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Resolution enhancement by dithering

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We show that the uncertainty determined by the minimum division of a measuring instrument can be diminished by using dithering. We present a numerical example to introduce the technique and two experiments that show how the precision is enhanced.
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I. INTRODUCTION

The evaluation of uncertainties and errors is an important topic in introductory physics and chemistry laboratory courses. Errors are usually classified into two groups depending on their source:[1,2] systematic or bias errors, and random or statistical errors. The first group includes, for instance, miscalibrated scales or bad zero settings. They can be avoided if the meter is checked against a standard, or realistically, against another and hopefully better meter. Random errors may be due, for example, to the inability of the observer or apparatus to determine a quantity exactly (usually called instrumental uncertainty), to the fact that the quantity being measured is not precisely defined, to the essentially random nature of the phenomenon, or to fundamental limits of measurement (for example, the uncertainty principle). The result of these kinds of errors is a spread of readings on repetition of the measurement under apparently the same conditions. This spread can be used to estimate the inaccuracy or uncertainty of the observation due to random errors. This kind of errors is usually reduced when the experiment is repeated many times and they can be analyzed by using statistical techniques. It also is important to verify that the fluctuations observed in the measurements are random, and not biased or correlated in time. Note that when we talk about errors, we do not include "mistakes," such as the misreading of a scale, the incorrect recording of a number or a mistaken calculation.

In some cases each measurement is *scale limited*, that is, its statistical uncertainty is smaller than the smallest increment we can read on the instrument scale. Such a measurement will yield exactly the same value for repeated measurements of the same quantity. It is usually stated that in these cases, the uncertainty must be quoted as the smallest increment that can be read on the scale (or some fraction of it).[1,2] Most of the instruments that are used in experiments involve a discretization of the measured quantity. This is true, not only for common instruments like rulers, where the measurement is limited by a minimum division, but also for more sophisticated instruments in which analog to digital conversion is involved and a least significant bit (LSB) always exists. In other cases, where statistical fluctuations are observed in the measured data, it is said that the statistical uncertainty can be reduced to the bound given by the smallest increment that can be read on the scale. This bound is due to the fact that the total error is the square root of the sum of the squares of the individual errors, that is, the instrumental and the statistical ones.[3] Our aim is to show that the scale-limited or LSB uncertainty (also known as reading error) can be diminished by using a well-known technique developed for the quantization of sound and images called *dithering*.[4,5]

The article is organized as follows. In Sec. II we introduce the dithering theory and explain how it can be used to reduce the scale limited uncertainty. In Sec. III we describe a numerical example and two experiments to show how dithering can be used to diminish the instrumental error. Some conclusions are presented in Sec. IV.

II. DITHERING

Digital signal processing is a powerful collection of techniques in signal analysis that are commonly used in scientific research.[6] Usually the digital sequence to be processed is obtained by sampling a bandlimited signal at discrete intervals of time. The sequence of samples thus obtained is stored in finite word length registers. The conversion from a continuous (analog) signal to a digital one consists of two operations: time discretization and amplitude quantization. Time discretization, if properly applied, can be shown to be error free. However, the effects of amplitude quantization always are present and manifest themselves in several ways (for example, distortion due to nonlinear response characteristics and loss of signal detail that is comparable in size to the quantization step).

To treat such effects, a method has been developed in which the notion of quantization error plays a central role.[4,5,7–9] This error is introduced by the coarseness of the amplitude quantization in the analog to digital conversion, and is treated as additive noise whose statistical properties depend upon the input analog signal.[4,7] It has been shown that the minimum loss of statistical data from the input occurs when the quantization error can be made independent of the input.[8,9] The technique proposed to realize such an independence condition is to use an added external signal (dither) before quantization.

