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Euler's rigid rotators, Jacobi elliptic functions, and the Dzhanibekov or tennis racket effect

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In this paper, the torque-free rotational motion of a general rigid body is developed analytically and is applied to the flipping motion of a T-handle spinning in zero gravity that can be seen in videos on the internet. This flipping motion is known both as the Dzhanibekov effect (after the cosmonaut who reported it) and more recently the tennis racket effect. The presentation is self-contained, accessible to students, and is complementary to the treatment found in most texts in that it involves a time-dependent analytical solution in terms of elliptic functions as opposed to a development based on conservation laws. These two complementary approaches are interesting and useful in different ways. In the present approach, the Euler rigid-body equations are derived and then solved as differential equations that are satisfied by Jacobi elliptic functions. This is analogous to solving the spring–mass harmonic oscillator problem by turning Newton's laws into differential equations that are satisfied by sine and cosine functions. The Jacobi functions are closely related to these trigonometric functions and are only slightly more complicated. They are defined as geometrical ratios on a reference ellipse and developed geometrically without reference to power series or complex variables. However, because these functions are less familiar, they are introduced in a short Appendix where their main properties are derived. Also, a link is provided to a Mathematica script for animating the analytical solution to the present problem.

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I. INTRODUCTION

Rigid-body motion is an interesting area of classical mechanics, and surprising rotational behavior arises in the motion of a freely rotating object without axial symmetry. For instance, one can observe the periodic flipping of the axis of rotation of a T-handle rotating freely in zero gravity as seen in popular videos of the "dancing T-handle" available on the internet (https://www.youtube.com/watch?v=1n-HMSCDYtM). This unstable motion is known as the Dzhanibekov effect after cosmonaut Vladimir Dzhanibekov who reported observing it during the 1985 Soyuz T-13 mission. It is treated below in some detail, where it emerges as an analytical solution of the rotational dynamics problem to be described presently.

Figure 1 shows a frame from a NASA video of a T-handle intended for moving an instrument module. The handle screws into the panel face and unscrews easily with little friction. By giving it a quick twist, an astronaut sets it spinning along the screw axis so that it unscrews itself and is launched into the air while spinning and drifting outward. For three or four revolutions, it spins steadily around the original axis and moves outward with the spinning crossbar of the T preceding its trunk. Then, suddenly the T flips so that, while still spinning, the trunk of the T is now preceding the crossbar. At first glance, it appears as if the angular velocity may have reversed, thereby violating the conservation of angular momentum. However, on closer inspection, one sees that the angular velocity points in the same direction along the axis before and after the flip. After another three or four rotations, the T flips again, and the flipping motion repeats periodically. More recently, this flipping effect has been associated with a spinning tennis racket and the tennis racket (or intermediate axis) theorem, which asserts that free rotation of a rigid object about a principal axis with an intermediate moment of inertia is unstable. This leads to a flipping motion of the body axis for both a tennis racket and a T-handle.[1–3]

In Secs. II and III, an explicit solution is worked out in some detail for the torque-free motion of a freely rotating rigid object. An immediate purpose for doing so is to consider the spinning T-handle and determine how Dzhanibekov flipping arises in the analytical solution. A broader purpose is simply to provide a complete but succinct analytical treatment of the motion of a general rigid object rotating freely in space (i.e., not subject to any forces or torques) to anyone who might want access to the calculation or to use it as a starting point for some other project. A link to a Mathematica notebook shared on the Wolfram Cloud is provided in Appendix C. The torque-free case is sometimes called the Euler problem because he was the first to reduce it to quadrature.[4–6] Whittaker has referred to it as one of the most important problems in the dynamics of systems with three degrees of freedom.[7]

While the closed-form solution of the Euler problem has been known since the work of Jacobi,[8] and while modern treatments addressing the stability and additional subtleties of the solution in a form rather like the result in the current paper continue to appear, a simple presentation addressed at students seems to us to be unavailable. While some texts describe the complete solution, we are yet to see one that provides a complete and self-contained treatment that includes the detailed time evolution of the rigid body. Synge and Griffith[9] come close to this in that they show a solution and provide a lengthy discussion that includes material on elliptic functions. However, they do not reduce the final integration for the time dependence to a simple Legendre form, nor do they express the solution in terms of initial conditions that are appropriate for comparison with Dzhanibekov flipping. Of the two branches that comprise the complete

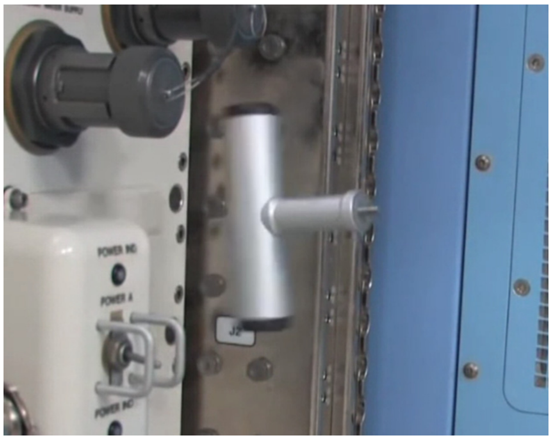

Fig. 1. T-Handle object on the International Space Station.

solution, the one related to Dzhanibekov flipping is presented below, while the other branch and the derivations of both are presented in Appendix B.

