CLASE TRANSFORMACIONES CANÓNICAS.

Ecuaciones de Hamilton.

$$H(q_i, p_i, t) = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L} .$$

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\} \text{ 2n ecuaciones de 1}^{\text{er}} \text{ orden.}$$

Si $\mathcal{L}$ o H no dependen explícitamente del tiempo, H se conserva.

TRANSFORMACIONES CANÓNICAS! preservan el volumen en el espacio de fases. (Liouville).
$\int dp\,dq = \int dP\,dQ$.

Una transformación canónica $(q, p) \to (Q, P)$ es tal que mantiene la forma de la acción.

$$S = \int \sum_{\mu} p_\mu dq_\mu - H dt = \int \sum_{\mu} P_\mu dQ_\mu - \bar{H} dt + dF \quad (*)$$

término de borde en la acción sin consecuencias para el principio variacional.

$$dF = \sum_{\mu} p_\mu dq_\mu - P_\mu dQ_\mu + (\bar{H} - H) dt$$

$\Rightarrow F = F(q_\mu, Q_\mu, t)$ tal que

$$\left.\begin{aligned} \frac{\partial F}{\partial q_\mu} &= p_\mu \\ \frac{\partial F}{\partial Q_\mu} &= -P_\mu \\ \frac{\partial F}{\partial t} &= \bar{H} - H . \end{aligned}\right\}$$

puedo despejar Q_μ, P_μ en función de q_μ, p_μ
$\to$ obtuve una transformación canónica!

Ej: $F = \sum_{\mu} q_\mu Q_\mu$ tipo 1.

$$\frac{\partial F}{\partial q_\mu} = p_\mu \Rightarrow p_\mu = Q_\mu$$

$$\frac{\partial F}{\partial Q_\mu} = -P_\mu \Rightarrow -P_\mu = q_\mu .$$

La transformación canónica generada por esta F es

$$\begin{cases} Q_\mu = p_\mu \\ P_\mu = -q_\mu . \end{cases}$$

→ INTERCAMBIA LOS ROLES DE LAS q Y LAS p.

Funciones generatrices que dependen de (q, P) (TIPO 2).

En ⊛ escribimos $P_\mu dQ_\mu = d(P_\mu Q_\mu) - Q_\mu dP_\mu$.

$$\sum_\mu p_\mu dq_\mu - Hdt = \sum_\mu -Q_\mu dP_\mu - \bar{H}dt + d(\underbrace{F_1 + \sum_\mu P_\mu Q_\mu}_{=F_2}).$$

$$\Rightarrow dF_2 = \sum_\mu p_\mu dq_\mu + Q_\mu dP_\mu + (\bar{H} - H)dt$$

$$\Rightarrow F_2 = F_2(q_\mu, P_\mu, t).$$

$$\begin{cases} p_\mu = \dfrac{\partial F_2}{\partial q_\mu} \\ Q_\mu = \dfrac{\partial F_2}{\partial P_\mu} . \\ \bar{H} - H = \dfrac{\partial F_2}{\partial t} . \end{cases}$$

Generatrices de tipo 3.

$$p_\mu dq_\mu = d(p_\mu q_\mu) - q_\mu dp_\mu$$

$$\sum_\mu p_\mu dq_\mu - Hdt = \sum_\mu P_\mu dQ_\mu - \bar{H}dt + dF.$$

$$\sum_\mu d(p_\mu q_\mu) - q_\mu dp_\mu - P_\mu dQ_\mu + (\bar{H} - H)dt = dF.$$

$$\sum_\mu -q_\mu dp_\mu - P_\mu dQ_\mu + (\bar{H} - H)dt = d(\underbrace{F - p_\mu q_\mu}_{=F_3}).$$

$$\Rightarrow F_3 = F_3(p_\mu, Q_\mu, t).$$

$$\begin{cases} \dfrac{\partial F_3}{\partial p_\mu} = -q_\mu \\ \dfrac{\partial F_3}{\partial Q_\mu} = -P_\mu \\ \dfrac{\partial F_3}{\partial t} = \bar{H} - H . \end{cases}$$