It has been shown that the addition of a proper dither signal can cause the independence and whitening of the quantization error, resulting in both a reduction of the signal distortion and an improvement of the system dynamic range.[4,6–9] Also, it has been shown that the best choice for the dither signal is a random dither uniformly distributed within an interval of amplitude equal to the quantization

Table I. Values of $\langle M(y)\rangle = \langle \text{int}(x+0.5+u)\rangle$, S, S_N, the result of the dithered measurement, and the difference between the result of the experiment and the true value for different values of N. $x=145.583\,98$, u a random number with a uniform $[-0.5,0.5]$ distribution. The seed of the random number generator was changed in each simulation.

N	$\langle M(y)\rangle$	S	S_N	Result	$\langle M(y)\rangle - x$
10	145.70	0.48	0.15	145.70 ± 0.15	0.11 ± 0.15
100	145.630	0.48	0.048	145.630 ± 0.048	0.046 ± 0.048
1000	145.577	0.49	0.015	145.577 ± 0.015	-0.007 ± 0.015
10 000	145.580	0.494	0.005	145.580 ± 0.005	-0.004 ± 0.005

step. Recently, the quantization error for Gaussian and uniformly distributed dither of different amplitudes has been studied,[10] and a connection between quantization with dithering and stochastic resonance has been established.[10–12]

The dithering technique is not only well known in image and sound processing (for which this technique was originally developed), but it is used in oscilloscopes and acquisition cards. In modern oscilloscopes the enhanced resolution of 12 bits for single shot waveforms is obtained using an 8 bit analog to digital converter based on the principle of averaging successive points sampled at a higher rate. If the noise present in the measured quantity (which in these applications is not added on purpose) is statistically independent of this magnitude, then an excellent resolution improvement is obtained based on the dithering principle.[13] A similar method is used by the MPLI program by Vernier when data is sampled at rates lower than 800 Hz.[14] (At higher sample rates, the acquisition rate does not allow the averaging of points.)

III. EXAMPLES

A. Numerical simulation of measurement with dithering

We introduce the following simulation to introduce the dithering technique. We want to measure a quantity x, that will be represented by a real number. We choose an arbitrary value for x, for example,

$$x=145.583\,98. \tag{1}$$

In order to measure x, we use an instrument that will be modeled by the nonlinear operation $M(x)=\text{int}(x+0.5)$, that is, our instrument measures the integer part of $(x+0.5)$. The scale limit or reading error in this case will be 1. If the quantity x is measured, the result, $M(x)$, would be

$$M(x)=146\pm0.5. \tag{2}$$

Of course, the same result will be obtained for repeated measurements. In this case, each measurement is scale limited and the uncertainty is quoted as the smallest division of the instrument. We are assuming that the measuring process is free of systematic errors. The finite number of decimal digits of x represent the fact that there always exists a finite natural limit to the number of significant figures of a quantity that is determined by changes in variables that are out of experimental control (for example, temperature drifts and atomic size) and in this example is the precision of the variables used in the simulation. Dithering will be done by adding a random number, u, with a uniform distribution over $[-0.5,0.5]$, to x. Now, the results of our measurement will be either

$$M(y)=(145\pm0.5) \quad \text{or} \quad M(y)=(146\pm0.5), \tag{3}$$

where $y\equiv x+u$ is the dithered variable. $M(y)$ is now measured N times, and the mean value,

$$\langle M(y)\rangle = \sum_{i=1}^{N} \frac{M(y_i)}{N} = \sum_{i=1}^{N} \frac{M(x+u_i)}{N}, \tag{4}$$

the estimate of the standard deviation of $M(y)$,

$$S=\left[\frac{1}{N-1}\sum_{i=1}^{N} (M(y_i)-\langle M(y)\rangle)^2\right]^{1/2}, \tag{5}$$

and the estimate of the standard deviation of $\langle M(y)\rangle$,

$$S_N=\frac{S}{\sqrt{N}}, \tag{6}$$

for different values of N are evaluated. The results are shown in Table I.