Another text that takes a similar approach is that by Dixon.[10] He begins by solving the Euler equations by comparison with the differential relations among the Jacobi elliptic functions, as we do in Sec. II herein, and then he transforms the solution to the inertial frame using direction cosines of a line fixed in the body frame with respect to the angular momentum vector $\vec{L}$ that is fixed in space. However, as with the treatment by Synge and Griffith,[9] the form of the solution is not convenient for comparison with the Dzhanibekov case. The solution presented in Sec. III uses an Euler-angle rotation matrix to transform to the inertial frame and carefully avoids the complex arguments, expansions, and transformations that are implicit in the solutions of Dixon and others to unify different branches of the solution by analytic continuation. The difference between Dixon's approach and the one presented below, which avoids complex manipulations, is in the fact that for us there are two branches to the complete solution of the general motion rather than one solution expressed in terms of (implicitly) complex variables. For the solution presented below, the development remains real and elementary.

Landau and Lifshitz[11] report a result in terms of the Jacobi theta functions, which Jacobi used to construct the elliptic functions and then applied them to solve the Euler equations. However, Landau and Lifshitz do not describe the behavior of the solution, nor do they offer a method for reducing the time integral.[8] In his popular textbook on mechanics, Goldstein[12] gives the alternative geometrical treatments of Poinsot[13] and of Binet and says that the full analytical solution is more complicated and less instructive. In a paper published some time ago in the present journal, Lock[14] takes the geometrical approach based on conservation laws in the tradition of Poinsot, as described also in Goldstein, which involves constructing a path in the configuration space of the solid object that conserves both the kinetic energy and the squared angular momentum. Both the stability of free rotations (the Euler problem) and the nutation of a top are treated geometrically in that paper. In the case of the free-rotation stability, Lock focuses on the separatrices between rotations about the third principal axis (with the highest moment of inertia) and about the first principal axis (with the lowest moment of inertia). That paper illustrates very well the alternative geometrical methods that are often used. The geometrical construction provides intuition and a fairly quantitative description of the path taken by the angular momentum vector ($\vec{L}$) over the constant-energy surface, but it lacks any information on the time dependence of its position along this path. We consider these geometrical developments as conceptually useful and complementary to the analytical treatments such as the one given below.

It is often observed that mathematical challenges can distract students from the essential physical concepts and that while such challenges are important, they should not be allowed to interfere with a qualitative understanding.[15] This assertion is true, but we feel that while geometrical constructions give a good qualitative understanding, there is substantial additional benefit in pursuing the full analysis. In textbooks, the relevant Euler differential equations are often derived and solved for the symmetrical case in which two moments of inertia are equal. Solving this special symmetrical case, or even just solving for the motion of a simple harmonic oscillator by matching undetermined coefficients in a trigonometric substitution, is not very different from solving the full time-dependent set of Euler equations by using the Jacobi elliptic functions. The full solution is an option that can be pursued, and in our experience, many students have found it interesting. It is not particularly difficult, and it opens other possibilities for addressing a broader range of mechanics problems. It is supplementary to the geometrical approach based on conservation laws.

In the remainder of this introduction, we summarize some concepts that a student would normally encounter as part of a junior- or senior-level mechanics course. Building on these, a student with access to one of the standard mechanics texts should be able to follow the main development presented below.

We begin with the fact that Newton's laws apply to the motion of objects in an inertial frame of reference. By considering a Cartesian coordinate frame rotating about the z axis with respect to an inertial frame and then generalizing, one can show that the time rate of change of any vector $\vec{A}$ in the inertial frame is related to the rate of change in the rotating frame by the standard formula as follows:[12]

$$\left(\frac{d\vec{A}}{dt}\right)_{inertial} = \left(\frac{d\vec{A}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{A}. \tag{1}$$

The angular velocity vector $\vec{\omega}$ of the rotating frame with respect to the inertial frame can, like any vector, be expressed in either frame, though it is defined in the inertial frame. The rate of change of the angular velocity $\vec{\omega}$ itself is the same in either frame, as one can see from Eq. (1). Because the parts of a rigid object remain at a fixed distance from one another, the velocity of any point in the body is determined completely at any instant by $\vec{\omega}$ and its position $\vec{r}$ in the body coordinate system. Thus, through some algebra, one finds that the kinetic energy T and the angular momentum $\vec{L}$ are related to $\vec{\omega}$ via the inertia tensor $\underline{I}$, which is a 3×3 symmetric matrix with nine time-dependent entries in the inertial frame. In any frame, one has the vector equations $\vec{L} = \underline{I}\cdot\vec{\omega}$ and $T = \frac{1}{2}\vec{\omega}\cdot\underline{I}\cdot\vec{\omega}$, expressing $\vec{L}$ and T in terms of the symmetric matrix $\underline{I}$. The components of the angular momentum vector $\vec{L}$ are constant in the inertial frame because there is no external torque, but they are not constant in the rotating frame as seen in Eq. (1).[4,16] One can choose body coordinate axes passing through the center of mass in which the matrix $\underline{\underline{I}}$ becomes diagonal, namely, the principal axes lying along the orthogonal eigenvectors of $\underline{\underline{I}}$. Euler hailed this discovery as an important simplification.[5]

Because $\vec{L}$ is constant in the inertial frame, we have the following in the body frame:

$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = 0 \;\Rightarrow\; \underline{\underline{I}} \cdot \frac{d\vec{\omega}}{dt} + \vec{\omega} \times (\underline{\underline{I}} \cdot \vec{\omega}) = 0. \tag{2}$$

