Correcting a widespread error concerning the angular velocity of a rotating rigid body

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In obtaining the instantaneous angular velocity of a rotating rigid body, a large number of standard textbooks uses an incorrect argument by only considering the rate of change of the Euler angles but disregarding the simultaneous rate of change of the corresponding time-dependent rotation axes. Therefore, a correct and concise derivation of this quantity for a rather general case is given, which also reveals that the usual simple formula does not at all arise as the erroneous textbook reasoning suggests, but as a consequence of certain generalized commutation relations satisfied by Euler rotations.

In order to obtain the expression for the instantaneous angular velocity of a rotating rigid body in terms of the Euler angles and their derivatives,[1]

$$\boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_3^{(0)} + \dot{\theta}\mathbf{e}_1^{(1)} + \dot{\psi}\mathbf{e}_3^{(2)}, \tag{1}$$

a surprisingly large number of well-known textbooks employ an incorrect argument in trying to avoid a tedious analytical derivation of (1) from the set of Euler rotations, which connect the lab frame $\{\mathbf{e}_i^{(0)}\}$ with the body frame $\{\mathbf{e}_i^{(3)}\}$ according to[1]

$$\mathbf{e}_i^{(3)} = \mathbf{R}_{\text{tot}} \cdot \mathbf{e}_i^{(0)} = \mathbf{R}(\mathbf{e}_3^{(2)},\psi) \cdot \mathbf{R}(\mathbf{e}_1^{(1)},\theta) \cdot \mathbf{R}(\mathbf{e}_3^{(0)},\varphi) \cdot \mathbf{e}_i^{(0)}. \tag{2}$$

The superscript (k) on the vectors appearing in (1) and (2) denotes the frame that results from carrying out the preceeding k rotations.

This incorrect argument says that[1-9] "in consequence of the vector property of infinitesimal rotations, the vector $\boldsymbol{\omega}$ can be obtained as the sum of three separate angular velocity vectors, $\boldsymbol{\omega} = \boldsymbol{\omega}'_\varphi + \boldsymbol{\omega}'_\theta + \boldsymbol{\omega}'_\psi$, whose magnitudes are evidently[4] given by $\dot{\varphi}$, $\dot{\theta}$, and $\dot{\psi}$ and which are directed along the $\mathbf{e}_3^{(0)}$ – axis, the $\mathbf{e}_1^{(1)}$ – axis, and the $\mathbf{e}_3^{(2)}$ – axis, respectively," thus apparently leading directly to the (correct) expression (1).

This seems to imply that in general the following simple recipe would permit the derivation of $\boldsymbol{\omega}$: given N rotations about axes $\mathbf{a}_1(t), \ldots \mathbf{a}_N(t)$, through angles $\alpha_1(t), \ldots, \alpha_N(t)$, connecting the body frame with the lab frame, then the corresponding angular velocity of the body frame is simply be given by $\boldsymbol{\omega} = \Sigma_i [d\alpha_i(t)/dt]\mathbf{a}_i(t)$.

However, if we rewrite the total rotation $\mathbf{R}_{\text{tot}}$ in (2) in the equivalent forms[11]

$$\mathbf{R}_{\text{tot}} = \mathbf{R}(\mathbf{e}_3^{(0)},\varphi) \cdot \mathbf{R}(\mathbf{e}_1^{(0)},\theta) \cdot \mathbf{R}(\mathbf{e}_3^{(0)},\psi) \tag{3a}$$

$$= \mathbf{R}(\mathbf{e}_3^{(3)},\varphi) \cdot \mathbf{R}(\mathbf{e}_1^{(3)},\theta) \cdot \mathbf{R}(\mathbf{e}_3^{(3)},\psi), \tag{3b}$$

we find that there must be something wrong, for the same reasoning as before would now yield the incorrect relations $\boldsymbol{\omega} = (\dot{\varphi} + \dot{\psi})\mathbf{e}_3^{(0)} + \dot{\theta}\mathbf{e}_1^{(0)}$ and $\boldsymbol{\omega} = (\dot{\varphi} + \dot{\psi})\mathbf{e}_3^{(3)} + \dot{\theta}\mathbf{e}_1^{(3)}$.

