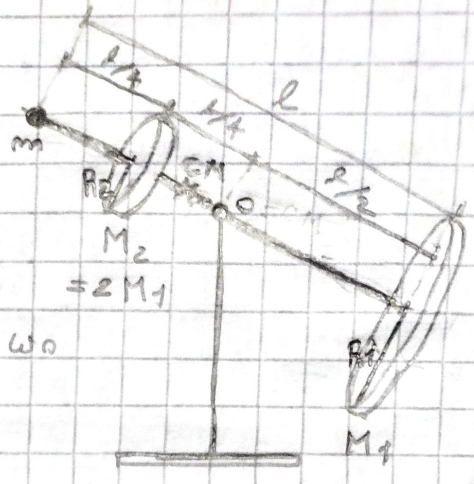
Entrega 5



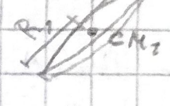
Antes de agregar la masa m, hay balance de torques gravitatorios respecto del centro del eje o

CI: $\theta(0)=\theta_0$, $\varphi(0)=\psi(0)=\dot\theta(0)=0$, $\dot\varphi(0)=\omega_0$
con $\Omega_3=\Omega_0>>1$, y $\ddot\varphi<<1$

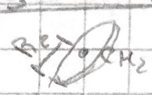
a) $I_1^o=I_2^o=I=?$, $I_3^o\equiv I_3=?$

Los momentos de inercia de los dos discos y de la masa m respecto de sus respectivos CM son:

$M_1)\quad I_1^{CM_1}=I_2^{CM_1}=\boxed{I^{CM_1}=\frac{M_1R_1^2}{4}}$, $\boxed{I_3^{CM_1}=\frac{M_1R_1^2}{2}}$



$M_2)\quad \boxed{I^{CM_2}=\frac{M_2R_2^2}{4}}$, $\boxed{I_3^{CM_2}=\frac{M_2R_2^2}{2}}$



$m)\quad \boxed{I^{cm}=0}$, $\boxed{I_3^{cm}=0}$ •cm

Ahora los escribo en o usando el Teo de Steiner:
(como o y los CM's están en el eje principal $\hat 3$, los I_3 van a ser iguales respecto de o y de los CM.)

$M_1)\quad \boxed{I^{o_1}}=\frac{M_1R_1^2}{4}-M_1\left(\frac{l}{2}\right)^2=\boxed{M_1\frac{R_1^2-l^2}{4}}$, $\boxed{I_3^{o_1}=\frac{M_1R_1^2}{2}}$

$M_2)\quad \boxed{I^{o_2}}=\frac{M_2R_2^2}{4}+M_2\left(\frac{l}{4}\right)^2=\boxed{M_2\frac{R_2^2+\left(\frac{l}{2}\right)^2}{4}}$, $\boxed{I_3^{o_2}=\frac{M_2R_2^2}{2}}$

$m)\quad \boxed{I^{om}}=0+m\left(\frac{l}{2}\right)^2=\boxed{\frac{ml^2}{4}}$, $\boxed{I_3^{om}=0}$

Ahora los momentos ppales de inercia respecto de o van a ser la suma de los momentos de cada masa:

además $\boxed{M_2=2M_1}$ $\Longrightarrow$

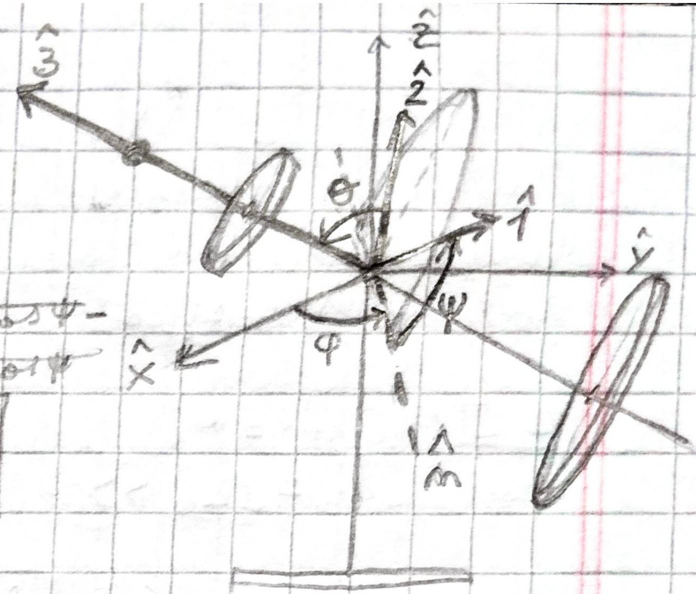
$$\begin{cases}\boxed{I=\frac{M_1}{4}\left(R_1^2+2R_2^2-\frac{l^2}{2}\right)+\frac{ml^2}{4}}\\ \boxed{I_3=\frac{M_1}{2}\left(R_1^2+2R_2^2\right)}\end{cases}$$