In the principal-axis coordinates, this simplifies to the familiar Euler dynamical system for components of $\vec{\omega}$, as seen in Sec. II. These equations appear in most mechanics books. Interestingly, Euler neither used them nor wrote them down in his original papers on the subject.[4] Solving for the motion of a freely rotating object, then, becomes the problem of finding the time-dependent transformation between the inertial frame and the principal-axis frame moving with the object. Having found it difficult to locate a complete set of formulas for this transformation in the literature, in the treatment below, we express it using a set of Euler angles. The Euler angles and the related transformation matrix for a general rotation are developed in most mechanics texts, though the convention for defining them varies from author to author. We follow the one in Goldstein,[12] which comprises three rotations and involves the line of nodes, which is defined to be along the cross product of (i) a unit vector along the z axis in the inertial frame and (ii) the principal z' axis in the rotating frame (denoted W in Fig. 2) and is the intersection between the $x'y'$-plane in the rotating frame and the xy-plane in the fixed frame. First, the inertial system is rotated by an angle ϕ about its z axis to bring the initial x axis into coincidence with the line of nodes; this is rotation R_1. Then, a second rotation R_2 by θ about the line of nodes brings the original z direction into the final z' direction in the rotating frame. Finally, a third rotation R_3 by ψ moves the x axis from the line of nodes to its final position in the rotating frame. Curiously, just as Euler did not use the Euler dynamical system in his treatment of the problem, neither did he use a set of Euler angles. Rather, he used several sets of coordinates, mainly the three projection cosines between the fixed and rotating sets of axes, that is, he used the components of a unit vector along the rotation axis and the magnitude ω.[4,6,10]

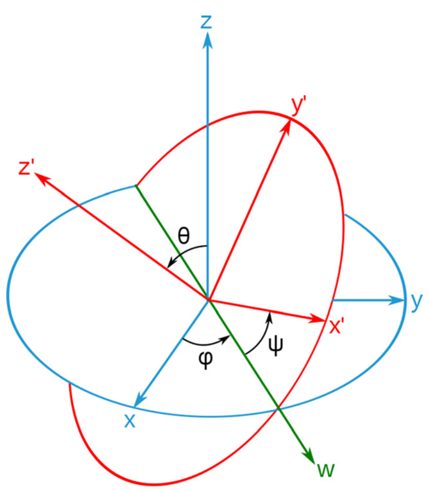

Fig. 2. Euler angle diagram. Line of nodes is along $\hat{z} \times \hat{z}' = \hat{w}$. The rotation R is a product of three factors: (1) rotation R_1 about z by ϕ, (2) rotation R_2 about x by θ, (3) rotation R_3 about z by ψ. Thus $R = R_3 R_2 R_1$.

In Sec. II, we solve the Euler equations of motion in the body principal-axis frame in terms of elliptic functions. There are two separate branches of the solution depending on the initial conditions. In Sec. III, we use the Euler angles to transform the solution to the inertial frame, and in Sec. IV, we apply the solution to the Dzhanibekov effect by comparing the first branch of the solution to a video of a dancing T-handle from the International Space Station. Finally, we summarize and discuss the main points in Sec. V.

II. SOLUTION IN THE BODY FRAME

From Eq. (2), the equation of motion for $\vec{\omega}$ in the body frame of reference is

$$\underline{\underline{I}} \cdot \frac{d\vec{\omega}}{dt} = -\vec{\omega} \times (\underline{\underline{I}} \cdot \vec{\omega}), \tag{3}$$

where the inertia tensor is a time-independent symmetric matrix with constant entries. By choosing the principal-axes coordinate system in the body frame, $\underline{\underline{I}}$ becomes diagonal so that

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \tag{4}$$

$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1, \tag{5}$$

$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2, \tag{6}$$

where I_1, I_2, and I_3 are the principal moments of inertia. We number the axes such that

$$I_2 < I_3 < I_1,$$

thereby choosing the intermediate axis of inertia to be the body z axis to simplify the transformation of the solution from the body frame to the inertial frame later on. The three Euler equations are the angular velocity component equations

$$\dot{\omega}_1 = -\left(\frac{I_3 - I_2}{I_1}\right)\omega_2\omega_3, \tag{7}$$

$$\dot{\omega}_2 = -\left(\frac{I_1 - I_3}{I_2}\right)\omega_3\omega_1, \tag{8}$$

$$\dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3}\right)\omega_1\omega_2, \tag{9}$$

in which each parenthetical quantity is positive. Now, we introduce the Jacobi elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, and $\mathrm{dn}(u,k)$. These are trigonometric functions defined using ratios on an ellipse, just as the usual trigonometric functions are ratios on a circle, and their properties are easily derived (see Appendix A).[17] The argument u is analogous to but not the same as an angle variable in the usual trigonometric case, and a new argument k denotes the eccentricity of the ellipse. The functions are periodic like ordinary trigonometric functions. The periods of both $\mathrm{sn}(u,k)$ and $\mathrm{cn}(u,k)$ are $4K$, where K is the complete elliptic integral, the first kind (see the Appendix A), and the period of $\mathrm{dn}(u,k)$ is $2K$. The complete elliptic integral K is a function of the eccentricity k, and as k of the reference ellipse defining the elliptic functions tends to zero, $K(k)$ tends to $\pi/2$. The functions sn and cn become the usual sine and cosine in this limit. The derivatives of the elliptic functions are

$$\frac{d}{du}\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\mathrm{dn}(u,k), \tag{10}$$

$$\frac{d}{du}\mathrm{cn}(u,k) = -\mathrm{sn}(u,k)\mathrm{dn}(u,k), \tag{11}$$

$$\frac{d}{du}\mathrm{dn}(u,k) = -k^2\mathrm{sn}(u,k)\mathrm{cn}(u,k). \tag{12}$$

