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Inverse-square orbits: Three little-known solutions and a novel integration technique

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Three methods, none of them widely known, are presented for determining the orbit of a particle subject solely to an inverse-square central force: One is by Laplace, another by Jacobi; the third may be here making its first appearance in print. All three differ markedly in major thrust; all of them culminate, of course, in conic-section orbits. Also included is use of a novel integration technique in the execution of a standard textbook solution of the same inverse-square orbit problem.

I. INTRODUCTION

A particle subject solely to an inverse-square central force pursues an orbit which, if not in a straight line, must be along one of the conic sections having a focus at the force center; there are many valid ways to prove this by solving the equations of motion so as to derive an algebraic equation of the orbit. All of this is widely known. The major purpose here is to present three such solutions, all of which are evidently barely known today, and one of which may have been totally unknown before February 1991. A subordinate purpose is to exhibit application of a narrowly known technique[1,2] to solving a differential equation that arises in a standard textbook treatment[3] of the inverse-square orbit problem. A sketch of a closely related technique[4,5] for accomplishing the inverse-square solution with reference to the same basic idea[1] is also presented.

The oldest solution of the three, accomplished by Pierre Simon Laplace circa 1798, exhibits an inspired leap onto a well concealed intellectual pinnacle.[6] Why it is generally unknown today is something of a mystery.

II. PRELIMINARIES

All three solutions, as presented below, make immediate use of the readily proved fact that motion under the sole influence of any central force must take place in a single plane—herein the xy plane. Thus the equations of motion from which an orbit equation must be derived read[7]

$$\ddot{x} = -\frac{\gamma x}{r^3}, \quad \ddot{y} = -\frac{\gamma y}{r^3}, \tag{1}$$

where γ is constant—positive for attraction, negative for repulsion, of the orbiting particle at (x,y) to or from the force center at the coordinate origin—and

$$r^2=x^2+y^2 \quad (r>0). \tag{2}$$

In standard fashion, we derive from (1)

$$0=x\ddot{y}-\ddot{x}y=\frac{d}{dt}(x\dot{y}-\dot{x}y),$$

whence

$$x\dot{y}-\dot{x}y=b, \tag{3}$$

the familiar angular-momentum integral, in which the integration constant b is the component of angular momentum per unit mass in a direction perpendicular to the xy plane, computed with respect to the origin. In terms of the plane-polar coordinates r, ϕ, (3) reads

$$r^2\dot{\phi}=b, \tag{4}$$

as is well known.

The reader will recall that a conic section in the xy plane is most generally described as a set of points $\{(x,y)\}$ for which the ratio (eccentricity) of the distance of (x,y) from a given point (focus) to its distance from a given line (directrix), both lying in the xy plane, is constant. Thus, if the conic has its focus at the origin, the line $Ax+By+C=0$ as its directrix, and ϵ (>0) as its eccentricity, then it is described by the equation

$$(x^2+y^2)^{1/2}=\epsilon\left(\frac{Ax+By+C}{(A^2+B^2)^{1/2}}\right). \tag{5}$$

The reader should also recall that (i) the coefficient of ϵ is, when non-negative, the distance from (x,y) to the line designated; and (ii) the conic is an ellipse when $0<\epsilon<1$, a parabola when $\epsilon=1$, a hyperbola when $\epsilon>1$.[8] Except in a well known limiting sense not useful in what follows, the form (5) does not take account of the circle, generally regarded as an ellipse having eccentricity zero.

In terms of the polar coordinates r,ϕ, (5) reads

$$r^{-1}=(\epsilon C)^{-1}(A^2+B^2)^{1/2}[1-\epsilon\cos(\phi-\delta)], \tag{6}$$

in which the constant δ satisfies

$$\cos\delta=\frac{A}{(A^2+B^2)^{1/2}}, \quad \sin\delta=\frac{B}{(A^2+B^2)^{1/2}}.$$

Thus a solution of the inverse-square orbit problem[9] consists of a mathematical process that starts with (1)—or an equivalent—and culminates with (5) or its equivalent (6), for appropriate values of the constants A, B, C, ϵ, δ.