If we compare the results shown in Table I with the value of x (see the last column of Table I), we notice that by performing 10 000 measurements, we are able to determine two decimal digits of the value of x using an instrument that can measure only integers. Because we are doing a numerical example, we know the real value of x. [Of course, we would not have this knowledge in a real experiment where only $M(x)$, the measured value, can be obtained.] Thus, we can determine whether our simulated results are consistent with the actual value of x. We observe that in all cases the differences between our results and x are smaller than the error obtained from the estimate of the standard deviation of the measured data. Notice that the probability of obtaining $\langle M(y)\rangle$ in the interval $[\langle M(y)\rangle - S_N, \langle M(y)\rangle + S_N]$ is 68%, provided that S_N is a good estimator of the standard deviation of $\langle M(y)\rangle$.

We also have done 100 series of 10 000 measurements to verify if the statistical hypothesis needed in order to use dithering techniques is fulfilled. In Fig. 1(a) we show the result of these 100 measurements in the order that they were obtained (in all cases $S_i\approx0.5$ and $S_{N_i}\approx0.005$). By using linear regression, we observe that there is no linear correlation between the points. In Fig. 1(b) we show a histogram of the 100 measured dithered data. A Gaussian fit gives a mean of 145.585 with a standard deviation of 0.005. These two tests are the way in which one can determine if the improved resolution is consistent with the experiment itself. We observed that the different series of 10 000 measurements seem to be statistically independent and that the histogram has a Gaussian distribution. A similar example can be done in the laboratory by using a ruler with a resolution of 1 cm to measure the size of an object, provided that the appropriate noise is added to the measurement process.[15]

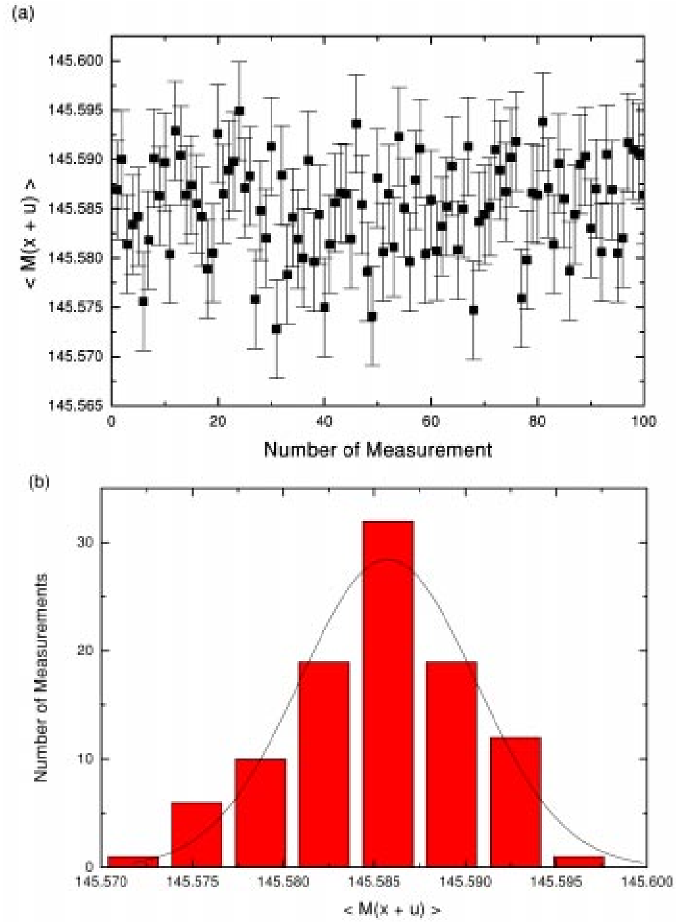

Fig. 1. (a) 100 measurements, $\langle M(x+u)\rangle$, obtained by taking the integer part of a real number after adding noise and averaging over 10 000 points, in the order that they were obtained. (In all cases $S_i\approx 0.5$ and $S_{Ni}\approx 0.005$.) (b) Histogram of the 100 measured dithered data. The mean value is 145.585 and the standard deviation is 0.005.