Further evidence against the above stated argument is born by the fact that the instantaneous angular velocity corresponding to a rotation about a time-dependent axis such as $\mathbf{R}(\mathbf{e}_1^{(1)},\theta)$ is given by the sum of three terms,[12,13]

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{e}_1^{(1)} + \sin\theta\dot{\mathbf{e}}_1^{(1)} + (1 - \cos\theta)\dot{\mathbf{e}}_1^{(1)} \times \mathbf{e}_1^{(1)}, \tag{4}$$

of which the disputed argument uses but the first.

To date, these deficiencies have not been pointed out in the educational literature, although a few recent textbooks have abandoned the criticized reasoning,[14,15] but the result (1) is only stated with the remark that its deduction from rotation matrices is cumbersome.[14] It is therefore the purpose of this paper to give first a simple analytical derivation of the instantaneous angular velocity in the general case of N arbitrary rotations and then to show that only in the particular case of an Euler sequence (by which we mean that the kth rotation is about an axis that has been generated by the application of the k-1 preceding rotations to the original lab frame), this general expression reduces to (1) as a consequence of certain generalized commutation relations satisfied by Euler rotations.[11] Finally, we will investigate two special cases directly, which will be very illuminating as to where the textbook reasoning goes wrong.

As usual in dealing with rotations in Euclidean three-space, the appropriate mathematical tools for the present investigation are coordinate-free rotation operators as introduced by Gibbs,[16,17]

$$\mathbf{R}(\mathbf{a},\alpha) = \mathbf{1}\cos\alpha + (1 - \cos\alpha)\mathbf{aa} + \sin\alpha\mathbf{a} \times \mathbf{1}, \tag{5}$$

because the absence of coordinates emphasizes the essentials and reduces the necessary algebra considerably. The only prerequisites that we require are three well-known properties of rotations (Refs. 10, 13, 18, and 19), namely,

$$\dot{\mathbf{R}} \cdot \mathbf{R}^T = \boldsymbol{\omega} \times \mathbf{1}, \tag{6a}$$

which unambiguously links the angular velocity corresponding to a time-dependent rotation to the rotation itself, the dot denoting as before a derivative with respect to time and $\mathbf{R}^T$ the transpose of $\mathbf{R}$;

$$\mathbf{R}(\mathbf{R}(\mathbf{a},\alpha) \cdot \mathbf{b},\beta) = \mathbf{R}(\mathbf{a},\alpha) \cdot \mathbf{R}(\mathbf{b},\beta) \cdot \mathbf{R}^T(\mathbf{a},\alpha)^{11,17}; \tag{6b}$$

and

$$\mathbf{R} \cdot (\boldsymbol{\omega} \times \mathbf{1}) \cdot \mathbf{R}^T = (\mathbf{R} \cdot \boldsymbol{\omega}) \times \mathbf{1},^{17} \tag{6c}$$

where $\mathbf{R}$ in (6c) is an arbitrary rotation, generally different from the one to which $\boldsymbol{\omega}$ corresponds.

Suppose that the transformation from the lab frame $\{\mathbf{e}_i^{(0)}\}$ to the body frame $\{\mathbf{e}_i^{(N)}\}$ is brought about by a sequence of N Euler rotations

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$$\mathbf{e}_j^{(N)} = \mathbf{R}_{\rm tot}\cdot\mathbf{e}_j^{(0)} = \mathbf{R}(\mathbf{e}_{i_{N-1}}^{(N-1)},\alpha_{N-1})\cdot\,\cdots\,\cdot\mathbf{R}(\mathbf{e}_{i_1}^{(1)},\alpha_1)\cdot\mathbf{R}(\mathbf{e}_{i_0}^{(0)},\alpha_0)\cdot\mathbf{e}_j^{(0)}. \quad (7)$$

The meaning of the superscripts is as in (2) and the subscript i_k may take on any of the values, 1, 2, 3, specifying the particular axis of the frame $\{\mathbf{e}_i^{(k)}\}$ about which we rotate in the kth step. Obviously, the set of three Euler rotations in (2) is a special case of (7), with $i_0 = 3$, $i_1 = 1$, and $i_2 = 3$. Admittedly, there is little physical interest in generating the body frame from the lab frame by more than three independent rotations, but the proof we seek is no more complicated for a general N than it is for $N = 3$, yet it brings out more clearly the simple mechanism which gives rise to (1).