b) $T(\varphi,\theta,\psi,\dot\varphi,\dot\theta,\dot\psi)=?$ como o es punto fijo, T es solo de rotación

$$\Rightarrow \boxed{T=T_{rot}=\frac{I}{2}\left(\Omega_1^2+\Omega_2^2\right)+\frac{I_3}{2}\Omega_3^2}\quad \left(\bar\Omega=(\Omega_1,\Omega_2,\Omega_3)\right)$$

ya que $\bar{\bar{I}}^o$ es diagonal

$$\rightarrow \begin{cases} \boxed{\Omega_1 = \dot{\varphi} \operatorname{sen}\theta \operatorname{sen}\psi + \dot{\theta}\cos\psi} \\ \boxed{\Omega_2 = \dot{\varphi} \operatorname{sen}\theta \cos\psi - \dot{\theta} \operatorname{sen}\psi} \\ \boxed{\Omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}} \end{cases}$$



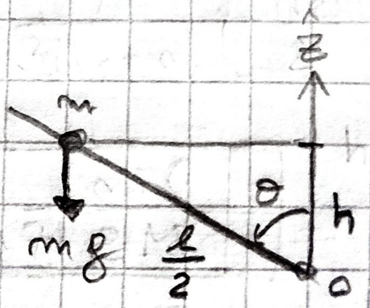
$$\Rightarrow \boxed{\Omega_1^2 + \Omega_2^2 = \dot{\varphi}^2 \operatorname{sen}^2\theta + \dot{\theta}^2} + \cancel{2\dot{\varphi}\dot{\theta} \operatorname{sen}\theta \operatorname{sen}\psi \cos\psi} - \cancel{2\dot{\varphi}\dot{\theta} \operatorname{sen}\theta \operatorname{sen}\psi \cos\psi}$$

$$\Rightarrow \boxed{T = \frac{I}{2}\left(\dot{\varphi}^2 \operatorname{sen}^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2}$$

Como el sistema estaba en equilibrio de torques antes de agregar la masa m, la fuerza gravitatoria va a estar sobre la misma.

V está dada por ésta fuerza gravitatoria:

$$\boxed{V = m g \frac{l}{2} \cos\theta}$$



c) $$\boxed{\mathcal{L} = \frac{I}{2}(\dot\varphi^2 \operatorname{sen}^2\theta + \dot\theta^2) + \frac{I_3}{2}(\dot\varphi\cos\theta + \dot\psi)^2 - mg\frac{l}{2}\cos\theta}$$

EL

φ) $\frac{\partial\mathcal{L}}{\partial\varphi} = 0 \rightarrow \varphi$ cíclico

$\Rightarrow \boxed{I\dot\varphi\operatorname{sen}^2\theta + I_3(\dot\varphi\cos\theta + \dot\psi)\cos\theta \equiv P_\varphi}$ cte