III. C.G.J. JACOBI'S SOLUTION

Although notation and details differ from the original, the following is in its essentials a procedure carried out by Carl Gustav Jacob Jacobi (1804–1851).[10] Dividing each of (1) by (4), substituting

$$x=r\cos\phi, \quad y=r\sin\phi, \tag{7}$$

and multiplying through by r^2, he obtained

$$\frac{\ddot{x}}{\dot{\phi}}=-\frac{\gamma}{b}\cos\phi, \quad \frac{\ddot{y}}{\dot{\phi}}=-\frac{\gamma}{b}\sin\phi.$$

These become, on application of the chain rule,

$$\frac{d\dot{x}}{d\phi}=-\frac{\gamma}{b}\cos\phi, \quad \frac{d\dot{y}}{d\phi}=-\frac{\gamma}{b}\sin\ \phi;$$

and easy integrations yield

$$\dot{x}=-(\gamma/b)(\sin\phi-B),$$
$$\dot{y}=(\gamma/b)(\cos\phi-A), \tag{8}$$

in which B and A are arbitrary integration constants. Then (3) converts (8), after elimination of ϕ and r via (7) and (2), into[11]

$$(x^2+y^2)^{1/2}=Ax+By+(b^2/\gamma). \tag{9}$$

Comparison of (9) with (5) shows that the orbit must be a conic whose eccentricity is

$$\epsilon=(A^2+B^2)^{1/2}, \tag{10}$$

and having a focus at the coordinate origin.

A well known fact relating eccentricity to the algebraic sign of the total energy is quickly derived from (8): Solve for A and B respectively, then add the squares and eliminate ϕ via (7); application of (3) and (10) then yields

$$\epsilon^2=A^2+B^2=1+\frac{2b^2}{\gamma^2}\left(\frac{1}{2}(\dot{x}^2+\dot{y}^2)-\frac{\gamma}{r}\right). \tag{11}$$

Since $(\dot{x}^2+\dot{y}^2)/2$ is the orbiting particle's kinetic energy per unit mass and $(-\gamma/r)$ its potential energy per unit mass (when taken as zero "at infinity", the usual convention), we read (11) as expressing the correlation that a negative-energy orbit has $0\leqslant\epsilon^2<1$ and so is an ellipse; a zero-energy orbit, with $\epsilon^2=1$, is parabolic; and positive total energy, giving $\epsilon^2>1$, prevails if and only if the orbit is hyperbolic.

It should be noted that the content of (8) appears in the context of another, fairly well known, approach to the inverse-square orbit problem: (8) expresses the constancy of the so-called Lenz vector—namely, $\alpha[(\mathbf{r}/r)-(b/\gamma)\dot{\mathbf{r}}\times\hat{\mathbf{b}}]$—in which α is a variously chosen constant and $\hat{\mathbf{b}}$ is a unit vector having the direction of the orbiting particle's angular momentum.[12]

Also embodied in (8) is the fact discovered by William Rowan Hamilton[13] (1805–1865) that the locus of $\mathbf{v}=\dot{x}\hat{\mathbf{i}}+\dot{y}\hat{\mathbf{j}}$ in the velocity plane is a circle having its center at $(\gamma/b)(-B\hat{\mathbf{i}}+A\hat{\mathbf{j}})$, with radius (γ/b). This circumstance leads to yet another solution[14]—closely related to the Lenz-vector method, obviously, yet distinct from it—of our orbit problem.

IV. A NEW SOLUTION, PERHAPS

With x and y still, as above, the respective Cartesian coordinates of the orbiting particle in its plane of motion with origin at the force center, we introduce the complex radius vector

$$z=x+iy \quad (i^2=-1). \tag{12}$$

With x and y real, we also have the conjugate complex

$$z^*=x-iy \tag{13}$$

and the modulus square

$$|z|^2=|z^*|^2=zz^*=x^2+y^2=r^2. \tag{14}$$

Thus the equations of motion (1) for our inverse-square orbiting particle read, in terms of (12),

$$\ddot{z}=-\gamma z/|z|^3. \tag{15}$$

Using (14), we rewrite (15) as

$$\ddot{z}=-\gamma/|z|z^*; \tag{16}$$

and, taking the complex conjugate of both sides, we get the equivalent

$$\ddot{z}^* = -\gamma/|z|z, \tag{17}$$

since γ and $|z|$ are real.

