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Some Lagrangians for systems without a Lagrangian

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Abstract
We demonstrate how to construct many different Lagrangians for two famous examples that were deemed by Douglas (1941 *Trans. Am. Math. Soc.* **50** 71–128) not to have a Lagrangian. Following Bateman's dictum (1931 *Phys. Rev.* **38** 815–9), we determine different sets of equations that are compatible with those of Douglas and derivable from a variational principle.

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1. Introduction

A perennial question in classical mechanics is the determination of a Lagrangian for a given system of ordinary differential equations. There are considerable practical implications for such an existence. Firstly one can apply Noether's theorem [1] to determine symmetries and the corresponding integrals in a straightforward fashion. Secondly one can then construct a Hamiltonian and proceed to investigate the corresponding problem in quantum mechanics [2].

In the case of a scalar second-order ordinary differential equation the existence of a Lagrangian is guaranteed, indeed an infinite number of them [3, 4]. For systems of second-order equations or equations of higher order [5] the existence of a Lagrangian is not guaranteed. Consequently there is a considerable body of literature devoted to the solution of the question about the existence of a Lagrangian in these cases. In this paper we consider a paradigmatic example in considerable detail. The classical paper of Douglas [6], which builds upon a number of earlier papers [7–10], provides several examples of two-dimensional (2D) systems of second-order equations that do not possess a Lagrangian.

Since the time of Douglas there have been a number of contributions to the problems of the existence of a Lagrangian and of quantization. Marmo and Saletan [11] demonstrated that even with a simple system such as the isotropic linear oscillator there can be inconsistencies between different Lagrangian formulations that do not preserve the symmetry properties expected on physical grounds. Santilli [12] devoted a large part of his treatise to the question of the inverse problem of Lagrangian mechanics. Morandi *et al* [13] dealt extensively with the formulation of the inverse problem in the context of the geometry of the tangent bundle, and *the possibility of giving alternative Lagrangian formulations of the same dynamical system, with Lagrangians not simply differing by the usual total time derivative*. Hojman and Shepley [14] gave some examples of classical equations that do not come from a Lagrangian and cannot be quantized consistently. Cortese and García [15] studied the question of consistency between a given set of equations of motion in configuration space and a Poisson bracket and found conditions for both the case in which the symplectic structure is commutative and the case in which the symplectic structure is noncommutative. Gitman and Kupriyanov [16] examined the canonical quantization of systems with linear equations of motion that are traditionally regarded as non-Lagrangian systems by a reduction to a set of linear first-order equations. The same authors [17] extended their considerations to determine the necessary and sufficient conditions for the existence of a multiplier matrix that would give to a set of second-order equations the structure of Euler–Lagrange equations. Here we are looking for an expansion of the possibilities for the determination of a Lagrangian of a given system. In this respect, we are motivated by the possibility of existence of systems that do not possess a Lagrangian according to the criteria of Douglas. Following Bateman's dictum [18] we look for different sets of equations compatible with those of Douglas and derivable from a variational principle.

In this paper we examine the properties of two of the examples given by Douglas [6] of a system that does not possess a Lagrangian. The first 2D system is a special instance of the class Case I in Douglas and has been mentioned by Gitman and Kupriyanov [17]:

$$\begin{aligned} \ddot{x} + \dot{y} &= 0, \\ \ddot{y} + y &= 0, \end{aligned} \tag{1}$$

in which the overdot represents differentiation with respect to the independent variable, t. Actually a system quite similar to (1), namely

$$\begin{aligned} \ddot{x} - \dot{y} &= 0, \\ \ddot{y} - y &= 0, \end{aligned}$$

was proposed in 1923 by E T Whittaker as stated by Bateman in his famous paper [18]. This system does not possess a Lagrangian and defeats Bateman's method of adding a mirror system to it.

The second system is

$$\begin{aligned} \ddot{x} &= x^2 + y^2, \\ \ddot{y} &= 0, \end{aligned} \tag{2}$$

and is an example of the class Case IIIB in Douglas [6]. In both cases we demonstrate the existence of several Lagrangians by means of the simple expedient of a reformulation of the systems. The existence of a Lagrangian depends upon representation. For a theoretical treatment of existence one is constrained to the given representation, but when one has an explicit example standard methods of transformation open the way to the determination of a Lagrangian of the transformed system. An equation may not have any point symmetry but have some (many) if set in another form[3] [20–23]. Moreover in [24, 25] it has been shown that Noether's theorem applied to different Lagrangians yields different[4] Noether's symmetries and thus different conservation laws. In [26, 27] an alternative way to quantization based on the preservation of symmetries was presented. Therefore as many Lagrangians as possible for an equation must be found and then Noether's theorem can identify those physical Lagrangians that either yield the missing conservation laws or successfully lead to quantization.