ψ) $\frac{\partial\mathcal{L}}{\partial\psi} = 0 \rightarrow \psi$ cíclico

$\Rightarrow \boxed{I_3(\dot\varphi\cos\theta + \dot\psi) \equiv P_\psi}$ cte $\Rightarrow \boxed{P_\varphi = I\dot\varphi\operatorname{sen}^2\theta + P_\psi\cos\theta}$

$$\Rightarrow \boxed{\dot\varphi = \frac{P_\varphi - P_\psi\cos\theta}{I\operatorname{sen}^2\theta}} \longrightarrow \text{derivo:}$$

$$\Rightarrow \ddot\varphi = -\frac{2P_\varphi\dot\theta\cancel{\operatorname{sen}\theta}\cos\theta}{I\operatorname{sen}^{\cancel{4}3}\theta} - \frac{P_\psi\dot\theta(-\operatorname{sen}^2\theta - 2\cos^2)\cancel{\operatorname{sen}\theta}}{\operatorname{sen}^{\cancel{4}3}\theta}$$

$$= \frac{P_\psi(\operatorname{sen}^2\theta + 2\cos^2\theta) - 2P_\varphi\cos\theta}{I\operatorname{sen}^3\theta}\dot\theta \quad P_\varphi = I\dot\varphi\operatorname{sen}^2\theta + P_\psi\cos\theta$$

$$= \frac{P_\psi\cancel{\operatorname{sen}^2\theta} + \cancel{2P_\psi\cos^2\theta} - 2I\dot\varphi\cancel{\operatorname{sen}^2\theta}\cos\theta - \cancel{2P_\psi\cos^2\theta}}{I\cancel{\operatorname{sen}^3\theta}}\dot\theta$$

$$= \boxed{\frac{P_\psi - 2I\dot\varphi\cos\theta}{I\operatorname{sen}\theta}\dot\theta} = \frac{I_3(\dot\varphi\cos\theta + \dot\psi) - 2I\dot\varphi\cos\theta}{I\operatorname{sen}\theta}\dot\theta$$

$$\Rightarrow \boxed{\boxed{\ddot\varphi = \frac{(I_3 - 2I)\dot\varphi\cos\theta + I_3\dot\psi}{I\operatorname{sen}\theta}\dot\theta}}$$

Para $\ddot\psi$ derivo $P_\psi = I_3(\dot\varphi\cos\theta + \dot\psi)$:

$$\Rightarrow 0 = \cancel{I_3}(\ddot\varphi\cos\theta - \dot\varphi\dot\theta\operatorname{sen}\theta + \ddot\psi) \Rightarrow \boxed{\ddot\psi = \dot\varphi\dot\theta\operatorname{sen}\theta - \ddot\varphi\cos\theta}$$

$$\Rightarrow \boxed{\boxed{\ddot\psi = \dot\varphi\dot\theta\operatorname{sen}\theta - \frac{(I_3 - 2I)\dot\varphi\cos^2\theta + I_3\dot\psi\cos\theta}{I\operatorname{sen}\theta}\dot\theta}}$$

$$\theta)\ \frac{d}{dt}\underbrace{[I\dot\theta]}_{=I\ddot\theta} \overset{EL}{=} I\dot\varphi^2\operatorname{sen}\theta\cos\theta - I_3(\dot\varphi\cos\theta + \dot\psi)\dot\varphi\operatorname{sen}\theta + mg\frac{l}{2}\operatorname{sen}\theta$$

$$\Rightarrow \boxed{\boxed{\ddot\theta = \frac{I - I_3}{I}\dot\varphi^2\operatorname{sen}\theta\cos\theta - \frac{I_3}{I}\dot\psi\dot\varphi\operatorname{sen}\theta + \frac{mgl}{2I}\operatorname{sen}\theta}}$$

d) $\theta \simeq \theta_0$, $\dot\theta, \dot\varphi, \ddot\varphi \ll$ también $\dot\psi = \Omega_3 \equiv \Omega_0$ cte $\gg 1$ y $m \ll$

def $\varepsilon = \theta - \theta_0 \ll \Rightarrow \theta = \theta_0 + \varepsilon \Rightarrow \ddot\theta = \ddot\varepsilon$

$$\begin{cases} \operatorname{sen}\theta \simeq \operatorname{sen}\theta_0 + \varepsilon\cos\theta_0 \\ \cos\theta \simeq \cos\theta_0 - \varepsilon\operatorname{sen}\theta_0 \end{cases} \rightarrow$$

cuando tenga $\dot\theta, \dot\varphi, \ddot\varphi$ o m, por un seno o un coseno, solo va a sobrevivir el término sin ε, ya que el termino con ε queda de 2do orden

$$\left(\text{Ej: } \dot\varphi\operatorname{sen}\theta \simeq \dot\varphi\operatorname{sen}\theta_0 + \cancelto{0}{\dot\varphi\varepsilon\cos\theta_0} = \dot\varphi\operatorname{sen}\theta_0\right)$$