From (16) and (17) we obtain

$$0 = \ddot{z}z^* - z\ddot{z}^* = \frac{d}{dt}(\dot{z}z^* - z\dot{z}^*),$$

from which immediately follows

$$\dot{z}z^* - z\dot{z}^* = 2ib, \tag{18}$$

in which the integration constant b is real, because $z\dot{z}^*$ is the conjugate complex of $\dot{z}z^*$. Indeed, detailed computation of the left-hand side of (18) identifies b as the like-designated quantity introduced in (3).

It requires little perspicacity to observe that (18) can be rewritten

$$2ib = (z^*)^2\frac{d}{dt}\left(\frac{z}{z^*}\right), \tag{19}$$

but perhaps a bit more to notice that fruitful use of (19) in conjunction with (16) would become a likelihood if somehow $(z^*)^2$ were to be replaced by $|z|z^*$ as coefficient of an exact time derivative. Such a replacement can be effected, obviously, through multiplication of $(z^*)^2$ by

$$\frac{|z|}{z^*} = \frac{|z|z}{z^*z} = \frac{z}{|z|}, \tag{20}$$

according to (14); but how can the feat be accomplished without destruction of the *constancy* of the left-hand member of (19)?

Eventually the opaque becomes transparent: In (19) we use the trivial fact that

$$\frac{z}{z^*} = \frac{z}{z^*}\cdot\frac{z}{z} = \frac{z^2}{|z|^2} = \left(\frac{z}{|z|}\right)^2 \tag{21}$$

and so obtain

$$\begin{aligned} 2ib &= (z^*)^2\frac{d}{dt}\left(\frac{z}{|z|}\right)^2 \\ &= 2(z^*)^2\left(\frac{z}{|z|}\right)\frac{d}{dt}\left(\frac{z}{|z|}\right) \\ &= 2z^*|z|\frac{d}{dt}\left(\frac{z}{|z|}\right), \end{aligned} \tag{22}$$

with the final step taken via (20). And so we can accomplish what was targeted when (21) was invoked:

Multiplying the two members of (16) by the corresponding extreme members of (22), we obtain

$$2ib\ddot{z} = -2\gamma\frac{d}{dt}\left(\frac{z}{|z|}\right);$$

from this we go directly to

$$\frac{d}{dt}\left(\frac{z}{|z|} + \frac{ib}{\gamma}\dot{z}\right) = 0 \tag{23}$$

and thence to

$$\frac{z}{|z|} + \frac{ib}{\gamma}\dot{z} = A + iB, \tag{24}$$

where A and B are real constants of integration—the same roles played by A and B in Sec. III above. From (24) follows, since b, γ, and $|z|$ are also real and $i^* = -i$,

$$\frac{z^*}{|z|} - \frac{ib}{\gamma}\dot{z}^* = A - iB. \tag{25}$$

Multiplying (24) by z^*, (25) by z, and adding the results, we obtain, with the aid of (14),

$$2|z| + \frac{ib}{\gamma}(\dot{z}z^* - z\dot{z}^*) = A(z+z^*) - iB(z-z^*)$$

—whence, because of (12), (13), (14), and (18),

$$(x^2+y^2)^{1/2} - (b^2/\gamma) = Ax + By. \tag{26}$$

We note the identity of (26) with (9); again the solution sought is achieved.

V. LAPLACE'S AMAZING SOLUTION

One finds in the collected works of Pierre Simon Laplace (1749–1827)[15] two solutions of our inverse-square orbit problem. Exposition of the more remarkable of these is simplified in what follows in three ways: (i) The orbit is assumed from the outset (as in both solutions above) to lie in the xy plane. (ii) A somewhat cumbersome computation indicated by Laplace is replaced here by one that is appreciably less awkward. (iii) Appeal is made here to a well known elementary differential-equations theorem rather than to the somewhat complicated more general theorem used by Laplace.

