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Some Divergent Trigonometric Integrals

Erik Talvila

1. INTRODUCTION. Browsing through an integral table on a dull Sunday afternoon some time ago, I came across four divergent trigonometric integrals. I wondered how these divergent integrals ended up in a respectable table. Tracing their history, it turned out they were originally "evaluated" when some convergent integrals were differentiated under the integral sign with respect to a parameter. We give a simple proof that these integrals diverge, look at their history in print and then make some final remarks about necessary and sufficient conditions for differentiating under the integral sign. We have no intent to defame either the well known mathematician who made the original error, or the editors of the otherwise fine tables in which the integrals appear. We all make mistakes and we're not out to point the finger at anyone; in this regard see the last two exercises of Chapter 2 in [**28**]. Maple and Mathematica also have considerable difficulties with these integrals.

2. FOUR DIVERGENT INTEGRALS. Here they are. Throughout, a and b are positive real numbers. Purported values appear on the right:

$$\int\limits_{x=0}^{\infty} x \left\{ \begin{matrix} \sin(ax^2) \\ \cos(ax^2) \end{matrix} \right\} \sin(bx)\, dx \text{ `}=\text{' } \frac{b}{4a}\sqrt{\frac{\pi}{2a}}\left[\sin\left(\frac{b^2}{4a}\right) \pm \cos\left(\frac{b^2}{4a}\right)\right] \tag{1}$$

$$\int\limits_{x=0}^{\infty} x \left\{ \begin{matrix} \sin(ax^2) \\ \cos(ax^2) \end{matrix} \right\} \cos(bx)\, dx \text{ `}=\text{' } \tag{2}$$

$$\frac{1}{2a}\left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} \mp \frac{b}{2a}\sqrt{\frac{\pi}{2a}}\left[\left\{ \begin{matrix} \sin[b^2/(4a)] \\ \cos[b^2/(4a)] \end{matrix} \right\} C\left(\frac{b^2}{4a}\right) \mp \left\{ \begin{matrix} \cos[b^2/(4a)] \\ \sin[b^2/(4a)] \end{matrix} \right\} S\left(\frac{b^2}{4a}\right)\right].$$

The two Fresnel integrals are

$$C(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \cos t \frac{dt}{\sqrt{t}} \quad \text{and} \quad S(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \sin t \frac{dt}{\sqrt{t}}. \tag{3}$$

In the literature, one sees the symbols C and S used to denote several different definitions of the Fresnel integrals.

Let's prove that (1) and (2) diverge.

Proposition. The integrals in (1) and (2) diverge.

Proof. Consider $A := \int_{-\infty}^{\infty} xe^{i(x^2+x)}\, dx$ (which, unfortunately, does not exist). Since the integrand is continuous, this integral exists if and only if the limits

$$\lim_{T\to\infty}\int_0^T xe^{i(x^2+x)}\, dx \qquad \text{and} \qquad \lim_{T\to\infty}\int_{-T}^0 xe^{i(x^2+x)}\, dx$$

both exist. Let $T_1, T_2 > 0$. Integrate by parts and complete the square:

$$\int_{-T_1}^{T_2} xe^{i(x^2+x)}\,dx = \frac{1}{2i}\left[e^{i(T_2^2+T_2)} - e^{i(T_1^2-T_1)}\right] - \frac{e^{-i/4}}{2}\int_{-T_1+1/2}^{T_2+1/2} e^{ix^2}\,dx. \tag{4}$$

Let's look at the convergence of $I := \int_{-\infty}^{\infty} e^{ix^2}\,dx$. Use the substitution $x^2 = t$. Then $I = \int_0^{\infty} e^{it}dt/\sqrt{t} = \sqrt{2\pi}(C(\infty) + iS(\infty))$. This can be seen to converge by applying Dirichlet's Test over $(1, \infty)$ [**24**, p. 261]. More properly, we should start with the 't' version of I, show that it converges and then transform back to the 'x' version. In fact, many ways have been found to evaluate I. One method is to use contour integration: Rotate the integral $\int_{-\infty}^{\infty} e^{-x^2}dx$ by $\pi/4$ in the complex plane; see [**27**, p. 184]. A second method is to use the gamma function [**24**, p. 272]. The result is $I = e^{i\pi/4}\sqrt{\pi}$. Now, as $T_1, T_2 \to \infty$ (independently) the final integral in (4) becomes I but the bracketed term fails to have a limit. Hence, the integral A diverges.