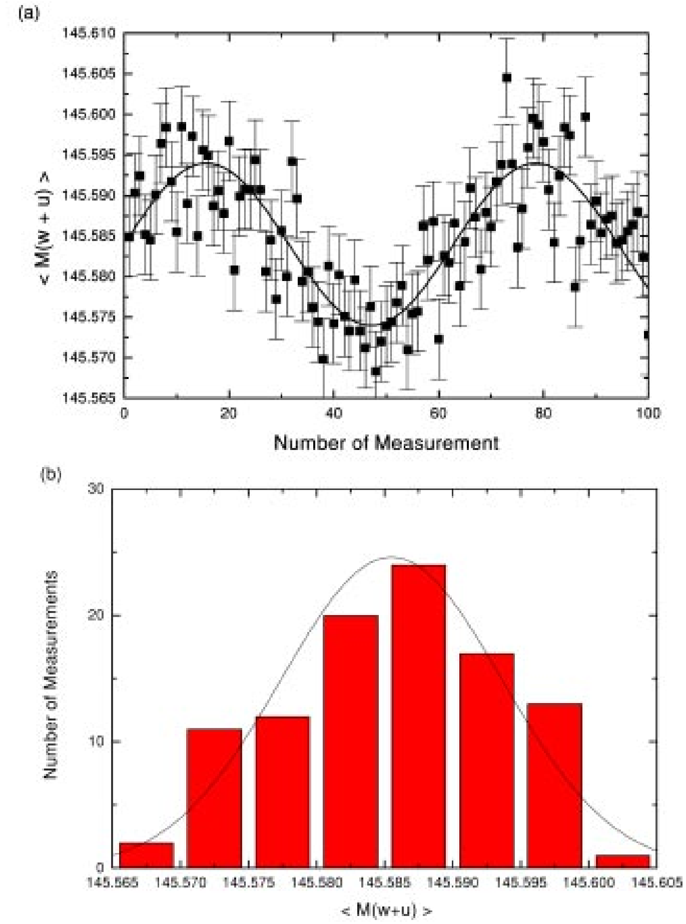

Fig. 2. (a) 100 measurements, $\langle M(w+u)\rangle$, obtained by taking the integer part of a real number with a nonstatistical fluctuation simulated by adding a $0.01\,\sin(0.1k)$ term (k is the measurement number) after adding noise and averaging over 10 000 points, in the order that they were obtained. (In all cases $S_i\approx 0.5$ and $S_{Ni}\approx 0.005$.) The line shows the simulated nonstatistical fluctuation. (b) Histogram of the 100 measured dithered data. The mean value is 145.590 and the standard deviation is 0.008.

Now we measure 100 series of 10 000 measurements of a variable $w=x+0.01\,\sin(0.1k)$, where $k=1,\ldots,100$ labels the measurements. The variable w differs from x by a magnitude that is 0.01 times the resolution of the instrument. The small sine component added to x simulates the presence of nonstatistical fluctuations. We want to verify whether or not the dithering technique is able to detect it. In Fig. 2(a) we show the result of these 100 measurements in the order that they were obtained (in all cases $S_i\approx 0.5$ and $S_{N_i}\approx 0.005$). From Figs. 1(a) and 2(a) the difference between x and w can be observed, although the resolution of the instrument is 1. In Fig. 2(b) we show a histogram of the 100 measured dithered w data. A Gaussian fit gives a mean of 145.590 with a standard deviation of 0.008. It can be seen that not only is the standard deviation is than the one expected $(0.5/\sqrt{10000}=0.005)$, but also the histogram does not look normally distributed. Thus, this statistical analysis is a proper tool that can be used to verify if the improved resolution is below the limit imposed by nonstatistical fluctuations.