Given (1), we record the immediate consequence

$$\frac{d}{dt}(r^3\ddot{x}) = -\gamma\dot{x}, \quad \frac{d}{dt}(r^3\ddot{y}) = -\gamma\dot{y}, \tag{27}$$

and proceed, using (1) and (2), to compute an analog to the left-hand members in (27)—namely, $(d/dt)(r^3\ddot{r})$: As preliminary, we use (2) with the aid of (1) to compute successively

$$r\dot{r} = x\dot{x} + y\dot{y}; \tag{28}$$

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = -\frac{\gamma}{r^3}(\dot{x}x + \dot{y}y) = -\frac{\gamma}{r^3}(\dot{r}r) = -\frac{\gamma\dot{r}}{r^2}; \tag{29}$$

$$x\ddot{x} + y\ddot{y} = -\frac{\gamma}{r^3}(x^2+y^2) = -\frac{\gamma}{r}; \tag{30}$$

and, finally, by means of the Leibniz product rule, for example,

$$\frac{d^3}{dt^3}(r^2) = \frac{d^3}{dt^3}(rr) = 2(r\dddot{r} + 3\dot{r}\ddot{r}). \tag{31}$$

First using (31), then (28), (30), and (29), we compute

$$\begin{aligned} \frac{d}{dt}(r^3\ddot{r}) &= r^3\dddot{r} + 3r^2\dot{r}\ddot{r} \\ &= \frac{1}{2}r^2\frac{d^3}{dt^3}(r^2) \\ &= r^2\frac{d^2}{dt^2}(r\dot{r}) \\ &= r^2\frac{d^2}{dt^2}(x\dot{x} + y\dot{y}) \\ &= r^2\frac{d}{dt}(x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2) \\ &= r^2\left[\frac{d}{dt}\left(\frac{-\gamma}{r}\right) - \frac{2\gamma\dot{r}}{r^2}\right]. \end{aligned} \tag{32}$$

In (32) we read, in brief,

$$\frac{d}{dt}(r^3\ddot{r}) = -\gamma\dot{r}. \tag{33}$$

Now all three equations in (27) and (33) are satisfied by the motion $x = x(t)$, $y = y(t)$, $z = 0$ of our particle that is solely influenced by the inverse-square force. And during this motion we have $r^3 = \psi(t)$, for the appropriate positive-valued ψ. Thus we are told that the second-order linear homogeneous differential equation

$$\frac{d}{dt}\left(\psi(t)\,\frac{du}{dt}\right) = -\gamma u \qquad (34)$$

possesses, according to (27) and (33), the *three* solutions $u=\dot{x}$, $u=\dot{y}$, and $u=\dot{r}$—from which we conclude via a widely known elementary theorem on linear homogeneous differential equations[16] that constants A and B exist such that

$$\dot{r} = A\dot{x} + B\dot{y}, \qquad (35)$$

since $\dot{x}$ and $\dot{y}$ are linearly independent for motion not in a straight line. Immediate integration of (35) yields

$$r = Ax + By + C,$$

from which, since $r=(x^2+y^2)^{1/2}$, we draw the orbit-shape conclusions already inferred from (9) and (26) above!

VI. A FAMILIAR SOLUTION SUBJECTED TO A NOVEL TREATMENT

(a) Starting with (1) and (2), a standard procedure[3] derives a familiar equation expressing conservation of total energy per unit mass, written here in terms of the plane polar coordinates r, ϕ, and first time derivatives thereof:

$$\tfrac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - (\gamma/r) = h \ (=\text{const}). \qquad (36)$$

Application of $\dot{r} = \dot{\phi}(dr/d\phi)$ and (4) eliminates the time variable from (36); multiplication by $(2/b^2)$ then gives

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} - \frac{2\gamma}{b^2 r} = \frac{2h}{b^2}. \qquad (37)$$

The standard straightforward way of dealing with (37) is to solve for $\pm(d\phi/dr)$ and then to perform the rather messy integration (or to use a table) in order to express ϕ in terms of r; then one inverts so as to obtain, eventually, (6) or its equivalent, with suitable identification of the various constants. There happens, however, to be a tidier procedure based on an idea published in this Journal in 1961[1] and again, with extended application, in 1989.[2] It springs from the following elementary fact:

> If u and v are real numbers such that $u^2+v^2=1$, then there exists a real θ (unique to within an integral multiple of 2π) such that $u=\sin\theta$ and $v=\cos\theta$. (When u and v are nonconstant, then so also, of course, is θ.)