$\longrightarrow$

φ) $I\ddot{\varphi}\,\text{sen}\,\theta_0 = (I_3 - 2I)\overset{\simeq 0}{\dot{\varphi}\dot{\theta}}\cos\theta + I_3\overset{=\Omega_0}{\dot{\psi}}\dot{\theta}$

$$\Rightarrow \boxed{\ddot{\varphi} - \frac{I_3\Omega_0}{I\,\text{sen}\,\theta_0}\dot{\theta} = 0} \quad (1)$$

ψ) $\ddot{\psi} = \overset{\simeq 0}{\dot{\varphi}\dot{\theta}}\,\text{sen}\,\theta - \ddot{\varphi}\cos\theta = -\ddot{\varphi}\cos\theta_0$

$$\Rightarrow \boxed{\ddot{\psi} + \frac{I_3\Omega_0\cos\theta_0}{I\,\text{sen}\,\theta_0}\dot{\theta} = 0} \quad (2)$$

m sen θ₀

θ) $\ddot{\theta} = \frac{I - I_3}{I}\overset{\simeq 0}{\dot{\varphi}^2}\,\text{sen}\,\theta\cos\theta - \frac{I_3}{I}\underset{\Omega_0}{\dot{\psi}}\dot{\varphi}\underset{\dot{\theta}\,\text{sen}\,\theta_0}{\text{sen}\,\theta} + \frac{mg\ell}{2I}\text{sen}\,\theta$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{I_3\Omega_0\,\text{sen}\,\theta_0}{I}\dot{\varphi} = \frac{mg\ell}{2I}\text{sen}\,\theta_0} \quad (3)$$

e) Hago (3) − i (1):

$$I\ddot{\theta} - iI\ddot{\varphi}\,\text{sen}\,\theta_0 + I_3\Omega_0\dot{\varphi}\,\text{sen}\,\theta_0 + iI_3\Omega_0\dot{\theta} = \frac{mg\ell}{2}\text{sen}\,\theta_0$$

$$\Rightarrow I(\ddot{\theta} - i\ddot{\varphi}\,\text{sen}\,\theta_0) + I_3\Omega_0\frac{i}{i}(\dot{\varphi}\,\text{sen}\,\theta_0 + i\dot{\theta}) = \frac{mg\ell}{2}\text{sen}\,\theta_0$$

$$\Rightarrow \boxed{I(\ddot{\theta} - i\ddot{\varphi}\,\text{sen}\,\theta_0) + iI_3\Omega_0(\dot{\theta} - i\dot{\varphi}\,\text{sen}\,\theta_0) = \frac{mg\ell}{2}\text{sen}\,\theta_0}$$

defino $\boxed{\lambda \equiv \theta - i\varphi\,\text{sen}\,\theta_0} \Rightarrow \boxed{\dot{\lambda} = \dot{\theta} - i\dot{\varphi}\,\text{sen}\,\theta_0} \Rightarrow \boxed{\ddot{\lambda} = \ddot{\theta} - i\ddot{\varphi}\,\text{sen}\,\theta_0}$

$$\Rightarrow \boxed{I\ddot{\lambda} + iI_3\Omega_0\dot{\lambda} = \frac{mg\ell}{2}\text{sen}\,\theta_0}$$

La solución es $\lambda = \lambda_h + \lambda_p$

◦) Para λ_p pido $\boxed{\ddot{\lambda}_p = 0} \Rightarrow 0 + iI_3\Omega_0\dot{\lambda}_p = \frac{mg\ell}{2}\text{sen}\,\theta_0$

$$\Rightarrow \boxed{\dot{\lambda}_p = -i\frac{mg\,\ell/2}{I_3\Omega_0}\text{sen}\,\theta_0} = \text{cte}$$