These derivatives arise geometrically, as seen in Appendix A, and we take them as essential properties of the Jacobi functions. An important point is that the derivatives of the elliptic functions as seen in Eqs. (10)–(12) exhibit the same coupled structure as do the Euler differential equations of Eqs. (7)–(9) for the angular velocity components ω_1, ω_2, and ω_3. That is to say, the time derivative of each one equals a positive or negative constant multiple of the other two, and there are one positive and two negative coefficients. This suggests expressing the $\vec{\omega}$ components in terms of elliptic functions. To solidify the thinking, consider for a moment a simple harmonic oscillator comprising a block of mass m sliding over a frictionless horizontal surface and attached to the end of a spring with force constant k. Newton's second law of motion gives

$$m\ddot{x} + kx = 0, \tag{13}$$

and we know physically that the solution should be periodic. In this case, the differential equation can be solved in a linear manner. Sometimes, though, instead of following the usual procedure for solving a linear homogeneous differential equation with constant coefficients, it is useful to solve for $x(t)$ simply by trying a sine or cosine solution. This approach does not depend on linearity but rather relies on the uniqueness of the solution modulo the initial conditions. The trigonometric form is substituted into the differential equations, and then, the undetermined constants are found, which reduce the differential equation to an identity and are such that the initial conditions are satisfied. The solution process that follows is analogous to the second solution process for the oscillator. We can make the same sort of substitution in the rotating-object problem except using elliptic functions rather than sine or cosine. The complete solution has two branches, each of which corresponds to a different set of initial conditions. The first branch of the complete solution, the one that we use to describe the motion of the T-handle, is parameterized as

$$\omega_1 = A\mathrm{cn}(bt + K(k), k), \tag{14}$$

$$\omega_2 = B\mathrm{dn}(bt + K(k), k), \tag{15}$$

$$\omega_3 = C\mathrm{sn}(bt + K(k), k). \tag{16}$$

The function $K(k)$ gives the quarter period of sn and cn as a function of the eccentricity k and reduces to $\pi/2$ in the circular limit as $k \to 0$, where sn $\to$ sin and cn $\to$ cos. The time translation by $K(k)$ is done to apply a particular set of initial conditions at $t = 0$.

Because $\mathrm{cn}(K(k),k) = 0$, $\mathrm{sn}(K(k),k) = 1$, and $\mathrm{dn}(K(k),k) = \sqrt{1-k^2}$, we can solve to find $B = \omega_{20}/\sqrt{1-k^2}$ and $C = \omega_{30}$, where ω_{20} and ω_{30} are the initial angular velocities around the y and z axes, respectively, in the body frame. The remaining three free parameters A, b, and k can, then, be solved for by substituting the derivatives of Eqs. (14)–(16) into Eqs. (7)–(9) and solving the system of algebraic equations. The details of the calculation and the other branch of the complete solution are presented in Appendix B. The parameters for the first branch of the solution are found to be

$$k = \omega_{30}\sqrt{\frac{I_3(I_1 - I_3)}{I_2(I_1 - I_2)\omega_{20}^2 + I_3(I_1 - I_3)\omega_{30}^2}}, \tag{17}$$

$$A = \omega_{30}\sqrt{\frac{I_3(I_3 - I_2)}{I_1(I_1 - I_2)}}, \tag{18}$$

$$b = \sqrt{\frac{(I_3 - I_2)(I_2(I_1 - I_2)\omega_{20}^2 + I_3(I_1 - I_3)\omega_{30}^2)}{I_1 I_2 I_3}}. \tag{19}$$

An important point to note here that we discuss further below regarding the loitering of $\vec{\omega}$ in the T-handle motion near the unstable intermediate axis is that in the limit as $\omega_{20} \to 0$ (i.e., when the initial rotation is about the intermediate axis of inertia), the eccentricity of the elliptic solutions, Eq. (17), approaches unity.

III. TRANSFORMATION OF SOLUTION TO THE INERTIAL FRAME

We now solve Euler's equations in the body frame, from which we already know about the period τ of the flipping motion in the case of the dancing T-handle from b and $K(k)$, namely, $\tau = 4K(k)/b$. However, the body-frame solution is not very enlightening to an observer in the inertial frame of reference. Therefore, we transform the solution from the body frame into the inertial frame. To do this, we use the Euler angles ϕ, θ, and ψ and the corresponding rotation matrix, according to the convention of Goldstein as described in Sec. I.[12] If the rotation matrix operates on the angular momentum vector $\vec{L}$, then by choosing the angular momentum to be along the z axis in the inertial frame, we can simplify the transformation to the body frame as

$$L\sin\theta\sin\psi = I_1\omega_1, \tag{20}$$

$$L\sin\theta\cos\psi = I_2\omega_2, \tag{21}$$

$$L\cos\theta = I_3\omega_3, \tag{22}$$

where ω_1, ω_2, and ω_3 are the elliptic-function solutions described in Sec. II and Appendix B. The problem, then, is to find the time evolution of each Euler angle. The angle θ is known directly from Eq. (22), and ψ can be found by dividing Eq. (20) by Eq. (21) to give

$$\cos\theta = \frac{I_3\omega_3}{L}, \tag{23}$$

$$\tan\psi = \frac{I_1\omega_1}{I_2\omega_2}. \tag{24}$$

Because we can associate a vector with an infinitesimal rotation, we can associate the time derivatives of the rotation angles (ϕ, θ, ψ) with the angular velocity $\vec{\omega}$.[12,18] Expressing the components of $\vec{\omega}$ in the body frame, we have

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi, \tag{25}$$

$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi. \tag{26}$$

Multiplying Eq. (25) by $\sin\psi$ and Eq. (26) by $\cos\psi$ and then adding the two give

$$\dot{\phi} = \frac{\omega_1 \sin\psi + \omega_2 \cos\psi}{\sin\theta}. \tag{27}$$