Generatrices de tipo 4

$p_\mu dq_\mu = d(p_\mu q_\mu) - q_\mu dp_\mu.$

$P_\mu dQ_\mu = d(P_\mu Q_\mu) - Q_\mu dP_\mu.$

$\Rightarrow \sum_\mu -q_\mu dp_\mu + Q_\mu dP_\mu + (\bar{H}-H)dt = d(\underbrace{F + P_\mu Q_\mu - p_\mu q_\mu}_{F_4}).$

$\Rightarrow F_4 = F_4(p_\mu, P_\mu, t).$

$$\begin{cases} \dfrac{\partial F_4}{\partial p_\mu} = -q_\mu. \\ \dfrac{\partial F_4}{\partial P_\mu} = Q_\mu. \\ \dfrac{\partial F_4}{\partial t} = \bar{H} - H. \end{cases}$$

Función generatriz	Derivadas	Caso trivial.
$F = F_1(q, Q, t).$	$p_i = \frac{\partial F_1}{\partial q_i}$ $P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$; $Q_i = p_i$; $P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i}$ $Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$; $Q_i = q_i$; $P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i}$ $P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$; $Q_i = -q_i$; $P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i}$ $Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$; $Q_i = p_i$; $P_i = -q_i$.

- Composición:

$(q, p) \to (q', p') \to (q'', p'').$

$\sum p dq - H dt = \sum p' dq' - H' dt + dF.$

$\sum p' dq' - H' dt = \sum p'' dq'' - H'' dt + dG.$

$\sum p dq - H dt = \sum p'' dq'' - H'' dt + d(F+G).$

$\Rightarrow$ La composición es una transformación canónica con generatriz $F+G$.

Transformaciones canónicas y corchetes de Poisson.

$q = (q_1, \dots, q_n)$

$p = (p_1, \dots, p_n)$ $\qquad \{q, p, t\} \longrightarrow \{Q, P, t\}$.

$Q = (Q_1, \dots, Q_n)$

$P = (P_1, \dots, P_n)$.

La transformación es canónica si se verifica:

$$[f, g]_{q,p} = [f, g]_{Q,P} \quad \forall f, g.$$

Tiene que verificarse que:

- $[Q_i, Q_j]_{q,p} = 0$
- $[P_i, P_j]_{q,p} = 0$.
- $[Q_i, P_j]_{q,p} = \delta_{ij}$

$$[f,g]_{q,p} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right)$$

PROBLEMA 11.

$$\begin{cases} Q = \ln\left(\frac{\sin p}{q}\right) \\ P = q \operatorname{cotg} p \end{cases}$$

Para ver si la transformación es canónica, miramos los corchetes de Poisson.

$$[Q, P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

- $\frac{\partial P}{\partial p} = \frac{\partial}{\partial p}(q \operatorname{cotg} p) = q \frac{\partial}{\partial p}\left(\frac{\cos p}{\sin p}\right) = q\left[-\frac{\sin p}{\sin p} - \frac{\cos p}{\sin^2 p}\cos p\right]$

$$= -q\left[1 + \frac{\cos^2 p}{\sin^2 p}\right] = -\frac{q}{\sin^2 p}.$$

- $\frac{\partial P}{\partial q} = \operatorname{cotg} p.$
- $\frac{\partial Q}{\partial p} = \frac{q}{\sin p}\frac{\cos p}{q} = \operatorname{cotg} p.$
- $\frac{\partial Q}{\partial q} = -\frac{q}{\cancel{\sin p}}\frac{\cancel{\sin p}}{q^2} = -\frac{1}{q}$

$$\Rightarrow [Q, P] = -\frac{1}{\cancel{q}}\cdot\left(-\frac{\cancel{q}}{\sin^2 p}\right) - \operatorname{cotg}^2 p = \frac{1}{\sin^2 p} - \frac{\cos^2 p}{\sin^2 p} = \frac{\sin^2 p}{\sin^2 p} = 1$$

$\Rightarrow$ la transformación es canónica.