By abbreviating the product of the first $N-1$ rotations in (7) by $\overline{\mathbf{R}}$, we have

$$\mathbf{R}_{\rm tot} = \mathbf{R}(\overline{\mathbf{R}}\cdot\mathbf{e}_{i_{N-1}}^{(0)},\alpha_{N-1})\cdot\overline{\mathbf{R}},$$

and further, using (6b),

$$\mathbf{R}_{\rm tot} = \overline{\mathbf{R}}\cdot\mathbf{R}(\mathbf{e}_{i_{N-1}}^{(0)},\alpha_{N-1})\cdot\overline{\mathbf{R}}^T\cdot\overline{\mathbf{R}} = \overline{\mathbf{R}}\cdot\mathbf{R}(\mathbf{e}_{i_{N-1}},\alpha_{N-1}).$$

Proceeding now with $\overline{\mathbf{R}}$ in the same way, we eventually arrive at the generalization of (3a),

$$\mathbf{R}_{\rm tot} = \mathbf{R}(\mathbf{e}_{i_0}^{(0)},\alpha_0)\cdot\mathbf{R}(\mathbf{e}_{k_1}^{(0)},\alpha_1)\cdot\,\cdots\,\cdot\mathbf{R}(\mathbf{e}_{i_{N-1}}^{(0)},\alpha_{N-1}). \quad (8)$$

In order to find $\boldsymbol{\omega}_{\rm tot}$ according to (6a), we differentiate $\mathbf{R}_{\rm tot}$ in (8) with respect to time and postmultiply by $\mathbf{R}_{\rm tot}^T$. Thus using selfexplanatory subscripts to simplify the notation,

$$\begin{aligned}\boldsymbol{\omega}_{\rm tot}\times\mathbf{1} = {}& \dot{\mathbf{R}}_0\cdot\mathbf{R}_1\cdot\,\cdots\,\cdot\mathbf{R}_{N-1}\cdot(\mathbf{R}_1\cdot\mathbf{R}_2\cdot\,\cdots\,\cdot\mathbf{R}_{N-1})^T\cdot\mathbf{R}_0^T\\ &+\mathbf{R}_0\cdot\dot{\mathbf{R}}_1\cdot\mathbf{R}_2\cdot\,\cdots\,\cdot\mathbf{R}_{N-1}\cdot(\mathbf{R}_2\cdot\mathbf{R}_3\cdot\,\cdots\,\cdot\mathbf{R}_{N-1})^T\cdot\mathbf{R}_1^T\cdot\mathbf{R}_0^T\\ &+\cdots+\mathbf{R}_0\cdot\mathbf{R}_1\cdot\,\cdots\,\cdot\mathbf{R}_{N-2}\cdot\dot{\mathbf{R}}_{N-1}\cdot\mathbf{R}_{N-1}^T\cdot\mathbf{R}_{N-2}^T\cdot\,\cdots\,\cdot\mathbf{R}_0^T,\end{aligned}$$

where $\mathbf{R}_{\rm tot}^T$ has been suitably factorized in each term. Omitting products of rotations that yield the unit operator $\mathbf{1}$ and again using (6a) and (6b), we are led to

$$\begin{aligned}\boldsymbol{\omega}_{\rm tot}\times\mathbf{1} &= \boldsymbol{\omega}_0\times\mathbf{1}+\mathbf{R}_0\cdot(\boldsymbol{\omega}_1\times\mathbf{1})\cdot\mathbf{R}_0^T+\cdots\\ &\quad+\mathbf{R}_0\cdot\mathbf{R}_1\cdot\,\cdots\,\cdot\mathbf{R}_{N-2}\cdot(\boldsymbol{\omega}_{N-1})\times\mathbf{1})\cdot(\mathbf{R}_0\cdot\mathbf{R}_1\cdot\,\cdots\,\cdot\mathbf{R}_{N-2})^T\\ &= \boldsymbol{\omega}_0\times\mathbf{1}+(\mathbf{R}_0\cdot\boldsymbol{\omega}_1)\times\mathbf{1}+\cdots+(\mathbf{R}_0\cdot\mathbf{R}_1\cdot\,\cdots\,\cdot\mathbf{R}_{N-2}\cdot\boldsymbol{\omega}_{N-1})\times\mathbf{1}.\end{aligned}$$