To get the integrals in (1) and (2) we do the following. Suppose $B := \int_{-\infty}^{\infty} xe^{i(x^2-x)}\,dx$ converges. The transformation $x \mapsto -x$ gives $B = -A$ so B diverges. The transformations $x \mapsto x/\sqrt{a} \pm (\sqrt{a} - b)/(2a)$ now show that $C := \int_{-\infty}^{\infty} xe^{i(ax^2 \pm bx)}\,dx$ diverges for all positive a and b. Finally, if the integrals in (1) and (2) converge then we can form the four convergent linear combinations

$$\int_{x=0}^{\infty} x\left[\cos(ax^2)\cos(bx) \mp \sin(ax^2)\sin(bx)\right]dx$$

$$\int_{x=0}^{\infty} x\left[\sin(ax^2)\cos(bx) \pm \cos(ax^2)\sin(bx)\right]dx.$$

But, the addition formulas for the sine and cosine functions followed by Euler's formula yields C. Hence, the integrals in (1) and (2) diverge. ∎

To see the manner in which the integrals diverge, let

$$\begin{aligned} A_T &:= \int_{-T}^{T} xe^{i(x^2+x)}\,dx \\ &= e^{iT^2}\sin T - \frac{e^{-i/4}}{2}\int_{-T+1/2}^{T+1/2} e^{ix^2}\,dx. \end{aligned} \tag{5}$$

As $T \to \infty$, the integral term in (5) has limit $\sqrt{\pi}\,e^{i(\pi-1)/4}/2$, whereas the term $e^{iT^2}\sin T$ oscillates rapidly with unit magnitude. This also shows that our integrals do not even exist as Cauchy principal values.

3. HISTORY OF THE DIVERGENT INTEGRALS. Now we look at the history of (1) and (2) in print. The thickest book of integrals (3500 pages in five volumes) is that of Prudnikov, Brychkov, and Marichev [**23**]. Our integrals appear in Volume I, 2.5.22. They also appear in the original Russian edition [**22**]; neither version references sources. It is interesting that they do not appear in the earlier book [**9**] by Ditkin and Prudnikov. Other major tables they are absent from include [**10**], [**16**], [**20**], and [**21**].

As the tables by Erdélyi and Oberhettinger are quite comprehensive, one suspects it was noticed that these integrals diverge and they were intentionally omitted. However, they are contained in the Gradshteyn and Ryzhik tome [**14**, 3.851]. They are not in the first few Russian editions, but the enlarged 1963 edition [**12**] includes them. All subsequent Russian editions and English translations beginning 1965 [**13**] contain the integrals in (1) and (2). Now, Gradshteyn and Ryzhik do give references. They say (in a garbled citation) that our integrals come from tables by Bierens de Haan.

David Bierens de Haan (1822–1895) was a Dutch mathematician noted for compiling tables of integrals, for actuarial work, for writing various essays in the history of science and mathematics, for producing an encyclopædic biography of Dutch scientists, and for being an early editor of the works of Christian Huygens—a mammoth task—it took until 1950 when the 22^{nd} volume was finally published. A complete list of de Haan's publications is given in [**18**]. There have been several papers on his life and work. See [**25**] for references, photos, and an interesting reproduction of the 1935 title page from a Japanese edition of his integral table. His 1858 *Tables d'intégrales définies* [**2**] was the first really substantial table of integrals. It was enlarged and corrected in an 1867 edition [**4**]. For nearly a century these were the preeminent integral tables. The 1867 edition was still being reprinted in 1957 [**5**], three years after the publication of the Bateman Manuscript tables [**10**]. The integrals in (1) appear in the 1858 table [**2**, formulas 193.17 and 193.18], in an 1862 companion volume that details the techniques used to compute integrals in the tables [**3**, p. 443], and in the 1867 table [**4**, formulas 150.4, 150.7].

