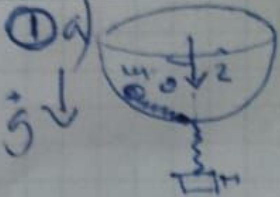
① a) 

$T = \frac{M}{2}\dot{z}^2 + \frac{m}{2}R^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)$; $U = -mgR\cos\theta - Mgz + \frac{k}{2}(R\theta + z)^2$

$\vec{g}\downarrow$

$$\Rightarrow \mathcal{L} = \frac{mR^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) + \frac{M}{2}\dot{z}^2 + mgR\cos\theta + Mgz - \frac{k}{2}(R\theta + z)^2$$

b) <u>Euler-Lagrange</u>:

$\theta)$ $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \boxed{mR^2\ddot{\theta} - mR^2\sin\theta\cos\theta\,\dot{\varphi}^2 + mgR\sin\theta + kR(R\theta + z) = 0}$ (1)

$\varphi)$ $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = \boxed{\frac{d}{dt}\left(mR^2\sin^2\theta\,\dot{\varphi}\right) = 0}$ (2) $\rightsquigarrow$ $mR^2\sin^2\theta\,\dot{\varphi} = cte$

$z)$ $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) - \frac{\partial \mathcal{L}}{\partial z} = \boxed{M\ddot{z} - Mg + k(R\theta + z) = 0}$ (3)

¡en este caso, θ también debe ser constante!

Si $z = d = cte \rightarrow \ddot{z} = 0 \Rightarrow Mg = k(R\theta_0 + d) \Rightarrow \frac{Mg}{k} = R\theta_0 + d \Rightarrow d = \frac{Mg}{k} - R\theta_0$

pero $\theta_0 \in [0, \pi/2] \Rightarrow d_{max} = \frac{Mg}{k}$ y $d_{mín} = \frac{Mg}{k} - \frac{R\pi}{2}$ → positivo por el dato del enunciado

$\Rightarrow$ el bloque podrá permanecer en reposo si $z \in \left[\frac{Mg}{k} - \frac{R\pi}{2}, \frac{Mg}{k}\right]$

c) $\theta = \pi/4 \Rightarrow \dot{\theta} = \ddot{\theta} = 0$ y $\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow$ (1) queda: $-\frac{mR^2\dot{\varphi}_0^2}{2} + \frac{mgR}{\sqrt{2}} + kR\left(\frac{R\pi}{4} + z_0\right) = 0$ (*)

(Sabemos q' $\dot{\varphi} = cte$ por la ec. (2) !!) ↗ (entonces $z = z_0 = cte$)

y (3) queda: $-Mg + k\left(\frac{R\pi}{4} + z_0\right) = 0 \Rightarrow z_0 = \frac{Mg}{k} - \frac{R\pi}{4}$

(*) $\Leftrightarrow \frac{m\cancel{R^2}\omega_0^2}{2} = \frac{mg\cancel{R}}{\sqrt{2}} + k\cancel{R}\left(\frac{Mg}{k}\right)$ (llamé $\dot{\varphi}_0 \equiv \omega_0$)

$$\Rightarrow \omega_0^2 = \frac{2g}{\sqrt{2}R} + \frac{2k}{m}\frac{Mg}{kR} = \left(\sqrt{2} + \frac{2M}{m}\right)\frac{g}{R}$$

$$\Rightarrow \omega_0 = \sqrt{\left(\sqrt{2} + \frac{2M}{m}\right)\frac{g}{R}} \Rightarrow \tau = \frac{2\pi}{\sqrt{\sqrt{2} + \frac{2M}{m}}}\sqrt{\frac{R}{g}}$$

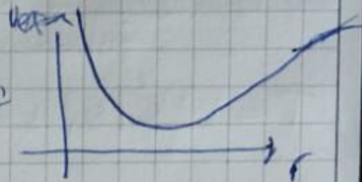
a) $\mathcal{L} = \frac{m}{2}\left(\dot r^2 + r^2\dot\varphi^2 + \dot z^2\right) - \beta \ln\left(r/r_0\right)$,