Función generatriz:

$F_1(q, Q)$

$$\begin{cases} p_i = \frac{\partial F_1}{\partial q_i} \\ P_i = -\frac{\partial F_1}{\partial Q_i} \end{cases}$$

$$Q = \ln\left(\frac{\sin p}{q}\right)$$

$$P = q \cdot \operatorname{cotg} p = q \cdot \frac{\cos p}{\sin p}$$

$$e^Q = \frac{\sin p}{q} \Rightarrow e^Q = \frac{\cancel{\sin p}}{P \cdot \cancel{\sin p}} \cos p = \frac{\cos p}{P}$$

$$q = P \cdot \frac{\sin p}{\cos p} \qquad \Rightarrow \cos p = P \cdot e^Q$$

$$p = \arccos(P \cdot e^Q)$$

$$\sin p = q \cdot e^Q$$

$$p = \arcsin(q\, e^Q) = \frac{\partial F_1}{\partial q}.$$

$$P = q \frac{\cos p}{\sin p} = \cancel{q} \cdot \frac{\sqrt{1 - q^2 e^{2Q}}}{\cancel{q} e^Q}$$

$$\cos p = \sqrt{\cos^2 p} = \sqrt{1 - \sin^2 p}$$

$$P = \frac{\sqrt{1 - q^2 e^{2Q}}}{e^Q} = -\frac{\partial F_1}{\partial Q}$$

$$-\frac{\partial F_1}{\partial Q} = \sqrt{e^{-2Q} - q^2 e^{2Q}}$$

$$\int \frac{1}{e^Q} \sqrt{1 - q^2 e^{2Q}}\, dQ = -e^{-Q}\sqrt{1 - q^2 e^{2Q}} - q \arcsin(q e^Q) + A(q)$$

Además,

$$p = \arcsin(q \cdot e^Q) = \frac{\partial F_1}{\partial q}.$$

$$\Rightarrow F_1 = e^{-Q}\sqrt{1 - e^{2Q} q^2} + q \cdot \arcsin(q \cdot e^Q).$$

$$\Rightarrow F_1(q, Q) = e^{-Q}\sqrt{1 - e^{2Q} q^2} + q \cdot \arcsin(q \cdot e^Q).$$

Si quiero $F_2(q, P)$,

$$F_2(q, P) = F_1(q, Q) + Q \cdot P.$$

Ya sabemos que $P = \frac{1}{e^Q}\sqrt{1 - q^2 e^{2Q}}$.

En general,

$$\begin{tikzcd}[column sep=huge, row sep=huge]
F_1(q,Q,t) \arrow[r, shift left, "+\sum_i Q_i P_i"] \arrow[d, shift right, "-\sum_i q_i p_i"'] & F_2(q,P,t) \arrow[l, shift left, "-"] \arrow[d, shift left, "-\sum_i q_i p_i"] \\
F_3(Q,p,t) \arrow[u, shift right, "+"'] \arrow[r, shift right, "+\sum_i Q_i P_i"'] & F_4(p,P,t) \arrow[l, shift right, "-"'] \arrow[u, shift left, "+"]
\end{tikzcd}$$

Ahora aplicamos la transformación al H del oscilador armónico.

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 . \qquad \frac{\partial F}{\partial t} = 0 \Rightarrow H = \overline{H} .$$

$$q = q(P,Q) .$$

$$p = p(P,Q) = \arccos(P \cdot e^{Q}) .$$

$$P = \frac{1}{e^{Q}} \sqrt{1 - q^2 e^{2Q}} \Rightarrow P^2 e^{2Q} = 1 - q^2 e^{2Q} .$$

$$q^2 = \frac{1 - P^2 e^{2Q}}{e^{2Q}} = \frac{1}{e^{2Q}} - P^2 = e^{-2Q} - P^2 . \Rightarrow \boxed{q^2 = e^{-2Q} - P^2}$$

$$\boxed{H(Q,P) = \frac{1}{2m} \arccos^2(P e^{Q}) + \frac{m\omega^2}{2} \left(e^{-2Q} - P^2\right) .}$$

PROBLEMA 12

$$H = \frac{p^2}{2m} + \frac{\omega^2 m}{2} q^2 .$$

$$F_1(q, Q) = \lambda q^2 \cot Q .$$

Con λ tal que $K(Q, P) = \omega P$.

