. (9) and (10) from the data shown in Fig. 4, namely the damping coefficient and the wave velocity, we should make sure

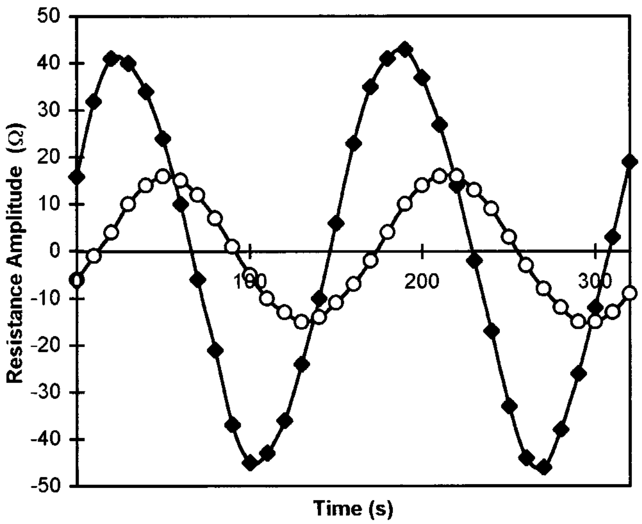

Fig. 4. Resistance amplitudes at 8 and 16 cm.

that the bar was actually isolated, or at least that there were negligible heat losses. Thus we calculated the mean temperatures at different measuring points and plotted them against their distances from the origin. Figure 5 shows these results, where we may see that the graph obtained follows a straight line corresponding to the steady state temperature distribution of an isolated bar, thus confirming the validity of Eqs. (9) and (10) that were obtained on the hypothesis of no heat losses along the bar.

The damping coefficient and the wave velocity calculated for these two thermal waves between both measuring points were $\epsilon = 13.2 \pm 0.7\ \mathrm{m}^{-1}$ and $v = (3.1 \pm 0.2) \times 10^{-3}\ \mathrm{m\ s}^{-1}$, respectively. According to Eqs. (9) and (10), these two quantities respectively give the thermal diffusivity of copper as $\kappa_\epsilon = (1.13 \pm 0.12) \times 10^{-4}\ \mathrm{m}^2\ \mathrm{s}^{-1}$ and $\kappa_v = (1.21 \pm 0.16) \times 10^{-4}\ \mathrm{m}^2\ \mathrm{s}^{-1}$. Thus the estimated value of the thermal diffusivity of copper could be obtained as the average of both κ_ϵ and κ_v, resulting in $\overline{\kappa} = (1.17 \pm 0.14) \times 10^{-4}\ \mathrm{m}^2\ \mathrm{s}^{-1}$. This result agrees with that obtained from Eq. (11), $\overline{\kappa} = (1.17 \pm 0.14) \times 10^{-4}\ \mathrm{m}^2\ \mathrm{s}^{-1}$, which could be considered as a physical average. Both results are also in good agreement with others found in the literature[6–8] and with the value obtained from measurements of thermal conductivity and heat capacity per unit volume.[9] The most recent reference found for the thermal diffusivity of copper is $\kappa = 1.020 \times 10^{-4}\ \mathrm{m}^2\ \mathrm{s}^{-1}$, which was obtained by periodic heating with a sinusoidally modulated laser beam.[10]

The value of $v = (3.1 \pm 0.2) \times 10^{-3}\ \mathrm{m\ s}^{-1}$ obtained for v, the propagation velocity, deserves some discussion. This value is, on the one hand, considerably larger than the drift velocity of conduction electrons in copper, which is of the order[11] of $4 \times 10^{-7}\ \mathrm{m\ s}^{-1}$, and on the other hand, a great deal smaller than the velocity of sound propagation in this material, of the order of $3000\ \mathrm{m\ s}^{-1}$, for usual values of the Young's modulus and the density of copper. These three different values should not surprise us since they proceed from processes that respond to different propagation mechanisms. In metals, heat propagates mainly by conduction through the free electrons that transfer energy from warmer to cooler regions, whereas the drift velocity of conduction electrons is due to the application of an external electric field. With regard to sound propagation, it is the lattice which transmits the perturbation. Furthermore, we should also consider the dependence of the velocity of propagation on the frequency, $v = \sqrt{2\kappa\omega}$, which means that the solid is a dispersive medium, and therefore the wave velocity increases with the heating frequency. Thus, if ω tends to infinity, v would approach infinity as well, which is physically inadmissible. The problem is that the conduction equation itself implies an infinite velocity of propagation and therefore is not valid in the whole frequency range. The problem of obtaining a conduction equation with finite propagation velocity has been widely studied in the literature.[12,13]

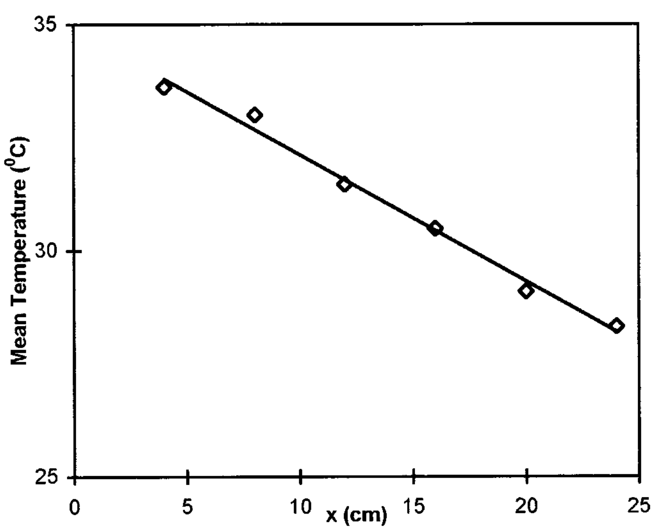

Fig. 5. Variation of the mean temperature along the bar.