Then, by using Eqs. (20) and (21), this reduces to

$$\dot{\phi} = \frac{(I_1\omega_1^2 + I_2\omega_2^2)L}{I_1^2\omega_1^2 + I_2^2\omega_2^2}. \tag{28}$$

Because there are no external torques, both the kinetic energy of the system $T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$ and the magnitude of the angular momentum $L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$ are conserved, and because these are scalars, they have the same numerical value in each frame. Thus, the equation for $\dot{\phi}$ simplifies to

$$\frac{d\phi}{dt} = \frac{(2T - I_3\omega_3^2)L}{L^2 - I_3^2\omega_3^2}. \tag{29}$$

Because ω_3 is proportional to the Jacobi sn function in both branches of the solution, the time-dependent ϕ can be found by integration.[19] This gives all three Euler angles in terms of the Jacobi elliptic functions as

$$\theta(t) = \arccos\left(\frac{I_3\omega_3(t)}{L}\right), \tag{30}$$

$$\psi(t) = \arctan\left(\frac{I_1\omega_1(t)}{I_2\omega_2(t)}\right), \tag{31}$$

$$\phi(t) = \frac{Lt}{I_3} + \frac{(L^2 - 2I_3T)\Pi\left[\frac{I_3^2C^2}{L^2}; \mathrm{Am}(bt,k), k\right]}{I_3bL}, \tag{32}$$

where $\Pi(n;x,k)$ is the incomplete elliptic integral of the third kind, $\mathrm{Am}(bt, k)$ is the Jacobi amplitude function (see Appendix A), and the $\omega(t)$ terms are those that were found as solutions in Sec. II and Appendix B. Equations (30)–(32), then, form the transformation of the complete solution in the body frame to the inertial frame in general. The final task is to match the initial data given in the inertial frame as seen by the observer.

For the first branch of the solution, at $t = 0$, $\psi = 0$, $\phi = 0$, and $\theta(0) = \theta_0$, we have

$$\omega_{30} = \frac{L}{I_3}\cos\theta_0. \tag{33}$$

From the magnitude of the angular momentum at $t = 0$, namely, $L^2 = I_2^2\omega_{20}^2 + I_3^2\omega_{30}^2$, we find that

$$\omega_{20} = \frac{L}{I_2}\sin\theta_0. \tag{34}$$

Similarly, for the second branch of the solution to the Euler equations in the body frame (see Appendix B), we find that $\phi(0) = 0$ and $\theta(0) = \theta_0$, but now $\psi(0) = \pi/2$. The initial condition for ω_{30} remains the same, and following the same argument, ω_{10} becomes

$$\omega_{10} = \frac{L}{I_1}\sin\theta_0. \tag{35}$$

The constant kinetic energy in Eqs. (29) and (32) is set by the initial conditions, namely,

$$T = \frac{1}{2}(I_1\omega_{10}^2 + I_2\omega_{20}^2 + I_3\omega_{30}^2), \tag{36}$$

where the appropriate component of $\vec{\omega}_0$ is zero depending on the particular branch that is being analyzed.

IV. DZHANIBEKOV EFFECT

The dancing motion of the T-handle known as the Dzhanibekov effect and more recently as an illustration of the tennis-racket theorem[1] can be ascribed to the first branch of the complete solution, in which the initial conditions of ω_{20} and ω_{30} are the rates of rotation around the x and y axes in the body frame at $t = 0$. The principle moments of inertia of the T-handle seen in Fig. 1 are calculated approximately using the moments of inertia of thick rods in conjunction with Steiner's parallel axis theorem.[18] Figures 3(a)–3(c) show consecutive times during one half cycle of a flip in the motion described by a particular solution with initial conditions that seem to be appropriate. By analyzing the solution, we see that the period for flipping, and thus the time that the T-handle lingers in a particular orientation before flipping, increases as $I_3 \to I_2$ or as the eccentricity $k \to 1$. This describes the observed motion assuming that the large cylinder on the T-handle in Figs. 1 and 3 is hollow. However, the most important initial factor is the value of θ_0, or the angle that the z axis in the body frame makes with $\vec{L}$ in the inertial frame at $t = 0$. For the particular solution illustrated in Fig. 3, we chose $\theta_0 = \pi/180$ rad. Although the behavior of our solution agrees qualitatively with the motion in the video of the T-handle, we have made no attempt to adjust the model to fit the period (τ) of the flipping motion because of the unknown moments of inertia of the T-handle and the extreme sensitivity of the solution to θ_0, of which we have no way of determining from the video. The situation is similar to that of a pencil balanced, hypothetically, on its point: the time it takes to fall down depends crucially on the initial displacement from its unstable equilibrium, and this would be very difficult to either control or measure. In Appendix C, we provide a link to the Wolfram Cloud and a Mathematica notebook that contains the simulation.

V. CONCLUSION

The general solution for $\vec{\omega}$ in the body frame is given in terms of the Jacobi functions sn, cn, and dn. The defining parameters A, b, and k for the first branch of the solution are reported in Eqs. (17)–(19), with the rest delegated to Appendix B. From the first branch, the flip period for the dancing T-handle is seen to be $\tau = 4K(k)/b$, which is approximately

$$\tau = 2\sqrt{\frac{I_1I_2I_3}{(I_3 - I_2)(I_2(I_1 - I_2)\omega_{20}^2 + I_3(I_1 - I_3)\omega_{30}^2)}} \times \log\left(\frac{I_2(I_1 - I_2)\omega_{20}^2}{I_2(I_1 - I_2)\omega_{20}^2 + I_3(I_1 - I_3)\omega_{30}^2}\right),$$

where we have expanded $K(k)$ for $k \to 1$.[20]

