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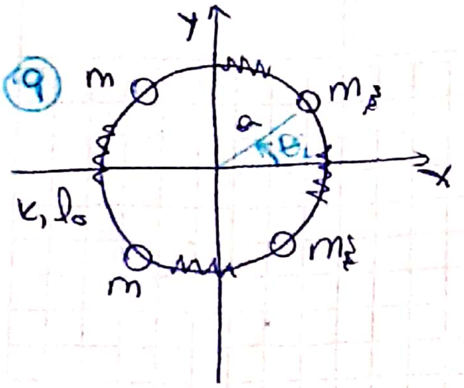

¿ frecuencias y modos normales?

mov plano y con $r = a$ (cte) $\Rightarrow$ 4 grados de libertad (uno por c/ masa)

- Escribo la energía cinética:

$$T = \frac{m}{2} a^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + \dot{\theta}_4^4 \right)$$

- Escribo la energía potencial:

$$V = \frac{k}{2}\Big[\left(|\theta_2-\theta_1| a - l_0\right)^2 + \left(|\theta_3-\theta_2| a - l_0\right)^2 + \left(|\theta_4-\theta_3| a - l_0\right)^2 + \left(|2\pi+\theta_1-\theta_4| a - l_0\right)^2 \Big]$$

- Ahora lo escribo en pequeñas oscilaciones:

$$\left.\begin{aligned} \theta_1 &= \eta_1 \\ \theta_2 &= \eta_2 + \frac{\pi}{2} \\ \theta_3 &= \eta_3 + \pi \\ \theta_4 &= \eta_4 + \frac{3\pi}{2} \end{aligned}\right\} \Rightarrow V(\eta_i) = \frac{k}{2}\Big[\left(\left(\frac{\pi}{2}+\eta_2-\eta_1\right)a-l_0\right)^2 + \left(\left(\frac{\pi}{2}+\eta_3-\eta_2\right)a-l_0\right)^2 + \left(\left(\frac{\pi}{2}+\eta_4-\eta_3\right)a-l_0\right)^2 + \left(\left(\frac{\pi}{2}+\eta_1-\eta_4\right)a-l_0\right)^2\Big]$$

$$\Rightarrow V_1 = \frac{k}{2}\Big[\left(\frac{\pi}{2}a\right)^2 + (\eta_2-\eta_1)^2 a^2 + l_0^2 - \underbrace{2\frac{\pi}{2}a^2(\eta_1-\eta_2)}_{(i)} - 2\frac{\pi}{2}a l_0 - \underbrace{2(\eta_2-\eta_1) a l_0}_{(ii)}\Big]$$

análogo en las otras 3 sumas.

Los términos (i) e (ii) se cancelan cuando se suman todas las partes, quedando el potencial total:

$$V = \frac{k}{2} a^2 \pi^2 - 4\frac{k}{2}\pi a l_0 + \frac{k a^2}{2}\Big[(\eta_2-\eta_1)^2 + (\eta_3-\eta_2)^2 + (\eta_4-\eta_3)^2 + (\eta_1-\eta_4)^2\Big]$$

me puedo sacar de encima las constantes, y desarrollo los cuadrados:

$$\tilde{V} = \frac{k a^2}{2}\Big[2\eta_1^2 + 2\eta_2^2 + 2\eta_3^2 + 2\eta_4^2 - 2\eta_2\eta_1 - 2\eta_2\eta_3 - 2\eta_3\eta_4 - 2\eta_1\eta_4\Big]$$

Ahora escribo las matrices:

$$\mathbb{M} = \begin{pmatrix} ma^2 & 0 & 0 & 0 \\ 0 & ma^2 & 0 & 0 \\ 0 & 0 & ma^2 & 0 \\ 0 & 0 & 0 & ma^2 \end{pmatrix} = ma^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{V} = \begin{pmatrix} +2ka^2 & -ka^2 & 0 & -ka^2 \\ -ka^2 & +2ka^2 & -ka^2 & 0 \\ 0 & -ka^2 & +2ka^2 & -ka^2 \\ -ka^2 & 0 & -ka^2 & +2ka^2 \end{pmatrix} = ka^2 \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

$\Rightarrow$ Busco: $\det(\mathbb{V} - \omega^2 \mathbb{M}) = 0$ ~~[illegible]~~
donde ω son las frecuencias (autovalores):

$$\Rightarrow \det\left(ka^2 \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} - m a^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = 0$$

$$\Rightarrow \det\left(\frac{k}{m} \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} - \omega^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = 0$$

$$\Rightarrow \begin{cases} \omega_1^2 = 0 \rightarrow \text{traslación rígida de todas las masas} \\ \omega_2^2 = \omega_3^2 = \dfrac{2k}{m} \\ \omega_4^2 = \dfrac{4k}{m} \end{cases}$$

- Busco los autovectores:

$$\omega_1^2 = 0 \Rightarrow \frac{k}{m} \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \bar{A}_1 = a_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega_4^2 = 4\frac{k}{m} \Rightarrow \begin{pmatrix} -2 & -1 & 0 & -1 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ -1 & 0 & -1 & -2 \end{pmatrix} \bar{A}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \bar{A}_2 = a_4 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\omega_3^2 = \omega_2^2 = \frac{2k}{m} \Rightarrow \begin{pmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \bar{A}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (i)$$

Hay 2 grados de libertad, es decir que hay un plano de posibilidades; elijo 2 vectores que cumplan con (i) y que sean ortogonales.

$$\Rightarrow \quad \bar{A}_2 = a_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad y \quad \bar{A}_3 = a_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