2. Lagrangian formulation for (1)

The system (1) can be written as a single fourth-order equation by means of differentiating the second equation with respect to t and then substituting for x from the first to obtain

$$\ddot{\ddot{x}} + \ddot{x} = 0. \tag{3}$$

This equation satisfies the conditions of Fels [5] and its unique second-order Lagrangian is the obvious one, namely

$$L = \frac{1}{2}\left(\dddot{x}^2 - \dot{x}^2\right) + \frac{\mathrm{d}g}{\mathrm{d}t}, \tag{4}$$

where $g(t, x, \dot{x})$ is the arbitrary gauge function. We apply Noether's theorem to this Lagrangian and obtain the following point symmetries and associated integrals:

$$\begin{aligned} \Gamma_1 &= \partial_t, & I_1 &= \ddot{x}^2 - 2\dot{x}\,\dddot{x} - \dot{x}^2, \\ \Gamma_2 &= \partial_x, & I_2 &= \dddot{x} + \dot{x}, \\ \Gamma_3 &= t\partial_x, & I_3 &= x - \ddot{x} + t(\dot{x} + \dddot{x}), \\ \Gamma_4 &= \sin t\,\partial_x, & I_4 &= \ddot{x}\sin t + \dddot{x}\cos t, \\ \Gamma_5 &= \cos t\,\partial_x, & I_5 &= \ddot{x}\cos t - \dddot{x}\sin t. \end{aligned} \tag{5}$$

Note that $I_4^2 + I_5^2 = \dddot{x}^{\,2} + \ddot{x}^2$. Consequently $I_1 = I_4^2 + I_5^2 - I_2^2$. In addition (3) has a Lie point symmetry, $\Gamma_6 = x\partial_x$, which is not a Noether symmetry.

In terms of the method of reduction of order we rewrite the system (1) as the set of four first-order equations

$$\dot{w}_1 = w_2, \tag{6}$$

$$\dot{w}_2 = -w_4, \tag{7}$$

$$\dot{w}_3 = w_4, \tag{8}$$

$$\dot{w}_4 = -w_3, \tag{9}$$

where we have written $x = w_1$ and $y = w_3$.

The system (6)–(9) is autonomous and we can reduce its order by one with the choice of one of the dependent variables in (6)–(9) as a new independent variable. We choose w_1 and revert to the original symbol x. Then the system (6)–(9) becomes

$$\frac{\mathrm{d}w_2}{\mathrm{d}x} = -\frac{w_4}{w_2}, \tag{10}$$

$$\frac{\mathrm{d}w_3}{\mathrm{d}x} = \frac{w_4}{w_2}, \tag{11}$$

$$\frac{\mathrm{d}w_4}{\mathrm{d}x} = -\frac{w_3}{w_2}. \tag{12}$$

From (10) and (11) it is obvious that $w_3 + w_2 = r_3$ is a constant, $r_3 = r_0$. Therefore, $w_3 = r_0 - w_2$. We use (10) to eliminate w_4 through

$$w_4 = -\frac{\mathrm{d}w_2}{\mathrm{d}x} w_2.$$

Then (12) becomes

$$\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2} = -\frac{1}{w_2}\left(\frac{\mathrm{d}w_2}{\mathrm{d}x}\right)^2 - \frac{1}{w_2} + \frac{r_0}{w_2^2}, \tag{13}$$

which is a single second-order equation for which a Lagrangian exists[5]. It is[6] [3, 4]

$$L = \frac{1}{2} w_2^2 \left(\frac{\mathrm{d}w_2}{\mathrm{d}x}\right)^2 - \frac{1}{2} w_2^2 + r_0 w_2 + \frac{\mathrm{d}g}{\mathrm{d}x}, \tag{14}$$

where $g(x, w_2)$ is the gauge function.

[3] The form of the equations of motions are less important than their solutions. In fact even numerical algorithms do not preserve the form of the equations, and indeed it is claimed that the most efficient should preserve the first integrals [19].

[4] Even different in number.