Now, Bierens de Haan lists Cauchy as his source for (1). An examination of Cauchy's works (27 volumes!) shows that these integrals appear twice [**8**] (1815) and [**6**] (see also [**7**]) (1825). In both instances, Cauchy correctly obtains the convergent integrals

$$\int_{x=0}^{\infty} \left\{ \begin{matrix} \sin(ax^2) \\ \cos(ax^2) \end{matrix} \right\} \cos(bx)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} \left[\cos\left(\frac{b^2}{4a}\right) \mp \sin\left(\frac{b^2}{4a}\right) \right] \tag{6}$$

with $a = 1$. He then proceeds to differentiate under the integral sign with respect to b. Very bad! The functions defined by the integrals in (6) are certainly differentiable since the right side of (6) is differentiable. But, differentiating under the integral sign leads to our divergent integrals (1). It is therefore Cauchy, the "Father of rigour", who commits an error that has been copied for 185 years. (The appropriateness of this epithet is contested. One in favour is [**11**]. One against is [**15**].) Bierens de Haan repeats this argument in [**3**, p. 443].

When the two integrals

$$\int_{x=0}^{\infty} \left\{ \begin{matrix} \sin(ax^2) \\ \cos(ax^2) \end{matrix} \right\} \sin(bx)\, dx =$$

$$\sqrt{\frac{\pi}{2a}} \left[\left\{ \begin{matrix} \cos[b^2/(4a)] \\ \sin[b^2/(4a)] \end{matrix} \right\} C\left(\frac{b^2}{4a}\right) \pm \left\{ \begin{matrix} \sin[b^2/(4a)] \\ \cos[b^2/(4a)] \end{matrix} \right\} S\left(\frac{b^2}{4a}\right) \right] \tag{7}$$

are differentiated under the integral sign we get the divergent integrals (2). After some incorrect manipulations, Bierens de Haan obtains the value 0 for these integrals [**3**, p. 443]. He then differentiates under the integral sign to get the value 0 for (2).

The integrals in (6) and (7) may be evaluated using the methods in the proof of the Proposition.

The tables of Bierens de Haan had many errors, both mathematical and typographical. Two long works discussing the correctness of his tables are [**19**] and [**26**]. Neither mentions our divergent integrals, however. All of the integral tables that we have mentioned have received considerable scrutiny. The journal *Mathematics of Computation*, and its predecessor, *Mathematical Tables and Other Aids to Computation*, list numerous errata. However, despite over 300 published pages of errata regarding these tables there do not seem to be any references to (1) and (2). The article [**17**] compares the correctness of various integral tables; it is shocking how high the error rates are.

4. MAPLE AND MATHEMATICA. Here are how Maple (V.5) and Mathematica (4.0) fare. Maple correctly evaluates (6) and (7) for arbitrary a and b but falters when asked to perform the calculation with specific numerical values. For example, it gives $\int_0^\infty \sin(3.1x^2)\cos(2.2x)\,dx = 0$. Maple correctly says that the integrals (1) and (2) diverge. Mathematica fails in a different way. It correctly calculates (6) and (7) (considerable simplification is needed to obtain the form of (7)). But, Mathematica thinks (1) and (2) converge! It gives the same incorrect values that are in the tables.

5. DIFFERENTIATION UNDER THE INTEGRAL SIGN. Differentiating the convergent integrals (6) and (7) under the integral sign with respect to b yields the divergent integrals (1) and (2). This doesn't mean the functions defined by (6) and (7) aren't differentiable; it just means we cannot obtain their derivatives by differentiating under the integral. Could we have predicted this in advance? This is a difficult problem. Suppose we have $\int_a^b f(x, y)\,dy$. A sufficient condition for differentiating under a Riemann integral is that $\int_a^b f_1(x, y)\,dy$ converges uniformly in x; see [**24**, p. 260]. For Lebesgue integrals the dominating condition $|f_1(x, y)| \leq g(y)$ for some $g \in L^1$ suffices. For Riemann and Lebesgue integrals, necessary and sufficient conditions for differentiating under the integral sign are harder to come by. However, this is a much simpler problem when we use the Henstock integral. The solution depends on being able to integrate every derivative, a property not held by either the Riemann or the Lebesgue integral; see [**29**]. A good introduction to the Henstock integral is given in [**1**].

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