$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow H = E = \frac{m}{2}\left(\dot r^2 + r^2\dot\varphi^2 + \dot z^2\right) + \beta\ln\left(r/r_0\right) = cte$

$T = T_2$ y V NO DEPENDE DE LAS VELOCIDADES

$\frac{\partial \mathcal{L}}{\partial \varphi} = 0 \Rightarrow \boxed{p_\varphi = m r^2 \dot\varphi = cte}$; $\frac{\partial \mathcal{L}}{\partial z} = 0 \rightarrow \boxed{p_z = m\dot z = cte}$

$\Longrightarrow E = \frac{m\dot r^2}{2} + \underbrace{\frac{p_\varphi^2}{2mr^2} + \frac{p_z^2}{2m} + \beta\ln\left(r/r_0\right)}_{V_{eff}(r)} = cte$



Como $\lim_{r\to 0} V_{eff}(r) = \lim_{r\to\infty} V_{eff} = +\infty$, siempre tengo dos puntos de retorno

$\rightarrow$ todas las órbitas son ligadas

b) Por lo anterior, necesariamente tiene un mínimo ~~$\Rightarrow \exists$ órbitas circulares~~

$\Longrightarrow \exists$ órbitas con $\dot r = 0 \ \forall t$ (son espirales en general.) pues $\dot z = cte$

$\frac{dV_{eff}}{dr} = \frac{-p_\varphi^2}{mr^3} + \frac{\beta}{r} = 0 \Longleftrightarrow \frac{p_\varphi^2}{m} = \beta r^2 \Longleftrightarrow \frac{p_\varphi^2}{m\beta} = r^2$

$\Longleftrightarrow \frac{m^{\cancel{2}} r_0^4 \omega_0^2}{\cancel{m}\beta} = r_0^2 \Longleftrightarrow \frac{m r_0^2\omega_0^2}{\beta} = 1 \Longleftrightarrow \left(\frac{2\pi}{\tau_0}\right)^2 \overset{(*)}{=} \frac{\beta}{m r_0^2}$

$\Longleftrightarrow \frac{\tau_0}{2\pi} = r_0\sqrt{\frac{m}{\beta}} \Longleftrightarrow \tau_0 = 2\pi\sqrt{\frac{m}{\beta}}\, r_0$ $\rightarrow$ depende de la masa de la partícula $\Rightarrow$ no puede ser un potencial exclusivamente gravitatorio!,

c) si $r \approx r_0 \Rightarrow V_{eff} \approx \left.\frac{d^2V_{eff}}{dr^2}\right|_{r_0} (r-r_0)^2 + \cancel{cte}$

$\frac{d^2V_{eff}}{dr^2} = \frac{3p_\varphi^2}{mr^4} - \frac{\beta}{r^2} \rightarrow \left.\frac{d^2V_{eff}}{dr^2}\right|_{r_0} = \frac{3\cancel{m^2}\cancel{r_0^4}\omega_0^2}{\cancel{m}\cancel{r_0^4}} - \underbrace{\frac{\beta}{r_0^2}}_{m\omega_0^2\ (*)} = 2m\omega_0^2$

$\Longrightarrow E \approx \frac{m\dot r^2}{2} + \frac{2m\omega_0^2}{2}(r-r_0)^2 \rightarrow$ osc. armónico con $\omega_{rad} = \sqrt{\frac{2\cancel{m}\omega_0^2}{\cancel{m}}}$

$\Longrightarrow \omega_{rad} = \sqrt{2}\,\omega_0$ o bien $\frac{2\pi}{\tau_{rad}} = \sqrt{2}\,\frac{2\pi}{\tau_0} \rightarrow \tau_{rad} = \frac{\tau_0}{\sqrt{2}}$

$\therefore$ es imposible q' se cierren $\left(\frac{\tau_{rad}}{\tau_0} \notin \mathbb{Q}\right)$.