- Ahora normalizo los autovectores (en la métrica de M).

$$1 = \bar{A}_1^T \cdot M \cdot \bar{A}_1 = a_1^2 m (1\ 1\ 1\ 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = m a_1^2 \cdot 4 \Rightarrow a_1 = \frac{1}{\sqrt{4m}}$$

$$1 = \bar{A}_2^T M \bar{A}_2 = a_2^2 m (1\ 0\ -1\ 0) \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = m a_2^2 2 \Rightarrow a_2 = \frac{1}{\sqrt{2m}}$$

$$1 = \bar{A}_3^T M \bar{A}_3 = m a_3^2 (0\ 1\ 0\ -1) \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = m 2 a_3^2 \Rightarrow a_3 = \frac{1}{\sqrt{2m}}$$

$$1 = \bar{A}_4^T M \bar{A}_4 = m a_4^2 (1\ -1\ 1\ -1) \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = m 4 a_4^2 \Rightarrow a_4 = \frac{1}{\sqrt{4m}}$$

- La solución general queda:

$$\cancel{\eta(t) = \frac{(B_1 + B_1' t)}{2m} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{B_2}{\sqrt{2m}} \cos(\omega_2 t + \varphi_2) + \frac{B_3}{\sqrt{2m}} \cos(\omega_3 t + \varphi_3) + \frac{B_4}{2m} \cos(\omega_4 t + \varphi_4)}$$

$$\boxed{\bar{\eta}(t) = \frac{(B_1 + B_1' t)}{\sqrt{4m}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{B_2 \cos(\omega_2 t + \varphi_2)}{\sqrt{2m}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \frac{B_3 \cos(\omega_3 t + \varphi_3)}{\sqrt{2m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \frac{B_4 \cos(\omega_4 t + \varphi_4)}{\sqrt{4m}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}}$$

con B_i y φ_i dependientes de las condiciones iniciales

NOTA

· Para hallar las coord normales: $\bar{\xi} = A^T M \bar{\eta}$

$$A^T = \frac{1}{\sqrt{m}}\begin{pmatrix} 1/2 & 1/\sqrt{2} & 0 & 1/2 \\ 1/2 & 0 & 1/\sqrt{2} & -1/2 \\ 1/2 & -1/\sqrt{2} & 0 & 1/2 \\ 1/2 & 0 & -1/\sqrt{2} & -1/2 \end{pmatrix}^T = \frac{1}{\sqrt{m}}\begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/2 & 1/2 & -1/2 \end{pmatrix}$$

$$\Rightarrow A^T \cdot M = \frac{1}{\sqrt{m}}\, m \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/2 & 1/2 & -1/2 \end{pmatrix}.$$

$$\Rightarrow \bar{\xi} = \sqrt{m}\begin{pmatrix} \frac{1}{2}(\eta_1+\eta_2+\eta_3+\eta_4) \\ \frac{1}{\sqrt{2}}(\eta_1-\eta_3) \\ \frac{1}{\sqrt{2}}(\eta_2-\eta_4) \\ \frac{1}{2}(\eta_1-\eta_2+\eta_3-\eta_4) \end{pmatrix}$$

$$\begin{cases} \xi_1 = \dfrac{4\sqrt{m}}{2\sqrt{4m}}(B_1 + B_1' t) = B_1 + B_1' t \\[2ex] \xi_2 = \dfrac{2\sqrt{m}B_2\cos(\omega_2 t+\varphi_2)}{\sqrt{2}\sqrt{2m}} = B_2\cos(\omega_2 t + \varphi_2) \\[2ex] \xi_3 = \dfrac{\sqrt{m}}{\sqrt{2}}\dfrac{2}{\sqrt{2m}} B_3\cos(\omega_3 t+\varphi_3) = B_3\cos(\omega_3 t+\varphi_3) \\[2ex] \xi_4 = \dfrac{\sqrt{m}}{2}\dfrac{4B_4}{\sqrt{4m}}\cos(\omega_4 t+\varphi_4) = B_4\cos(\omega_4 t+\varphi_4) \end{cases}$$

· Impongo condiciones iniciales:

$$\left.\begin{array}{l} \xi_1(0) = \xi_2(0) = \xi_3(0) = 0 \qquad \xi_4(0) = b \\ \dot{\xi}_1(0) = \dot{\xi}_2(0) = \dot{\xi}_3(0) = \underbrace{\dot{\xi}_4(0) = 0}_{(*)} \end{array}\right\} \Rightarrow \xi_1 = \xi_2 = \xi_3 = 0 \text{ y } \overset{(*)}{\varphi_4 = 0}$$

$$\Rightarrow \xi_4(t) = b\cos(\omega_4 t)$$

• volviendo a las coordenadas $\bar{\eta}$: $\bar{\eta} = A\bar{\xi}$

$$\Rightarrow \bar{\eta} = \frac{1}{\sqrt{m}} \begin{pmatrix} 1/2 & 1/\sqrt{2} & 0 & 1/2 \\ 1/2 & 0 & 1/\sqrt{2} & -1/2 \\ 1/2 & -1/\sqrt{2} & 0 & 1/2 \\ 1/2 & 0 & -1/\sqrt{2} & -1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ b\cos(\omega_4 t) \end{pmatrix}$$

$$\bar{\eta} = \frac{1}{2\sqrt{m}} b\cos(\omega_4 t) \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

• vuelvo a $\bar{\theta}$:

$$\bar{\theta} = \bar{\eta} + \begin{pmatrix} 0 \\ \pi/2 \\ \pi \\ \frac{3\pi}{2} \end{pmatrix}$$