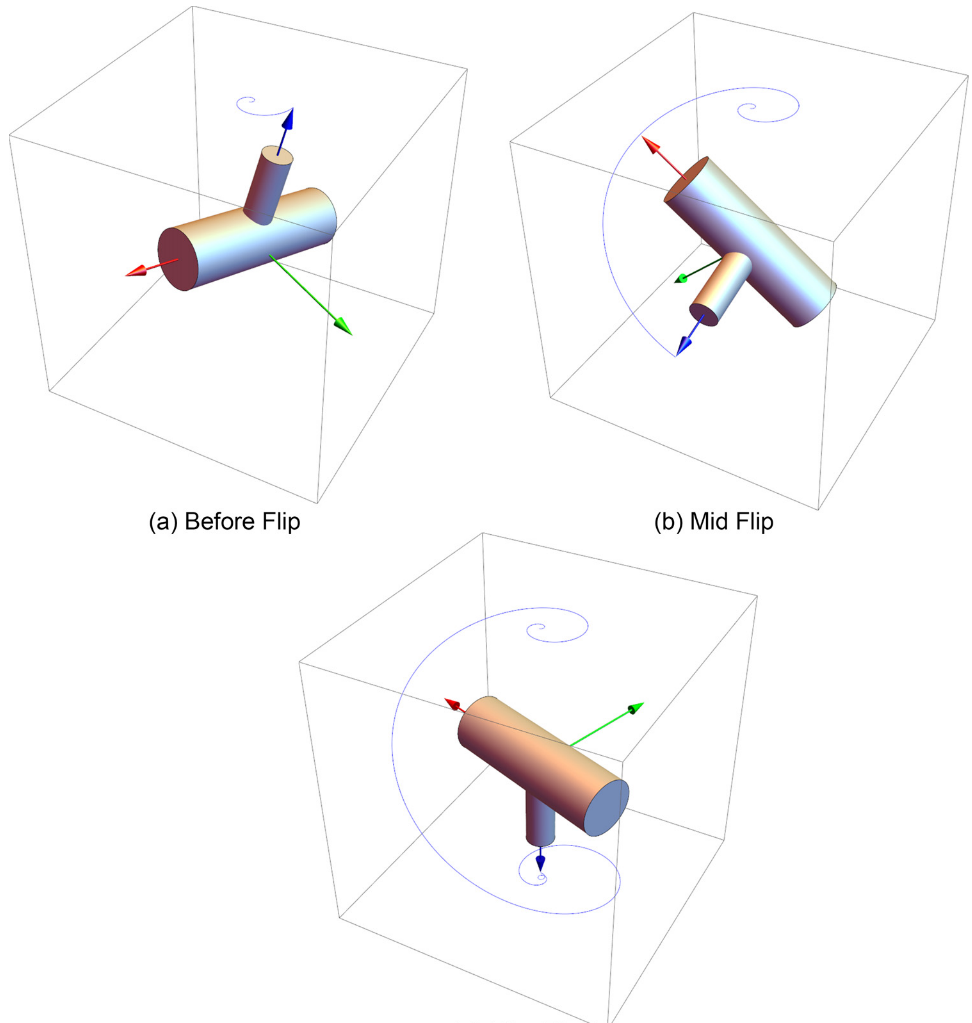

Fig. 3. Dzhanibekov effect Animation.

In Sec. III, the body-frame solution is transformed back to the inertial frame using Goldstein's convention for the Euler angles. The results in Eqs. (30)–(32) involve the body-frame solution and the additional Legendre elliptic integral of the third kind. In the other solutions presented elsewhere, it seems to us that the time dependence of ϕ is represented in a less transparent way. The theta functions used by Jacobi and others are usually introduced as complex series expansions, where the Legendre integral is seen to be an integration of a simple expression involving just the sn function.[8,9,11,12]

As noted in Sec. IV, we have not made a quantitative comparison between the solution and the video of the actual T-handle; however, the solution does qualitatively match the motion of the T-handle from the NASA video quite well.[17]

APPENDIX A: ELLIPTIC FUNCTIONS

This appendix is added to make the paper completely self-contained and to show that everything one needs to know about elliptic functions is relatively easy to develop. In particular, this development is trigonometric and makes no explicit use of complex analysis. The Jacobian elliptic functions are like trigonometric functions except that they are defined on the ellipse,

$$\frac{x^2}{a^2} + y^2 = 1, \tag{A1}$$

rather than on the unit circle. The shape of the ellipse is controlled by the eccentricity,

$$k = \sqrt{1 - 1/a^2}. \tag{A2}$$

For a given k, the sn and cn functions are defined by analogy to sine and cosine, namely,

$$\mathrm{sn}(u, k) = y \quad \text{and} \quad \mathrm{cn}(u, k) = x/a. \tag{A3}$$

In this context, k is called the modulus, and the argument u of the elliptic functions, namely,

$$u = \int_0^\theta r\, d\theta, \tag{A4}$$

is the integral along the ellipse from the x intercept $(a, 0)$ to the point (x, y). Note that u is neither the arc length nor the area subtended. In terms of the polar angle θ, the upper limit is such that $\sin\theta = y/r$. This is not to be confused with Jacobi's amplitude $\phi = \arcsin y = \mathrm{Am}(u, k)$ such that

$\sin(\mathrm{Am}(u,k)) = \mathrm{sn}(u,k)$. The latter equation will serve to define the elliptic amplitude function Am(u, a).