B. Measurement of voltage using an oscilloscope

In this example we measure the voltage of a 9 V battery using a TDS5052 Tektronix oscilloscope. This oscilloscope measures voltage using an 8 bit A/D converter, and in the 5 V/Div scale that we have used, the minimum division of the instrument is 0.2 V. We generate Gaussian noise with standard deviation of 0.4 V by using the noise function of a HP 33021A waveform generator that is added to the voltage of the battery. We record 200 series of 10 000 points in 800 μs by using the 2 million point record length of the oscilloscope and a resolution of 0.4 ns. The standard deviation of each measurement is 0.18 V and the standard deviation of the average over 10 000 points is 0.0018 V. The 200 dithered points are shown in Fig. 3(a) in the order that they were measured; in Fig. 3(b) we show a histogram to study the distribution of the data. The Gaussian fit gives a mean of 9.047 V with a standard deviation of 0.002 V. As we have said, the results of these tests determine that the dithering technique has been properly applied. Note that the values obtained from the oscilloscope are, for instance, 8.6, 8.8, 9.0, 9.2, etc. Thus, by adding noise and averaging over 10 000 points, we improve the precision of he oscilloscope by two orders of magnitude.

We next measure the voltage of the battery over 5 min when a 1 kΩ resistor is applied. For each measured time we record two series of 10 000 points in 0.8 s using a 25 000 point record length and a resolution of 40 ns. The results are

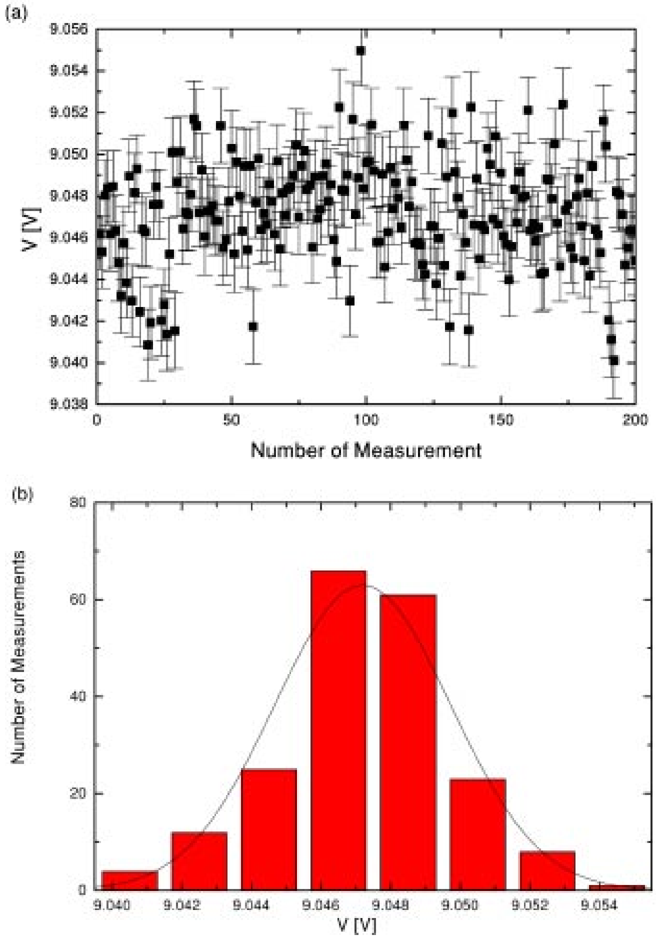

Fig. 3. (a) 200 measurements of the 9 V battery voltage after adding noise and averaging over 10 000 points, in the order that they were obtained. (In all cases $S_i = 0.18$ V and $S_{Ni} = 0.0018$ V.) (b) Histogram of 200 of the 9 V battery voltage dithered data. The mean value is 9.047 V and the standard deviation is 0.002 V.

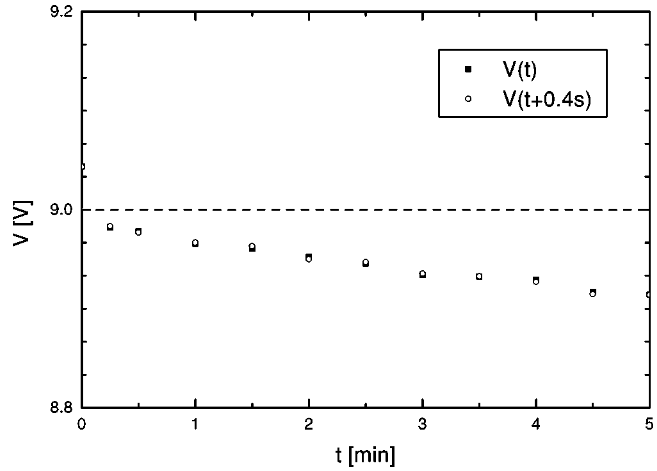

Fig. 4. Measurements of the voltage of the battery over 5 min when a 1 kΩ load is applied. We take two voltage series, $V(t)$ and $V(t+0.4s)$, to determine if their differences are significant. The dispersion of each point of both data series is 0.002 V and is not plotted in the figure.

