Corchetes de Poisson

10. Demuestre las siguientes propiedades de los corchetes de Poisson, siendo f, g, h funciones arbitrarias de p_i, q_i; $F(f)$ es una función de f y c es una constante.

(a) $[f, c] = 0$; $[f, f] = 0$; $[f, g] + [g, f] = 0$; $[f + g, h] = [f, h] + [g, h]$; $[fg, h] = f[g, h] + [f, h]g$; $\frac{\partial}{\partial t}[f, g] = [\frac{\partial f}{\partial t}, g] + [f, \frac{\partial g}{\partial t}]$; $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$; $[f, F(f)] = 0$

(b) $[q_i, q_j] = [p_i, p_j] = 0$; $[q_i, p_j] = \delta_{ij}$; $[f, q_i] = -\frac{\partial f}{\partial p_i}$; $[f, p_i] = \frac{\partial f}{\partial q_i}$

(c) $[f, g^n] = ng^{n-1}[f, g]$; $[g, F(f)] = F'(f)[g, f]$

$$f(q_1, \ldots, q_N, p_1 \ldots p_N)$$

$$g(q_1, \ldots, q_N, p_1 \ldots p_N)$$

$$[f, g]_{q_i, p_i} = \sum_{i=0}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

(a) $[f,c]=0$; $[f,f]=0$; $[f,g]+[g,f]=0$; $[f+g,h]=[f,h]+[g,h]$; $[fg,h]=f[g,h]+[f,h]g$; $\frac{\partial}{\partial t}[f,g]=[\frac{\partial f}{\partial t},g]+[f,\frac{\partial g}{\partial t}]$; $[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0$; $[f,F(f)]=0$

$$[f,g]=\sum_{i=0}^{N}\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

$$[g,f]=\sum_{i=0}^{N}\frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}-\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}=-[f,g]$$

Antisimetría

$$[f+g,h]=\sum_{i=0}^{N}\frac{\partial(f+g)}{\partial q_i}\frac{\partial h}{\partial p_i}-\frac{\partial(f+g)}{\partial p_i}\frac{\partial h}{\partial q_i}=[f,h]+[g,h]$$

$$[f.g,h]=\sum_{i=0}^{N}\frac{\partial(f.g)}{\partial q_i}\frac{\partial h}{\partial p_i}-\frac{\partial(f.g)}{\partial p_i}\frac{\partial h}{\partial q_i}=f[g,h]+[f,h]g$$

$$\frac{\partial}{\partial t}[f,g]=[\frac{\partial f}{\partial t},g]+[f,\frac{\partial g}{\partial t}]$$

Linealidad

$$\frac{\partial}{\partial t}[f,g]=[\frac{\partial f}{\partial t},g]+[f,\frac{\partial g}{\partial t}];$$

$$[\frac{\partial f}{\partial t},g]=\sum_{i=0}^{N}\frac{\partial^2 f}{\partial q_i\partial t}\frac{\partial g}{\partial p_i}-\frac{\partial^2 f}{\partial p_i\partial t}\frac{\partial g}{\partial q_i}$$

$$[f,\frac{\partial g}{\partial t}]=\sum_{i=0}^{N}\frac{\partial f}{\partial q_i}\frac{\partial^2 g}{\partial p_i\partial t}-\frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial q_i\partial t}$$

$$\frac{\partial}{\partial t}[f,g]=\frac{\partial}{\partial t}(\sum_{i=0}^{N}\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i})$$

$$\frac{\partial^2 f}{\partial q_i\partial t}=\frac{\partial^2 f}{\partial t\,\partial q_i}\qquad\frac{\partial^2 f}{\partial p_i\partial t}=\frac{\partial^2 f}{\partial t\,\partial p_i}\qquad\frac{\partial^2 g}{\partial q_i\partial t}=\frac{\partial^2 g}{\partial t\,\partial q_i}\qquad\frac{\partial^2 g}{\partial p_i\partial t}=\frac{\partial^2 g}{\partial t\,\partial p_i}$$

$$[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0;$$

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$$[f,F(f)]=0$$

$$[g,F(f)]=F'(f)[g,f]$$

$$[g,F(f)]=\sum_{i=0}^{N}\frac{\partial g}{\partial q_i}\frac{\partial F}{\partial p_i}-\frac{\partial g}{\partial p_i}\frac{\partial F}{\partial q_i}=\frac{dF}{df}[g,f]=F'(f)[g,f]$$

$$\frac{\partial F}{\partial p_i}=\frac{dF}{df}\frac{\partial f}{\partial p_i} \qquad \frac{\partial F}{\partial q_i}=\frac{dF}{df}\frac{\partial f}{\partial q_i}$$

(b) $[q_i, q_j] = [p_i, p_j] = 0; [q_i, p_j] = \delta_{ij}; [f, q_i] = -\frac{\partial f}{\partial p_i}; [f, p_i] = \frac{\partial f}{\partial q_i}$

$$[q_k, q_j] = \sum_{i=0}^{N} \frac{\partial q_k}{\partial q_i}\frac{\partial q_j}{\partial p_i} - \frac{\partial q_k}{\partial p_i}\frac{\partial q_j}{\partial q_i} = 0$$

$$\frac{\partial q_k}{\partial q_i} = \delta_{ki} \qquad \frac{\partial q_k}{\partial p_i} = 0$$

$$[p_k, p_j] = \sum_{i=0}^{N} \frac{\partial p_k}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial p_k}{\partial p_i}\frac{\partial p_j}{\partial q_i} = 0$$

$$\frac{\partial p_k}{\partial p_i} = \delta_{ki} \qquad \frac{\partial p_k}{\partial q_i} = 0$$

$$[q_k, p_j] = \sum_{i=0}^{N} \frac{\partial q_k}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial q_k}{\partial p_i}\frac{\partial p_j}{\partial q_i} = \delta_{ki}\delta_{ji} = \delta_{kj}$$

$$[f, q_j] = \sum_{i=0}^{N} \frac{\partial f}{\partial q_i}\frac{\partial q_j}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial q_j}{\partial q_i} = -\frac{\partial f}{\partial p_j}$$

$$[f, p_j] = \sum_{i=0}^{N} \frac{\partial f}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial p_j}{\partial q_i} = \frac{\partial f}{\partial q_j}$$

(c) $[f, g^n] = ng^{n-1}[f, g]$;

$$[f,g^n]=[f,g^{n-1}.g]=[f,g^{n-1}]g+g^{n-1}[f,g]$$

$$[f,g^{n-1}]=[f,g^{n-2}.g]=[f,g^{n-2}]g+g^{n-2}[f,g]$$

$$[f,g^n]=[f,g^{n-2}]g^2+2.g^{n-1}[f,g]=[f,g^{n-3}]g^3+3.g^{n-1}[f,g]$$

Hago hasta n-1:

$$[f,g^n]=[f,g]g^{n-1}+(n-1).g^{n-1}[f,g]=n\,g^{n-1}[f,g]$$