⇒ Propongo: ↪ tomo $\boxed{\tilde{C} = C - if\,\text{sen}\,\theta_0 \in \mathbb{C}}$

$$\lambda_p = -i\underbrace{\frac{mg\,\ell/2}{I_3\Omega_0}}_{\equiv\omega_p}\text{sen}\,\theta_0\, t + \tilde{C}$$

$\equiv \omega_p$: velocidad de precesión de trompo pesado

$$\Rightarrow \boxed{\lambda_p = -i\omega_p\,\text{sen}\,\theta_0\, t + \tilde{C}} \quad \boxed{\omega_p = \frac{mg\,\ell/2}{I_3\Omega_0}}, \quad \boxed{\tilde{C} = C - if\,\text{sen}\,\theta_0} \rightarrow$$

$\Rightarrow$ •) La homogénea la igualo a 0:

$$\boxed{I\ddot{\lambda}_h + i I_3 \Omega_0 \dot{\lambda}_h = 0} \Rightarrow \boxed{\ddot{\lambda}_h + i \underbrace{\frac{I_3}{I}\Omega_0}_{\omega_L} \dot{\lambda}_h = 0}$$

ω_L: velocidad de precesión de trompo libre

La solución es de la forma:

$$\boxed{\lambda_h = a\, e^{i(\omega_L t - \varphi_0)} \operatorname{sen}\theta_0}$$

ó $\boxed{\lambda_h = \lambda_0\, e^{i\omega_L t}}$ con $\boxed{\lambda_0 = a\, e^{-i\varphi_0} \operatorname{sen}\theta_0}$

$$\Rightarrow \lambda = a \operatorname{sen}\theta_0 \underbrace{e^{i(\omega_L t - \varphi_0)}}_{= \cos(\omega_L t - \varphi_0) + i \operatorname{sen}(\omega_L t - \varphi_0)\ \text{(Euler)}} - i\omega_p \operatorname{sen}\theta_0\, t + C - i f \operatorname{sen}\theta_0$$

$$\Rightarrow \lambda = \underbrace{a \operatorname{sen}\theta_0 \cos(\omega_L t - \varphi_0) + C}_{=\theta} - i \underbrace{(\omega_p t + a \operatorname{sen}(\omega_L t - \varphi_0) + f)}_{=\varphi} \operatorname{sen}\theta_0$$

y tenía $\lambda = \theta - i\varphi \operatorname{sen}\theta_0$

$$\Rightarrow \begin{cases} \boxed{\theta(t) = a \operatorname{sen}\theta_0 \cos(\omega_L t - \varphi_0) + C} \\ \boxed{\varphi(t) = \omega_p t + a \operatorname{sen}(\omega_L t - \varphi_0) + f} \end{cases}$$

Falta hallar las ctes $\{a, \varphi_0, C, f\}$ usando las CI: $\varphi(0) = 0$, $\dot{\varphi}(0) = \omega_0$

$\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$

$$\begin{cases} \theta(0) = a \operatorname{sen}\theta_0 \overbrace{\cos(-\varphi_0)}^{=\cos\varphi_0} + C = \theta_0 \Rightarrow \boxed{C = \theta_0 - a \operatorname{sen}\theta_0 \cos\varphi_0} \\ \varphi(0) = a \operatorname{sen}\varphi_0 + f = 0 \Rightarrow \boxed{f = -a \operatorname{sen}\varphi_0} \end{cases}$$

$$\begin{cases} \dot{\theta}(0) = \cancel{-} a \operatorname{sen}\theta_0\, \omega_L \overbrace{\operatorname{sen}(-\varphi_0)}^{= \cancel{-}\operatorname{sen}\varphi_0} = 0 \Rightarrow a\,\omega_L \operatorname{sen}\theta_0 \operatorname{sen}\varphi_0 = 0 \\ \dot{\varphi}(0) = \omega_p + a\,\omega_L \cos(-\varphi_0) = \omega_0 \Rightarrow \boxed{a = \frac{\omega_0 - \omega_p}{\omega_L \cos\varphi_0}} \end{cases}$$