shown in Fig. 4. Both series, $V(t)$ (evaluated with the data of the first 10 000 points) and $V(t+0.4s)$ (evaluated with the data of the second 10 000 points), have voltage values with differences that are not significant (the difference between both measurements is smaller than the statistical dispersion of each data series, 0.002 V). In this way we can ensure that the interval of time in which we are making the dithering average is consistent with the precision we are obtaining with this technique (that is, the voltage of the battery within this interval of time changes less than 0.002 V). As can be seen, we are able to detect the discharge of the battery, even though throughout the test, the voltage drops by no more than one-tenth of the instrument resolution.

C. Measurement of a complex signal using an A/D converter

To show another example of how dithering can be effectively used in an experimental situation, we have measured a complex signal through an A/D card. The measured signal was a sinc function, consisting of an unlimited repetition of a $\sin(x)/x$ waveform, with an overall period of about 22.3 s and an internal frequency of 0.72 Hz, directly obtained from a HP 33021A waveform generator. The signal amplitude was chosen to have both a very poor signal (less than a fifth of a LSB of the A/D card) and a larger one that slightly exceeds 2 LSBs for the 8 bit ADC. Figure 5(a) shows the digitized signal using an 8 bit A/D converter. The LSB of this A/D converter is 10 V/$2^8 \approx 0.039$ V. Between the times t_1 and t_3 shown in Fig. 5(a), each point in the plot is the mean value of 512 single measurements. Outside this time interval, this averaging was not done. Between $t_1 = 25$ s and $t_2 = 47$ s, the added noise (0.012 $V_{\text{peak-peak}}$) is negligible compared with the magnitude of the LSB of the 8 bit A/D converter. For these conditions, a correct measurement of the signal is impossible, and it is very easy to distinguish the three unique digitized levels in which the whole signal is comprised. Only the highest part of the waveform is detected due to the long distance between consecutive digital levels, which are depicted as thin dashed horizontal lines in the plot. At $t_2 = 47$ s, an uncorrelated noise signal of about 2 LSBs (0.19 $V_{\text{peak-peak}}$) is externally added to the input of the A/D converter.

Almost any noise generator with a frequency spectrum in the audio range can be used, and even a sinusoidal generator at a much higher frequency than the signal is adequate, if the beat of both the measured signal and the injection noise signal is low, in order to prevent moire or intermodulation aliasing. As an example, this experiment was performed using a homemade sinusoidal generator at approximately 16 kHz. The shift on the measured offset level shown in Fig. 5(a) is due to the fact that the offset dc value of the signal was not exactly coincident with the magnitude of the nearest LSB of the 8 bit digitization. Despite the fact that the digitization is only 8 bits, and the amplitude of the signal is about 2 LSBs, the measurement at $t > 47$ s shows very low distortion, which is almost impossible to detect with the naked eye in the time domain. A small-amplitude ripple, a product of the dithering through the noise injection, is apparent at the central peak of the sinc signal. The overall measurement system, with the 8 bit A/D converter and 512 single measurements per point, improves the precision by a factor of $\sqrt{512}$ or 22.6. This factor is equivalent to about 4.5 bits so the system behaves like a 12.5 bit digitizer. In Fig. 5(b) the same signal was digitized using a 12 bit A/D card, and thus we can compare it