We thus find the general result

$$\boldsymbol{\omega}_{\rm tot} = \boldsymbol{\omega}_0+\mathbf{R}_0\cdot\boldsymbol{\omega}_1+\mathbf{R}_0\cdot\mathbf{R}_1\cdot\boldsymbol{\omega}_2+\cdots+\mathbf{R}_0\cdot\mathbf{R}_1\cdot\,\cdots\,\cdot\mathbf{R}_{N-2}\cdot\boldsymbol{\omega}_{N-1}. \quad (9)$$

Note that no particular assumption on the kind of sequence of N rotations has entered the steps leading from (8) to (9) which is therefore quite generally valid. If, however, we confine ourselves to the special sequence (7), which we have shown to be expressible in the equivalent form (8) with all the rotation axes being time independent, a further simplification is possible, for now it follows from (4) that all the angular velocities in (9) are of the simple form $\boldsymbol{\omega}_k = \dot{\alpha}_k\mathbf{e}_{i_k}^{(0)}$. Furthermore, the kth term on the right-hand side of (9) is multiplied by such a product of rotations, so as to transform $\mathbf{e}_{i_k}^{(0)}$ into $\mathbf{e}_{i_k}^{(k)}$, which can be seen upon reversing the order of these products by (7) and (8). Hence, we arrive at the desired generalization of (1),

$$\boldsymbol{\omega}_{\rm tot} = \dot{\alpha}_0\mathbf{e}_{i_0}^{(0)}+\dot{\alpha}_1\mathbf{e}_{i_1}^{(1)}+\cdots+\dot{\alpha}_{N-1}\mathbf{e}_{i_{N-1}}^{(N-1)}. \quad (10)$$

With the general formulas (9) and (10) we are now able to specify our criticism raised in the introduction. First of all we note that although for three general rotations the result (9) gives three contributions to $\boldsymbol{\omega}_{\rm tot}$, there is little justification for denoting them by $\boldsymbol{\omega}_{\rm tot} = \boldsymbol{\omega}'_\psi+\boldsymbol{\omega}'_\theta+\boldsymbol{\omega}'_\varphi$, as it is usually done, because the various terms in this decomposition are related in different ways to the rotations to which they correspond. Indeed, with the conventional definition (2) of the Euler angles we infer from (9) the relation

$$\boldsymbol{\omega}_{\rm tot} = \boldsymbol{\omega}_\psi+\mathbf{R}(\mathbf{e}_3^{(2)},\varphi)\cdot\boldsymbol{\omega}_\theta+\mathbf{R}(\mathbf{e}_3^{(2)},\psi)\cdot\mathbf{R}(\mathbf{e}_1^{(1)},\theta)\cdot\boldsymbol{\omega}_\varphi, \quad (11)$$

where now all unprimed $\boldsymbol{\omega}$ vectors are "true" angular velocities in the sense of (6a), with $\boldsymbol{\omega}_\psi$ and $\boldsymbol{\omega}_\theta$ being of the general form (4), since the corresponding rotation axes are time dependent. Hence, not even one of the $\boldsymbol{\omega}'$ quantities coincides with its "true" counterpart.

Next, it is very instructive to study the effect of a variation of the conventional definition (2) of the Euler angles on the angular velocity (1). We take as one possible example

$$\mathbf{e}_j^{(3)} = \overline{\mathbf{R}}_{\rm tot}\cdot\mathbf{e}_j^{(0)} = \mathbf{R}(\mathbf{e}_2^{(1)},\bar{\psi})\cdot\mathbf{R}(\mathbf{e}_1^{(1)}\bar{\theta})\cdot\mathbf{R}(\mathbf{e}_3^{(0)},\bar{\varphi})\cdot\mathbf{e}_j^{(0)}. \quad (12)$$

With (6b) we find that $\mathbf{R}_{\rm tot}$ can be written equivalently as

$$\overline{R}_{\rm tot} = \mathbf{R}(\mathbf{e}_3^{(0)},\bar{\varphi})\cdot\mathbf{R}(\mathbf{e}_2^{(0)},\bar{\psi})\cdot\mathbf{R}(\mathbf{e}_1^{(0)},\bar{\theta}). \quad (13)$$