It therefore follows—most obviously when (37) is rewritten as

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \left(\frac{\gamma}{b^2} - \frac{1}{r}\right)^2 = \frac{2h}{b^2} + \frac{\gamma^2}{b^4}$$

—that there exists a θ for which

$$\frac{1}{r^2}\left(\frac{dr}{d\phi}\right) = \left(\frac{2h}{b^2} + \frac{\gamma^2}{b^4}\right)^{1/2}\sin\theta \qquad (38)$$

and

$$\frac{\gamma}{b^2} - \frac{1}{r} = \left(\frac{2h}{b^2} + \frac{\gamma^2}{b^4}\right)^{1/2}\cos\theta. \qquad (39)$$

Differentiating (39) with respect to ϕ, we get

$$\frac{1}{r^2}\left(\frac{dr}{d\phi}\right) = -\left(\frac{2h}{b^2} + \frac{\gamma^2}{b^4}\right)^{1/2}\left(\frac{d\theta}{d\phi}\right)\sin\theta.$$

Viewing this result in the light of (38), we immediately conclude that

$$\frac{d\theta}{d\phi} = -1$$

whence

$$\theta = -\phi + \delta, \qquad (40)$$

in which δ is a constant of integration. Setting (40) into (39) then gives, on rearrangement,

$$\frac{1}{r} = \left(\frac{\gamma}{b^2}\right)\left[1 - \left(1 + \frac{2hb^2}{\gamma^2}\right)^{1/2}\cos(\phi-\delta)\right]. \qquad (41)$$

Comparison with (6) identifies the orbit equation (41) with that of a conic section having eccentricity

$$\epsilon = [1 + 2(hb^2/\gamma^2)]^{1/2}.$$

(b) Application of the basic idea on which the foregoing is based was made in solving the same inverse-square orbit problem as early as 1976.[4,5] It should be noted, however, that in order to achieve a solution of (37) the 1976 authors apply the basic idea instead to the equation obtained from the elimination of $\dot{\phi}$ between (36) and (4)—that is, after appropriate square completion, to

$$\dot{r}^2 + \left(\frac{b}{r} - \frac{\gamma}{b}\right)^2 = 2h + \left(\frac{\gamma}{b}\right)^2. \qquad (42)$$

From (42) they infer the existence of a ψ for which

$$\dot{r} = [2h + (\gamma/b)^2]^{1/2}\sin\psi \qquad (43)$$

and

$$(b/r) - (\gamma/b) = [2h + (\gamma/b)^2]^{1/2}\cos\psi. \qquad (44)$$

Differentiating (44) with respect to ψ, they obtain

$$-\frac{b}{r^2}\frac{dr}{d\psi} = -\left[2h + \left(\frac{\gamma}{b}\right)^2\right]^{1/2}\sin\psi = -\dot{r}, \qquad (45)$$

on reference to (43). Then, using (4), they conclude from equality of the extreme members of (45) that $\dot{\psi} = \dot{\phi}$, whence $\psi = \phi - \delta$—substitution of which into (44) leads to the required solution (41) after mild manipulation.

VII. COMMENTS AND ACKNOWLEDGMENTS

(a) Had I not come upon, in late 1990 and early 1991, four quite different solutions of the inverse-square orbit problem—Jacobi's,[10] Laplace's,[6] one by Hart,[17] and another by Keill[18]—that were all new to me after more than 50 years of paying attention to such matters, I should be firmly confident that the one presented in Sec. IV above is original with me. Now, however, I shall not be astonished to learn, albeit unwelcomingly, that I was anticipated in its discovery. I hope, in any event, that the reader has been favorably served by my effort to indicate in Sec. IV the motivation that led to the crucial (22). Although (22) holds for *any* central-force motion, it appears to lead to a useful result only in the case of the inverse-square force law.

(b) The attempt to produce a proof starting from (15) *cum* (12) was provoked by a recent paper[19] in which (12) is used for a polar-coordinate paraphrase of the Lenz-vector approach[12] and a book[20] by Wintner in which almost simultaneously I came upon Hart's solution.[17] I am grateful to my correspondent Dr. Božidar A. Aničin of the University of Belgrade Engineering Faculty for directing attention to a set of several pages in a 1989 Russian-language

volume by the mathematician V. I. Arnol'd. After its translation into English—for which my hearty thanks go to Diana Holton-Hinshaw—one of the pages revealed mention of "the book of A. Vintner, 1941," in a connection unrelated to Hart's solution.