- $\frac{\partial F_1}{\partial q} = p.$
- $\frac{\partial F_1}{\partial Q} = -P$
- $\frac{\partial F_1}{\partial t} = K - H . \Rightarrow \boxed{K = H}$

$$\frac{\partial F_1}{\partial q} = 2\lambda q \cot Q = p . \Rightarrow \cot g\, Q = \frac{1}{tg\, Q} = \frac{p}{2\lambda q} .$$

$$\frac{\partial F_1}{\partial Q} = \cancel{-} \frac{\lambda q^2}{\sin^2 Q} = \cancel{-} P$$

$$tg\, Q = \frac{2\lambda q}{p}$$

$$Q = \operatorname{arctg}\left(\frac{2\lambda q}{p}\right)$$

$$q^2 = \frac{P}{\lambda} \sin^2 Q$$

$$q = \sqrt{\frac{P}{\lambda}} \sin Q$$

$$p = 2\lambda \sqrt{\frac{P}{\lambda}}\, \cancel{\sin Q}\, \frac{\cos Q}{\cancel{\sin Q}} \Rightarrow p = 2\sqrt{\lambda P} \cos Q$$

$$0 = \omega P - \frac{4\lambda P \cos^2 Q}{2m} - \frac{\omega^2 m}{2} \frac{P}{\lambda} \sin^2 Q .$$

$$\Rightarrow \frac{4\lambda}{2m} = \frac{\omega^2 m}{2\lambda} = \omega$$

$$\Rightarrow \frac{2\lambda}{m} = \omega \rightarrow \boxed{\lambda = \frac{m\omega}{2}}$$

$$\frac{\omega^2 m}{2\lambda} = \omega \rightarrow \boxed{\lambda = \frac{m\omega}{2}} \checkmark$$

$\Rightarrow$ En las coordenadas Q, P, tenemos

$K = \omega P$.

Las ecs. de Hamilton:

$$\dot{Q} = \frac{\partial K}{\partial P}$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \Rightarrow P = cte. = \frac{E}{\omega}.$$

$$\dot{Q} = \omega \Rightarrow Q(t) = \omega t + \alpha.$$

La transformación inversa es $G = -F_1$.

$G(Q, q) = -\lambda Q^2 \cot g\, q$.

$$\begin{cases} q = \sqrt{\dfrac{2P}{m\omega}} \sin Q \\ p = \sqrt{2Pm\omega} \cos Q. \end{cases}$$

$$\operatorname{tg} Q \left(\frac{\frac{2P}{m\omega}}{2Pm\omega} \right)^{1/2} = \frac{q}{p} \Rightarrow \operatorname{tg} Q = \frac{q}{p} \cdot m\omega$$

$$\left\{ Q = \operatorname{arctg}\left(\frac{m\omega}{p} q \right) \right\}$$

$$q^2 + p^2 = \frac{2P}{m\omega} \sin^2 Q + 2Pm\omega \cos^2 Q.$$

$$\Rightarrow p^2 + m^2\omega^2 q^2 = 2Pm\omega \cos^2 Q + 2Pm\omega \sin^2 Q = 2Pm\omega.$$

$$\Rightarrow \left\{ P = \frac{1}{2m\omega} \left(p^2 + q^2 m^2 \omega^2 \right) \right\} = cte.$$

$$\begin{cases} p = \sqrt{2mE} \cos(\omega t + \alpha). \\ q = \sqrt{\dfrac{2E}{m\omega^2}} \sin(\omega t + \alpha). \end{cases}$$

SOLUCIÓN GENERAL EN LAS COORDENADAS ORIGINALES.

b) Generatriz de tipo 2 que genera la misma transformación.

$$p = \frac{\partial F_2}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P}. \qquad F_2 = F_2(q, P).$$

$$F_1(q, Q) \xrightarrow{+Q.P} F_2(q, P)$$

$$F_1(q, Q) = \frac{m\omega}{2} q^2 \cot g\, Q$$

$$F_2(q, P) = \frac{m\omega}{2} q^2 \cot g\, Q(q,P) + Q \cdot P$$

$$tg\, Q = m\omega \frac{q}{p} \Rightarrow \cot g\, Q = \frac{p}{m\omega q}$$

¿$p(P, q)$?

$$P = \frac{1}{2m\omega}(p^2 + m^2\omega^2 q^2)$$

$$2m\omega P = p^2 + m^2\omega^2 q^2 \Rightarrow p^2 = 2m\omega P - m^2\omega^2 q^2.$$

$$\Rightarrow \left\{ F_2(q, P) = \frac{\cancel{m}\omega^2}{2} q^2 \frac{(2m\omega P - m^2\omega^2 q^2)^{1/2}}{\cancel{m\omega}\cancel{q}} + P.\,\mathrm{arctg}\left[\frac{m\omega\, q}{\sqrt{2m\omega P - m^2\omega^2 q^2}}\right] \right\}$$