[5] In terms of the original variables $w_2 = \dot{x}$.

[6] A Jacobi last multiplier of (13) is w_2^2.

Moreover we can transform the system (6)–(9) into a system of two second-order equations that admits a Lagrangian. We use (6) to eliminate w_2 through

$$w_2 = \dot{w}_1$$

and (9) to eliminate w_3 through

$$w_3 = -\dot{w}_4.$$

Then, (7) and (8) yield

$$\begin{aligned}\ddot{w}_1 &= -w_4,\\ \ddot{w}_4 &= -w_4,\end{aligned} \tag{15}$$

which is a 2D system for which a Lagrangian exists. In [28] it was shown that any system

$$\ddot{q}_1 = f_1(t, q_1, q_2), \quad \ddot{q}_2 = f_2(t, q_1, q_2) \tag{16}$$

admits a variational principle, with Lagrangian $L = L(t, q_1, q_2, \dot{q}_1, \dot{q}_2)$, if the relationships

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \quad (i, j = 1, 2) \tag{17}$$

satisfy the equation

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial q_1}(M\dot{q}_1) + \frac{\partial}{\partial q_2}(M\dot{q}_2) + \frac{\partial}{\partial \dot{q}_1}(f_1 M) + \frac{\partial}{\partial \dot{q}_2}(f_2 M) = 0, \tag{18}$$

associated with the Jacobi last multiplier, where M generically stands for each of the $M_{ij} (i, j = 1, 2)$.

Therefore, since a Jacobi last multiplier of the system (15) is just a constant, then

$$M_{11} = \frac{\partial^2 L}{\partial \dot{w}_1^2}, \quad M_{12} = \frac{\partial^2 L}{\partial \dot{w}_1 \partial \dot{w}_4}, \quad M_{22} = \frac{\partial^2 L}{\partial \dot{w}_4^2} \tag{19}$$

yield the following Lagrangian[7] for the system (15):

$$L = \frac{1}{2}\left(\dot{w}_1^2 - 2\dot{w}_1\dot{w}_4 + 2\dot{w}_4^2 - w_4^2\right) + \frac{\mathrm{d}g}{\mathrm{d}t}, \tag{20}$$

where $g(t, w_1, w_4)$ is the gauge function.

3. Lagrangian formulation for (2)

Since one can solve for y in (2) independently of x one can replace the system with

$$\begin{aligned}\ddot{x} &= x^2 + (\alpha t + \beta)^2,\\ y &= \alpha t + \beta,\end{aligned} \tag{21}$$

where α and β are constants of integration. It then follows that there is a Lagrangian for (2) given by

$$L = \tfrac{1}{2}\dot{x}^2 + x(\alpha t + \beta)^2 + \tfrac{1}{3}x^3. \tag{22}$$

[7] It turns out that $M_{12} = -M_{11}$. We assume that $M_{11} = a_1, M_{22} = a_2, a_1 \neq a_2$.

In terms of the method of reduction of order, we rewrite the system (2) as the set of four first-order equations

$$\begin{aligned}\dot{w}_1 &= w_2,\\ \dot{w}_2 &= w_1^2 + w_3^2,\\ \dot{w}_3 &= w_4,\\ \dot{w}_4 &= 0,\end{aligned} \tag{23}$$

where we have written $x = w_1$ and $y = w_3$.

The system (23) is autonomous and we can reduce its order by one with the choice of one of the dependent variables in (23) as a new independent variable. We choose w_3 and revert to the original symbol y. Then the system (23) becomes

$$\frac{\mathrm{d}w_1}{\mathrm{d}y} = \frac{w_2}{w_4}, \tag{24}$$

$$\frac{\mathrm{d}w_2}{\mathrm{d}y} = \frac{w_1^2 + y^2}{w_4}, \tag{25}$$

$$\frac{\mathrm{d}w_4}{\mathrm{d}y} = 0. \tag{26}$$

From (26) it is obvious that w_4 is a constant. We use (24) to eliminate w_2 through

$$w_2 = \frac{\mathrm{d}w_1}{\mathrm{d}y} w_4.$$

Then (25) becomes

$$\frac{\mathrm{d}^2 w_1}{\mathrm{d}y^2} = \frac{w_1^2 + y^2}{w_4^2}. \tag{27}$$

In terms of the original variable x, (27) is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}y^2} = \frac{x^2 + y^2}{w_4^2}, \tag{28}$$

which is a single second-order equation for which a Lagrangian exists. It is

$$L = \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 - \frac{x^3 + 3xy^2}{3w_4^2} + \frac{\mathrm{d}g}{\mathrm{d}y}, \tag{29}$$

where $g(y, x)$ is the gauge function.