(c) Of the many inverse-square orbit solutions I have seen, Laplace's is the only one that proceeds without deriving and using the angular-momentum conservation embodied, for example, in (3), (4), and (18).

(d) The computation that culminates in the remarkable (33) could have been accomplished with *slightly* greater economy if the scalar-product concept had been used for each of $x\dot{x} + y\dot{y}$, $\dot{x}\ddot{x} + \dot{y}\ddot{y}$, and $x\ddot{x} + y\ddot{y}$ in (28), (29), (30), and (32). In replacing Laplace's computation by a simplified procedure here, I choose, however, not to employ any device that would not in late 18th century have been directly available for Laplace's use.

(e) Laplace[6] employs (his generalization of) the differential-equation theorem introduced in Sec. V also to prove independently that the inverse-square orbit lies in a plane—although he fails to exploit the result to simplify his solution of the orbit problem. Here is how: When the motion is not *ab initio* assumed to be confined to a plane, the third equation $\ddot{z} = -\gamma(z/r^3)$ must be attended to along with (1). During the motion, therefore, the second-order linear homogeneous equation

$$\ddot{w} = -\gamma[w/\psi(t)] \qquad (46)$$

—in which $r^3 = \psi(t)$ during the motion, as in Sec. V—has the three solutions $w = x(t)$, $w = y(t)$, and $w = z(t)$, according to (1) and (46). By the theorem[16] referred to and used in Sec. V, there exist constants A, B, C—not all zero—such that $Ax + By + Cz = 0$: The orbit must lie in a plane containing the coordinate origin, the force center. (It should be clear that this method applies equally well for proof that motion under *any* central-force law is confined to a plane through the force center: The function ψ is different for the different central-force laws.)

(f) A traditional view places the first proof of the proposition that inverse-square force implies conic-section orbit in Newton's *Principia*[21]—a view that has, however, been refuted.[22] Contrary judgments in defense of the *Principia's* treatment of inverse-square orbits can also be found.[23] The earliest proof seems to have been constructed by John Keill.[24] What is evidently the latest prior to February 1991[25] uses a geometric approach that makes it totally different in character from any of the ten others known to this author.

[1]R. Weinstock, "An Unusual Method of Solving the Harmonic-Oscillator Equation," Am. J. Phys. **29**, 830–831 (1961).

[2]R. Weinstock, "A novel method for solving a class of differential equations," Am. J. Phys. 57, 1144–1147 (1989).

[3]See, for example, R. A. Becker, *Introduction to Theoretical Mechanics* (McGraw-Hill, New York, 1954), pp. 230–231.

[4]A. M. McLoughlin and G. J. Smith, "Obtaining Keplerian orbits," Mathematical Gazette **60**, 209–210 (1976).

[5]I was unaware of the 1976 application (Ref. 4) when I submitted the first version of this paper—including the material in Sec. VI (a)—to Am. J. Phys. It was made known to me by a friend, who sent me a copy of W. Robin, "Schwarzschild relativistic orbits," Eur. J. Phys. **12**, 204–206 (1991). Robin mentions Ref. 4 as making use of the idea propounded in Ref. 1 "independent[ly] of [its author]".

[6]P. S. Laplace, *Oeuvres* (Gauthier-Villars, Paris, 1878–1912), Vol. 1, pp. 175–177.

[7]R. B. Lindsay, *Physical Mechanics* (Van Nostrand, Princeton, NJ, 1961), 3rd ed., pp. 75, 78.

[8]See any early twentieth-century book on analytic geometry—e.g., E. S. Crawley and H. B. Evans, *Analytic Geometry* (F. S. Crofts, New York, 1935), 2nd ed., pp. 53–54, 75.

[9]Historians call the problem of deducing the orbit from the equations of motion the "inverse problem"—in contrast to their designating as the "direct problem" that of determining the force law from knowledge of the orbit and the force center. See, for example, E. J. Aiton, "The Solution of the Inverse-Problem of Central Forces in Newton's *Principia*," Arch. Int. d'Histoire Sci. **38**, 271–276 (1988).