Because the radius is not constant for an ellipse, there is a third elementary elliptic function, not corresponding to any trigonometric function, in addition to sn(u, k) and cn(u, k), namely,

$$\mathrm{dn}(u,k) = \frac{r}{a}. \tag{A5}$$

From the ellipse equation, we have

$$\mathrm{cn}(u,k)^2 + \mathrm{sn}(u,k)^2 = 1. \tag{A6}$$

From this result and the Pythagorean relation $x^2 + y^2 = r^2$, we have

$$\mathrm{dn}(u,k)^2 + k^2\mathrm{sn}(u,k)^2 = 1. \tag{A7}$$

In the normal trigonometry based on a circle, the latter two identities reduce to one.

It is relatively easy to show from the definitions that

$$\frac{d}{du}\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\mathrm{dn}(u,k), \tag{A8}$$

and because all the calculus properties of the Jacobi functions follow from this, we devote a few lines to showing it here.

The first step is to express $du = r\,d\theta$ in Cartesian form. Starting from $x = r\cos\theta$ and $y = r\sin\theta$, we find

$$x\,dy - y\,dx = r^2\,d\theta, \tag{A9}$$

so

$$du = \frac{x\,dy - y\,dx}{r}. \tag{A10}$$

We use the ellipse equation to eliminate x and dx, namely,

$$x\,dx + a^2 y\,dy = 0, \quad dx = -a^2\frac{y}{x}\,dy. \tag{A11}$$

Substituting this and multiplying by x in Eq. (A10) give

$$\frac{x^2dy - xy\,dx}{r} = x\,du. \tag{A12}$$

Now, we use the ellipse equation again in the form $x^2 + a^2y^2 = a^2$ to obtain

$$\frac{(x^2 + a^2y^2)dy}{r} = \frac{a^2}{r}\,dy = x\,du, \tag{A13}$$

which gives

$$\frac{dy}{du} = \frac{x}{a}\frac{r}{a} \quad \text{or} \quad \frac{d}{du}\,\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\,\mathrm{dn}(u,k), \tag{A14}$$

which is the desired result. By differentiating the two algebraic identities, Eqs. (A6) and (A7), shown above and substituting Eq. (A14), one arrives at

$$\frac{d}{du}\mathrm{cn}(u,k) = -\mathrm{sn}(u,k)\mathrm{dn}(u,k) \quad \text{and}$$
$$\frac{d}{du}\mathrm{dn}(u,k) = -k^2\,\mathrm{sn}(u,k)\mathrm{cn}(u,k). \tag{A15}$$

These three differential equations match the Euler equations almost perfectly.

Elliptic functions are related quite closely to the elliptic integrals of Legendre. From the separable differential equation

$$\frac{d}{du}\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\,\mathrm{dn}(u,k) = \sqrt{1 - \mathrm{sn}(u,k)^2}\sqrt{1 - k^2\mathrm{sn}(u,k)^2}, \tag{A16}$$

we have

$$u = \int_0^{\mathrm{sn}(u,k)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{A17}$$

This shows the elliptic integral of the first kind commonly denoted as the F function to be the inverse of the sn function. However, the argument of F has been taken historically to be the Jacobi amplitude angle ϕ such that $\sin\phi = \mathrm{sn}(u,k)$ rather than sn(u, k), so

$$F(\phi,k) = \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{A18}$$

Thus written, the first elliptic integral function $F(\phi,k)$ of Legendre is related to the inverse sn function by

$$\mathrm{sn}^{-1}(y,k) = F(\sin^{-1}y, k) \tag{A19}$$

over the standard domain of $\sin^{-1}$.

There are also the second and third types of Legendre elliptic integrals. These are integrals of simple elliptic function expressions. The second elliptic integral is the integral of Jacobi's dn squared, namely,

$$E(\phi,k) = \int_0^{\phi}\sqrt{1 - k^2\sin^2\phi}\,d\phi = \int_0^{u}(\mathrm{dn}(u,k))^2du, \tag{A20}$$

where in the second integral we made the elliptic substitution $\sin\phi = \mathrm{sn}(u,k)$.

Legendre's third elliptic integral contains an extra parameter n, which can be any real number, namely,

$$\Pi(n;\phi,k) = \int_0^{\phi}\frac{1}{1 - n\sin^2\phi}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}, \tag{A21}$$

and on making the same elliptic substitution, we obtain

$$\Pi(n;\phi,k) = \int_0^{u}\frac{du}{1 - n\,\mathrm{sn}^2(u,k)}. \tag{A22}$$

These three elliptic integral functions are standard, and their properties are tabulated on the internet and in most textbooks regarding Elliptic functions.

APPENDIX B: SOLUTIONS OF EULER'S EQUATIONS

The three Euler equations are the component equations

$$\dot{\omega}_1 = -\left(\frac{I_3 - I_2}{I_1}\right)\omega_2\omega_3, \tag{B1}$$

$$\dot{\omega}_2 = -\left(\frac{I_1 - I_3}{I_2}\right)\omega_3\omega_1, \tag{B2}$$

$$\dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3}\right)\omega_1\omega_2. \tag{B3}$$

1. Branch 1

The first branch of the complete solution is parameterized as

$$\omega_1 = A\mathrm{cn}(bt + K(k), k), \tag{B4}$$

$$\omega_2 = B\mathrm{dn}(bt + K(k), k), \tag{B5}$$

$$\omega_3 = C\mathrm{sn}(bt + K(k), k). \tag{B6}$$

Two of the amplitudes are related to the initial data by setting $t=0$. Because $\mathrm{cn}(K(k),k) = 0$, $\mathrm{sn}(K(k),k) = 1$, and $\mathrm{dn}(K(k),k) = \sqrt{1-k^2}$, we can solve to find $B = \omega_{20}/\sqrt{1-k^2}$ and $C = \omega_{30}$, where ω_{20} and ω_{30} are the initial angular velocities around the y and z axes, respectively, in the body frame. Then, by substituting these constants back in and taking the derivatives of Eqs. (B4)–(B6), we obtain

$$\dot{\omega}_1 = -(Ab)\mathrm{sn}(bt + K(k), k)\mathrm{dn}(bt + K(k), k), \tag{B7}$$

$$\dot{\omega}_2 = -\left(\frac{k^2 b\omega_{20}}{\sqrt{1-k^2}}\right)\mathrm{sn}(bt + K(k), k)\mathrm{cn}(bt + K(k), k), \tag{B8}$$

$$\dot{\omega}_3 = (\omega_{30} b)\mathrm{cn}(bt + K(k), k)\mathrm{dn}(bt + K(k), k). \tag{B9}$$