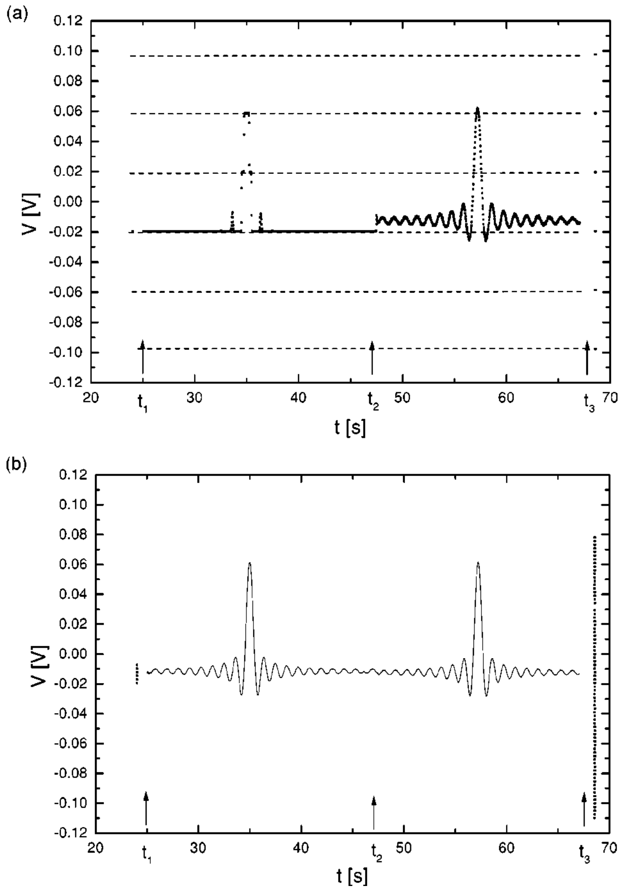

Fig. 5. Digitized sinc function, which consist of an unlimited repetition of a $\sin(x)/x$ waveform, with an overall period of about 22.3 s and an internal frequency of 0.72 Hz. Before t_1 and after t_3 the discreteness of the measured voltage shows both the magnitude of the noise present and the resolution of the A/D cards. (a) Digitized signal using an 8 bit A/D converter. Between times t_1 and t_3 each point in the plot is the mean value of 512 single measurements. Between $t_1=23$ s and $t_2=47$ s, the added noise is negligible. At $t_2=47$ s, an uncorrelated noise signal of about 2 LSBs is externally added. The dashed horizontal lines show digital levels of the A/D converter. (b) Digitized signal using a 12 bit A/D card.

with the 8 bit dithered signal shown in Fig. 5(a). The LSB of this 12 bit A/D converter is $10V/2^{12}\approx 0.0024$ V. Before t_1 and after t_3, the discreteness of the measured voltage shows both the size of the noise present before and after t_2 and the resolution of the 8 and 12 bit A/D.

IV. CONCLUSIONS

We have shown how increased resolution can be achieved through the attenuating effect of a dither signal on the error of the measurement. We presented a brief introduction of the theory involved in the dithering technique and three experiments that show how it can be presented and applied in introductory physics and chemistry laboratory courses. Even though dithering is an old and well-known tool used in fields like image and sound processing, it remains mostly unknown to physicists and chemists. In fact, in most introductory laboratory courses it is taught that the minimum division of an instrument is a lower bound to the precision of a measurement and cannot be improved by doing statistics. We have shown that dithering is a way in which this lower bound can be overcome.

There are three main questions to be considered before assuming that an experimental result can be improved by repeated measurements: (a) is there enough time and resources?; (b) what is the limit imposed by systematic errors; and (c) what is the limit imposed by nonstatistical fluctuations?[16] With present day technology making computer aided experiments possible, it is easy to evaluate a large amount of data and thus a statistical treatment can be done in the laboratory. Thus, the practical limit is given by the sample rate of the acquisition process. The dithering technique can be used to calibrate the instruments and assure that significant systematic errors are not present. When using dithering, it must be remembered that the hypothesis and the tests are to ensure that nonstatistical fluctuations are smaller than the improved resolution. If these tests cannot be done, it is always a conservative approach to take the minimum division of the instrument as the lower bound to statistics.

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