Consequently, the general formula (10) gives a corresponding angular velocity of

$$\boldsymbol{\omega}_{\rm tot} = \dot{\bar{\varphi}}\mathbf{e}_3^{(0)}+\dot{\bar{\psi}}\mathbf{e}_2^{(1)}+\dot{\bar{\theta}}\mathbf{e}_1^{(2)}. \quad (14)$$

Note that $\dot{\bar{\theta}}$ is multiplied by the vector $\mathbf{e}_1^{(2)}$, whereas no such axis appears in any of the rotations in (12). This demonstrates very clearly that the disputed "textbook reasoning" is inappropriate, since it does not explain why "the partial angular velocity that the rotation through θ about the axis $\mathbf{e}_1^{(1)}$ contributes to $\omega_{\rm tot}$" is directed along $\mathbf{e}_1^{(2)}$. On the other hand, an inspection of the general formula (9) together with the order of the rotations in (13), which as a consequence of our particular choice of successive rotation axes in the present example is no longer a straight reversal of (12), reveals the formal reason for the appearance of $\mathbf{e}_1^{(2)}$.

As a concluding piece of evidence in favor of our point, let us calculate directly the angular velocity (1) with the usual definition (2) of the Euler angles, but for the special case $\psi\equiv 0$, i.e., $\mathbf{R}(\mathbf{e}_3^{(2)},\psi) = \mathbf{1}$, in order to reduce the algebra. Again we find from the general formula (9) that

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\theta+\mathbf{R}(\mathbf{e}_1^{(1)},\theta)\cdot\boldsymbol{\omega}_\varphi,$$

with

$$\boldsymbol{\omega}_\varphi = \dot{\varphi}\mathbf{e}_3^{(0)} = \dot{\varphi}\mathbf{e}_3^{(1)}$$

and

$$\boldsymbol{\omega}_\theta = \dot{\theta}\mathbf{e}_1^{(1)}+\sin\theta\dot{\mathbf{e}}_1^{(1)}+(1-\cos\theta)\dot{\mathbf{e}}_1^{(1)}\times\mathbf{e}_1^{(1)}.$$

Since

$$\dot{\mathbf{e}}_1^{(1)} = \dot{\varphi}\mathbf{e}_2^{(2)}, \qquad \mathbf{e}_1^{(1)}\times\dot{\mathbf{e}}_1^{(1)} = \dot{\varphi}\mathbf{e}_3^{(1)} = \dot{\varphi}\mathbf{e}_3^{(0)}$$

and

$$\mathbf{R}(\mathbf{e}_1^{(1)},\theta)\cdot\mathbf{e}_3^{(1)} = \cos\theta\mathbf{e}_3^{(1)}-\sin\theta e_2^{(1)},$$

we obtain

$$\omega = \dot\theta e_1^{(1)} + \dot\varphi(\sin\theta e_2^{(1)} + e_3^{(0)} - \cos\theta e_3^{(1)}) + \dot\varphi(\cos\theta e_3^{(1)} - \sin\theta e_2^{(1)}) = \dot\theta e_1^{(1)} + \dot\varphi e_3^{(0)},$$

which is the result expected from (1) and (10). Note, however, that this final form of ω comes about in a way rather different from the picture that the erroneous "textbook reasoning" suggests. Indeed, the contribution $\dot\varphi e_3^{(0)}$ to the final expression is a part of ω_θ and arises from the time dependence of the corresponding rotation axis $e_1^{(1)}$, whereas the whole ω_φ contribution cancels out.

We hope to have convincingly demonstrated that it is incorrect to pretend to "derive" the instantaneous angular velocity of a rotating rigid body by the inappropriate arguments found in so many contemporary textbooks. Instead, physics teachers must revert to a proper analytical derivation, which in terms of coordinate-free rotation operators, as proposed above, is no longer tedious.

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PROBLEM

(a) Prove that the principal moments of inertia of a physical body satisfy triangular inequalities of the form $I_1 + I_2 \geq I_3$, and cyclic permutations of indices.

(b) *Corollary:* Find the upper and lower limits for the length of the third principal semiaxis of the ellipsoid of inertia, given that two of the semiaxes have the lengths a_1 and a_2. Let $a_1 = 20$ units, $a_2 = 19$ units as a specific example.

(Solution on page 264).

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