$$\Rightarrow \frac{\omega_0 - \omega_p}{\omega_L \cos\varphi_0} \operatorname{sen}\theta_0 \operatorname{sen}\varphi_0 = \underbrace{\frac{\omega_0 - \omega_p}{\omega_L}}_{\neq 0} \underbrace{\operatorname{sen}\theta_0}_{\neq 0} \operatorname{tg}\varphi_0 = 0$$

$$\Rightarrow \boxed{\operatorname{tg}\varphi_0 = 0} \Rightarrow \boxed{\varphi_0 = m\pi} \boxed{m \in \mathbb{Z}} \Rightarrow \boxed{f = 0}$$

$$\Rightarrow \boxed{\cos\varphi_0 = \pm 1}$$

$$\Rightarrow \boxed{a = \pm \frac{\omega_0 - \omega_p}{\omega_L}}, \boxed{C = \theta_0 \mp a \operatorname{sen}\theta_0}$$

$$\Rightarrow \begin{cases} \boxed{\theta(t) = \pm \frac{\omega_0 - \omega_p}{\omega_L} \operatorname{sen}\theta_0 \cos(\omega_L t - m\pi) + \theta_0 - \frac{\omega_0 - \omega_p}{\omega_L} \operatorname{sen}\theta_0} \\ \boxed{\varphi(t) = \omega_p t \pm \frac{\omega_0 - \omega_p}{\omega_L} \operatorname{sen}(\omega_L t - m\pi)} \end{cases}$$

elijo $n = 1 \Rightarrow \varphi_0 = \pi \Rightarrow \begin{cases} \cos\varphi_0 = -1 \\ \cos(\omega_L - \varphi_0) = -\cos(\omega_L) \\ \operatorname{sen}(\omega_L - \varphi_0) = -\operatorname{sen}(\omega_L) \end{cases}$

$$\Rightarrow \boxed{\theta(t) = \frac{\omega_p - \omega_0}{\omega_L} \operatorname{sen}\theta_0 \left(1 - \cos(\omega_L t)\right) + \theta_0}$$

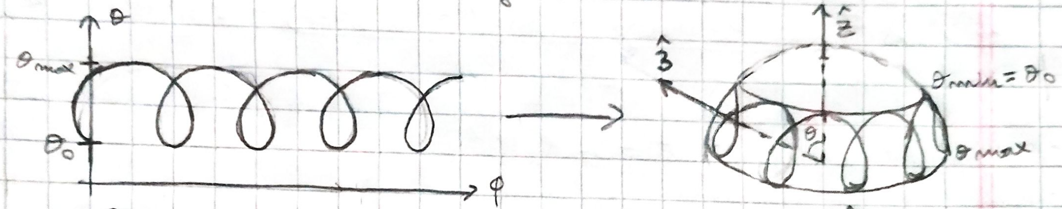
$$\boxed{\varphi(t) = \omega_p t - \frac{\omega_p - \omega_0}{\omega_L} \operatorname{sen}(\omega_L t)} \qquad \boxed{\dot\varphi(t) = \omega_p - (\omega_p - \omega_0)\cos(\omega_L t)}$$

Ahora grafico la variación de θ (nutación) cuando el giróscopo gira en φ.

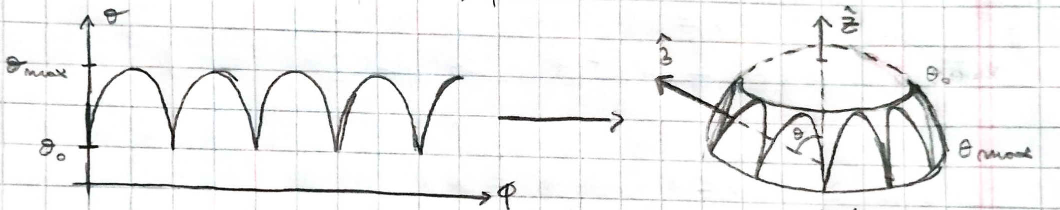
Observaciones:

- si $\omega_p - \omega_0 > 0$ el giróscopo sube inicialmente y $\theta_0 = \theta_{min}$
- si $\omega_p - \omega_0 < 0$ el giróscopo baja inicialmente y $\theta_0 = \theta_{max}$
- si $\omega_p - \omega_0 = 0$ no hay nutación (ni sube ni baja)
- si $|\omega_p - \omega_0| > \omega_p$ $\dot\varphi$ cambia de signo