By solving for the elliptic functions in Eqs. (B4)–(B6) and then substituting into Eqs. (B7)–(B9), we obtain

$$\dot{\omega}_1 = -\left(\frac{bA\sqrt{1-k^2}}{\omega_{20}\,\omega_{30}}\right)\omega_2\omega_3, \tag{B10}$$

$$\dot{\omega}_2 = -\left(\frac{k^2\omega_{20} b}{A\omega_{30}\sqrt{1-k^2}}\right)\omega_3\omega_1, \tag{B11}$$

$$\dot{\omega}_3 = \left(\frac{\omega_{30} b\sqrt{1-k^2}}{A\omega_{20}}\right)\omega_1\omega_2. \tag{B12}$$

Comparing Eqs. (B10)–(B12) to Eqs. (B1)–(B3), respectively, gives the algebraic system

$$\frac{I_3 - I_2}{I_1} = \frac{bA\sqrt{1-k^2}}{\omega_{20}\,\omega_{30}}, \tag{B13}$$

$$\frac{I_1 - I_3}{I_2} = \frac{k^2\omega_{20} b}{A\omega_{30}\sqrt{1-k^2}}, \tag{B14}$$

$$\frac{I_1 - I_2}{I_3} = \frac{\omega_{30} b\sqrt{1-k^2}}{A\omega_{20}}. \tag{B15}$$

To determine all the previously undetermined constants of the solution, we begin by multiplying Eq. (B14) by the reciprocal of Eq. (B15) to find k as

$$k = \omega_{30}\sqrt{\frac{I_3(I_1 - I_3)}{I_2(I_1 - I_2)\omega_{20}^2 + I_3(I_1 - I_3)\omega_{30}^2}}. \tag{B16}$$

By dividing Eq. (B13) by Eq. (B15), we find A as

$$A = \omega_{30}\sqrt{\frac{I_3(I_3 - I_2)}{I_1(I_1 - I_2)}}. \tag{B17}$$

From k and A, we find the final unknown b as

$$b = \sqrt{\frac{(I_3 - I_2)(I_2(I_1 - I_2)\omega_{20}^2 + I_3(I_1 - I_3)\omega_{30}^2)}{I_1 I_2 I_3}}. \tag{B18}$$

Equations (B4)–(B6) and (B16)–(B18) give a complete description for a set of initial conditions in which $\vec{\omega}$ is initially in the yz plane.

2. Branch 2

In the xz-plane case, we have

$$\omega_1 = B\mathrm{dn}(bt + K(k), k), \tag{B19}$$

$$\omega_2 = A\mathrm{cn}(bt + K(k), k), \tag{B20}$$

$$\omega_3 = C\mathrm{sn}(bt + K(k), k). \tag{B21}$$

At $t=0$, this allows for an initial rotation of the object around the x axis, or the axis with the greatest moment of inertia in the body frame, where now $B = \omega_{10}/\sqrt{1-k^2}$ and $C = \omega_{30}$. Solving the resulting algebraic system using a similar scheme as above gives

$$k = \omega_{30}\sqrt{\frac{I_3(I_3 - I_2)}{I_1(I_1 - I_2)\omega_{10}^2 + I_3(I_3 - I_2)\omega_{30}^2}}, \tag{B22}$$

$$A = \omega_{30}\sqrt{\frac{I_3(I_1 - I_3)}{I_2(I_1 - I_2)}}, \tag{B23}$$

$$b = \sqrt{\frac{(I_3 - I_2)(I_2(I_1 - I_2)\omega_{10}^2 + I_3(I_1 - I_3)\omega_{30}^2)}{I_1 I_2 I_3}}. \tag{B24}$$

APPENDIX C: MATHEMATICA SIMULATION OF SPINNING T-HANDLE

The Mathematica notebook used to simulate the T-Handle and by which Fig. 4 was created is published openly on the

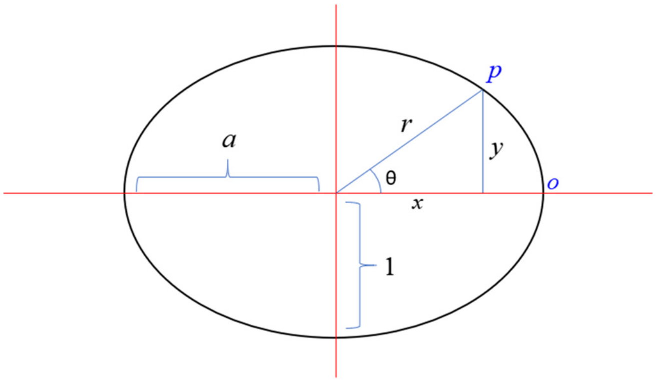

Fig. 4. Normalized ellipse.

Wolfram Cloud. It can be found at https://www.wolfram-cloud.com/obj/5b6d6507-cfe1-4d02-bb0f-35584f235871 or https://tinyurl.com/y3kdhkfd.

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