V. CONCLUSIONS

The heat equation is one of the simplest partial differential equations of physics, and this is why it may be considered as an ideal tool to introduce oneself to this type of equation. It could be the first step for the study of other equations such as Maxwell's or Schrödinger's that explain wave phenomena of great relevance in physics.

From our viewpoint, a laboratory experiment on thermal-wave propagation in solids at the undergraduate level provides students with the opportunity to get acquainted with heat conduction in a way that is essentially different from that of classical experiments on stationary heat transmission. This type of experiment also allows students to learn thermal diffusivity measuring techniques in a simple and pedagogical way.

We believe that it is good for students to make contact with nonstationary phenomena such as the heat conduction process, thus permitting a better understanding of Fourier's equation. In dynamical problem studies such as the one under consideration, the theoretical analysis along with its practical realization helps to show the student the essential differences between a conceptual setting of a problem and the final and complete achievement of an experimental result, by assuming some simplifications, making approximations, determining the accuracy of the result, etc. Thus, in our case on thermal wave propagation or determination of thermal properties of matter, we have tried to simplify the experimental equipment without detriment to obtaining acceptable results for the thermal diffusivity of copper. This is actually a matter that we consider of fundamental interest for our students, in that conducting this experiment in the laboratory not only gave an opportunity to make contact with the heat conduction process, but also the chance to teach (and learn) experimental techniques and measuring methods to determine solid properties, in this case thermal diffusivity, with a certain degree of accuracy.

We should finally point out that the accurate determination of thermal properties of solids should motivate more in-depth studies from a microscopic perspective, thereby providing an adequate interpretation of the measurements.

[a]Deceased 27 December 1997.
[b]Author to whom all correspondence should be addressed; electronic-mail: Ernesto.Lopez@uv.es

[1]F. Kreith and W. Z. Black, *Basic Heat Transfer* (Harper & Row, New York, 1980), p. 142.

[2]H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Oxford U.P., Oxford, 1969), 2nd ed., pp. 68–69.

[3]J. E. Parrott and A. D. Stuckes, *Thermal Conductivity of Solids* (Pion, London, 1975), pp. 24–28.

[4]E. Lopez-Baeza, J. de la Rubia, and H. J. Goldsmid, ''Angström's thermal diffusivity method for short samples,'' J. Phys. D: Appl. Phys. **20**, 1156–1158 (1987).

[5]E. Lopez-Baeza, J. de la Rubia, and H. J. Goldsmid, ''Angström's method for short samples with consideration for heat losses,'' in *The First European Conference on Thermoelectrics*, edited by D. M. Rowe (Peter Peregrinus, on behalf of the IEE, London, 1989), Chap. 10, pp. 120–134.

[6]C. Y. Ho, R. W. Powell, and K. Y. Wo, ''Thermal diffusivity of the elements,'' *Proceedings of the Eighth Conference on Thermal Conductivity*, Purdue University, 1969 (Plenum, New York, 1969), p. 971.

[7]Y. S. Touloukian, R. W. Powell, C. Y. Ho, and M. C. Nicolau, *Thermophysical Properties of Matter*, Thermal Diffusivity Vol. 10 (IFI/Plenum, New York, 1973).

[8]L. Verdini and A. Santucci, ''Propagation properties of thermal waves and thermal diffusivity in metals,'' Nuovo Cimento B **62**, 399–421 (1981).

[9]Thermal diffusivity, thermal conductivity and heat capacity per unit volume are related through the expression $\kappa=\lambda/\rho c_p$, where λ is the thermal conductivity, ρ the density, and c_p the specific heat at constant pressure. Values of these quantities can be found in the *Handbook of Chemistry and Physics*, edited by R. C. Weast (Chemical Rubber, Boca Raton, FL, 1986), 67th ed., providing a final thermal diffusivity value of $\kappa=1.17 \cdot 10^{-4}\ \mathrm{m^2\,s^{-1}}$.

[10]W. Czarnetzki, M. Wandelt, and W. Roetzel, ''Thermal wave analysis for measurements of thermal diffusivity,'' *Proceedings of the Joint Conference 1996: IEEE Instrumentation and Measurement Technology Conference and IMEKO Technical Committee 7*, Brussels, Belgium, 4–6 June 1996 (IEEE, New York, 1996), pp. 1195.

[11]This order of magnitude has been estimated by adapting an example of Tipler's to the conditions of our sample (cross-sectional area of the order of $175\ \mathrm{mm^2}$). See P. A. Tipler, *Physics for Scientists and Engineers* (Worth, New York, 1991), 3rd ed., p. 718.

[12]H. D. Weymann, ''Finite speed of propagation in heat conduction, diffusion, and viscous shear motion,'' Am. J. Phys. **35**, 488–496 (1967).

[13]K. Etori, ''Remarks on the temperature propagation and the thermal diffusivity of a solid,'' Jpn. J. Appl. Phys. **11**, 955–957 (1972).