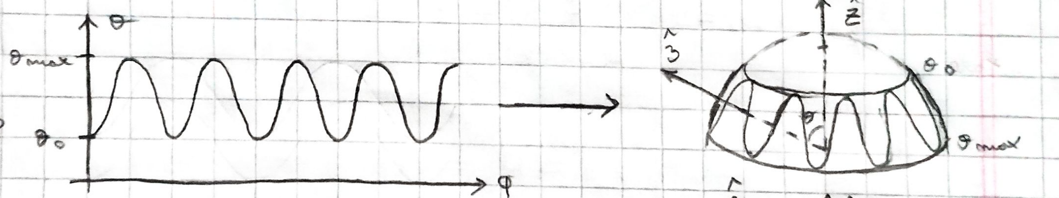
$\omega_0 = -\omega_p$:
$\begin{cases} \omega_p + \omega_p > 0 \\ |\omega_p + \omega_p| > \omega_p \end{cases}$



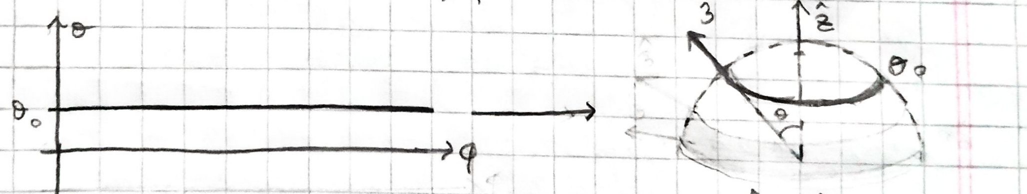
$\omega_0 = 0$:
$\begin{cases} \omega_p - 0 > 0 \\ |\omega_p - 0| = \omega_p \end{cases}$



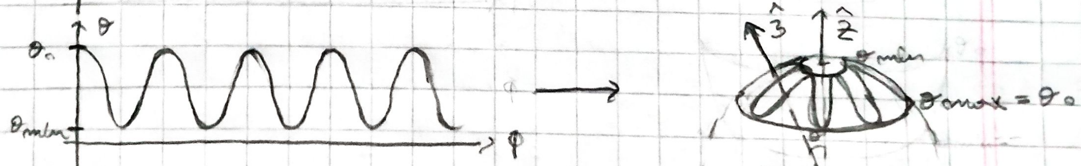
$\omega_0 = \omega_p/2$:
$\begin{cases} \omega_p - \omega_p/2 > 0 \\ |\omega_p - \omega_p/2| < \omega_p \end{cases}$



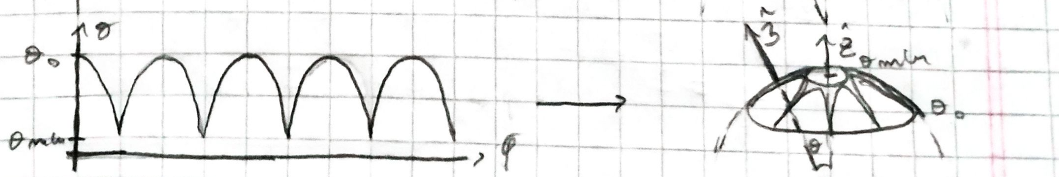
$\omega_0 = \omega_p$:
$\begin{cases} \omega_p - \omega_p = 0 \\ |\omega_p - \omega_p| < \omega_p \end{cases}$



$\omega_0 = \frac{3}{2}\omega_p$:
$\begin{cases} \omega_p - \frac{3}{2}\omega_p < 0 \\ |\omega_p - \frac{3}{2}\omega_p| < \omega_p \end{cases}$



$\omega_0 = 2\omega_p$:
$\begin{cases} \omega_p - 2\omega_p < 0 \\ |\omega_p - 2\omega_p| = \omega_p \end{cases}$



$\omega_0 = 3\omega_p$:
$\begin{cases} \omega_p - 3\omega_p < 0 \\ |\omega_p - 3\omega_p| > \omega_p \end{cases}$