We write w_1 in terms of its real and imaginary parts as $r_1 + \mathrm{i}r_2$ so that (27) can be written as the system

$$\frac{\mathrm{d}^2 r_1}{\mathrm{d}y^2} = \frac{r_1^2 - r_2^2 + y^2}{w_4^2}, \tag{30}$$

$$\frac{\mathrm{d}^2 r_2}{\mathrm{d}y^2} = \frac{2r_1 r_2}{w_4^2}. \tag{31}$$

The system (30)–(31) admits a Lagrangian. In fact its Jacobi last multiplier is a constant and therefore the relationships [28]

$$M_{11} = \frac{\partial^2 L}{\partial r_1'^2}, \quad M_{12} = \frac{\partial^2 L}{\partial r_1' \partial r_2'}, \quad M_{22} = \frac{\partial^2 L}{\partial r_2'^2}, \tag{32}$$

where the prime denotes differentiation with respect to the current independent variable y, yield the following Lagrangian[8] for the system (30)–(31):

$$L = \frac{1}{2}\left(r_2'^2 + 2r_1'r_2' - r_1'^2\right) + \frac{1}{3w_4^2}\Big(3y^2(r_2 - r_1) - (r_1 + r_2)(r_1^2 + r_2^2 - 4r_1r_2)\Big) + \frac{\mathrm{d}g}{\mathrm{d}y}, \qquad (33)$$

where $g(y, r_1, r_2)$ is the gauge function.

We may use (31) to eliminate r_1 from (30) to obtain a fourth-order equation in r_2, namely

$$r_2^{\mathrm{iv}} = \frac{1}{2r_2^2 w_4^4}\Big(-4r_2^5 + 4y^2r_2^3 + 4w_4^4 r_2 r_2' r_2''' + 3w_4^4 r_2 r_2''^2 - 4w_4^4 r_2'^2 r_2''\Big) \equiv R. \qquad (34)$$

Equation (34) satisfies the two conditions of Fels for the existence of a second-order Lagrangian [5]. In [29] it was shown that, if an equation of the fourth order satisfies the two conditions of Fels [5], the unique Lagrangian can be found by finding the Jacobi last multiplier M, namely a solution of the following equation:

$$\frac{\mathrm{d}\log M}{\mathrm{d}y} + \frac{\partial R}{\partial r_2'''} = 0, \qquad (35)$$

and imposing

$$\sqrt{M} = \frac{\partial^2 L}{\partial r_2''^2}.$$

In the case of equation (34), $M = r_2^{-2}$ and the corresponding Lagrangian is

$$L = \frac{r''^2}{2r_2} + 2r_2\frac{r_2^2 - 3y^2}{3w_4^4} + \frac{\mathrm{d}g}{\mathrm{d}y}, \qquad (36)$$

where $g(y, r_2, r_2')$ is the gauge function.

Equation (34) does not possess any Lie point symmetries.

4. Final remarks

The two systems, (1) and (2), that we have considered here have been known for many years. Douglas [6] demonstrated the lack of existence of a Lagrangian in each case for the given form. What we have done is to show that a Lagrangian can be found if the system is suitably transformed. In the case of (1) the process was quite simple and the equivalent formulation turned out to be supplied with many symmetries. In the case of (2) a subtler approach was required, but we were able to obtain a fourth-order equation that satisfied the two conditions of Fels [5] and had an obvious Jacobi last multiplier so that a second-order Lagrangian could be found. The important point is that we had specific systems and so could use their specific properties to determine representations of the respective systems for which Lagrangians could be found. This approach emphasizes the difference between a theoretical discussion and the process of the resolution of an actual problem.

We emphasize that in this paper

- we did not look for any of the many linear Lagrangians admitted by systems of first-order equations [17, 30, 31];
- all the derived Lagrangians can be classically transformed into Hamiltonians[9] and therefore their quantization can be dealt with by either known techniques or new methods [26, 32].

Finally, for both systems (1) and (2), we have solved the general problem as stated by Bateman in [18], namely 'finding a set of equations equal in number to a given set, compatible with it and derivable from a variational principle' without recourse to any additional set of equations.

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