[10]C. G. J. Jacobi, *Gesammelte Werke* (G. Reimer, Berlin, 1881–91), Vol. 4, pp. 281–282.

[11]In Jacobi's actual treatment (Ref. 10) the Cartesian coordinates x, y are completely abandoned, by means of (7), following (8); instead of (9) he emerges with a form essentially the same as (6) for the orbit equation.

[12]See, for example, H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980), 2nd ed. pp. 102–103. For a detailed history of the Lenz vector and its various names, see Herbert Goldstein, "Prehistory of the 'Runge–Lenz' vector," Am. J. Phys. **43**, 737–738 (1975); and "More on the prehistory of the Laplace or Runge–Lenz vector," Am. J. Phys. **44**, 1123–1124 (1976).

[13]W. R. Hamilton, Proc. R. Irish Acad. 3, p. 344ff, as listed in Goldstein's 1976 paper (Ref. 12).

[14]D. E. Christie, *Vector Mechanics* (McGraw-Hill, New York, 1964), 2nd ed., pp. 289–290.

[15]Ref. 6, pp. 125–129, 175–177.

[16]R. P. Agnew, *Differential Equations* (McGraw-Hill, New York, 1942), p. 96.

[17]H. Hart, "Integration of the rectangular equations of motion in the case of a central force varying inversely as the square of the distance," Messenger Math. **9**, 131–132 (1880). I thank Andrew Hyman for sending me a copy of this item.

[18]I am deeply grateful to my long-time correspondent Professor D. Thomas Whiteside for communicating to me in February 1991 a solution of the inverse-square orbit problem he attributes to John Keill as published in the 1708 volume of the Philosophical Transactions.

[19]R. W. Finkel, "Complex coordinates in central-force analyses," Am. J. Phys. **58**, 1085–1087 (1990).

[20]A. Wintner, *The Analytical Foundations of Celestial Mechanics* (Princeton U. P., Princeton, NJ, 1941), pp. 178. On p. 422, Wintner attributes the solution published under Hart's name (Ref. 17) to Laplace (Ref. 6) and, independently, to Jacobi (Ref. 10). There is, however, no evidence in the locations pointed to by Wintner that either Laplace or Jacobi was aware of the method submitted by Hart.

[21]*Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World*, edited by F. Cajori, translated from the Latin to English by Andrew Motte in 1729 (Greenwood, New York, 1969: reprint of 1962 edition published by the University of California), pp. 56–61.

[22]F. Rosenberger, *Isaac Newton und Seine Physikalischen Principien* (Barth, Leipzig, 1895), pp. 183–184; Wintner (Ref. 20), pp. 421–422; R. Weinstock, "Long-buried dismantling of a centuries-old myth: Newton's *Principia* and inverse-square orbits," Am. J. Phys. **57**, 846–849 (1989), and "Newton's *Principia* and Inverse-Square Orbits: the Flaw Reëxamined," Historia Mathematica **19**, 60–70 (1992).

[23]P. Stehle, "Comment on Weinstock's objection to Newton's logic," Am. J. Phys. **51**, 199–200 (1983); E. J. Aiton (Ref. 9); H. Erlichson, "Comment on 'Long-buried dismantling of a centuries-old myth'," Am. J. Phys. **58**, 882–884 (1990); B. H. Pourciau, "On Newton's Proof that Inverse-Square Orbits Must Be Conics," Ann. Sci. **48**, 159–172 (1991).

[24]See Ref. 18. The presumed priority of Keill's 1708 proof is independent of the controversy represented by Refs. 22 and 23. I know of no defender of the *Principia's* proffered proof as offered in the second (1713) and third (1726) editions who claims validity for the first-edition (1687) offering. In the first, what Newton offers as proof is an unadorned claim that the proposition proved several times in the foregoing article follows immediately from its converse. It is a two-sentence augmentation to the same claim in the second and third editions that is defended as justification of it by the authors listed in Ref. 23, among others.

[25]J. C. Rainwater and R. Weinstock, "Inverse-square orbits: a geometric approach," Am. J. Phys. **47**, 223–227 (1979).