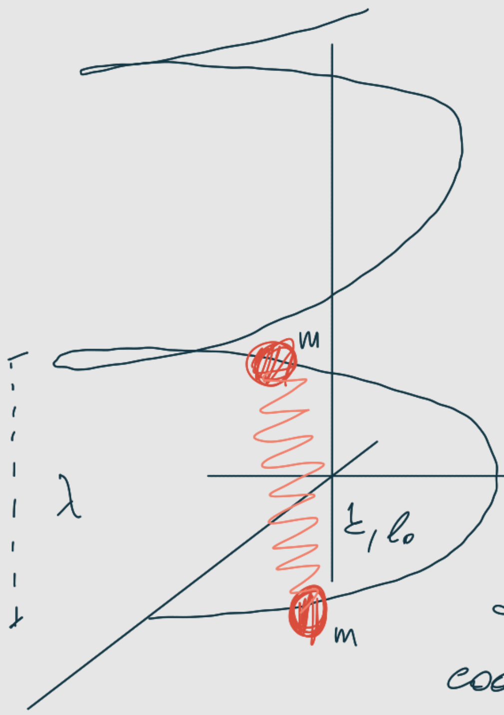
1º Parcial 1C 2024: Ej 3



$l_0 = \lambda$

datos: m, k, l_0, λ

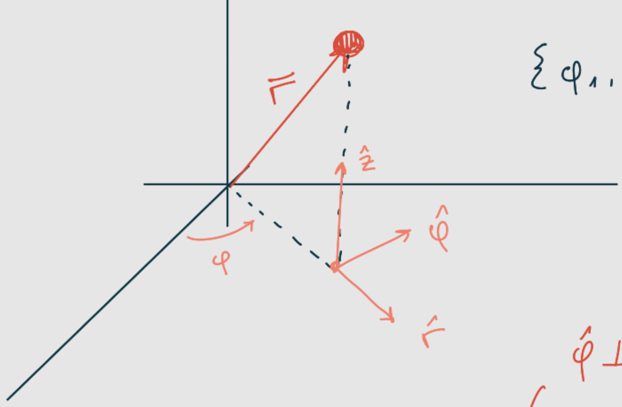
Como todo problema de partículas moviéndose en lugares, hay que comenzar describiendo los vectores $\bar{r}_{(q_i)}$ con unas buenas coord. generalizadas

Como la hélice es una superficie de dimensión 1 (una curva) alcanzará 1 coord. para cada partícula.

Coord. Cilíndricas

$$\bar{r}_{(\varphi,z)} = r\,\hat{r}_{(\varphi)} + z\,\hat{z} \longrightarrow r = a \;\&\; z = \frac{\lambda\varphi}{2\pi}$$

$$\rightarrow \bar{r}_{(\varphi)} = a\,\hat{r}_{(\varphi)} + \frac{\lambda\varphi}{2\pi}\hat{z}$$



$\{\varphi_1, \varphi_2\}$: Coord. gen. → 2 g.l.

$$\dot{\bar{r}} = a\dot{\varphi}\,\hat{\varphi} + \frac{\lambda}{2\pi}\dot{\varphi}\,\hat{z}$$

$\hat{\varphi} \perp \hat{z}$

$$\rightarrow |\dot{\bar{r}}|^2 = \dot{\bar{r}} \cdot \dot{\bar{r}} = a^2\dot{\varphi}^2 + \frac{\lambda^2}{4\pi^2}\dot{\varphi}^2 = \underbrace{\left(a^2 + \frac{\lambda^2}{4\pi^2}\right)}_{\equiv R^2,\ \text{por comodidad}}\dot{\varphi}^2$$

La energía cinética es, entonces:

$$\boxed{T = \frac{mR^2}{2}\left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2\right)}$$

Para el potencial es fácil ver que el estiramiento del resorte es

$$\Delta x = |\bar{r}_2 - \bar{r}_1| = |a\hat{r}_{(\varphi_2)} - a\hat{r}_{(\varphi_1)} + \frac{\lambda}{2\pi}(\varphi_2-\varphi_1)\hat{z}|$$

$$= \sqrt{(a\hat{r}_{(\varphi_2)} - a\hat{r}_{(\varphi_1)} + \frac{\lambda}{2\pi}(\varphi_2-\varphi_1)\hat{z})\cdot(a\hat{r}_{(\varphi_2)} - a\hat{r}_{(\varphi_1)} + \frac{\lambda}{2\pi}(\varphi_2-\varphi_1)\hat{z})}$$

$$= \sqrt{2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2}$$

$\hat{r}(\varphi_i)\cdot\hat{r}(\varphi_i) = 1$, $\hat{r}(\varphi_2)\cdot\hat{r}(\varphi_1) = \cos(\varphi_2-\varphi_1)$ $\hat{r}\cdot\hat{z} = 0$

Luego

$l_0 = \lambda$

$$V(\varphi_1,\varphi_2) = \frac{k}{2}\left(\sqrt{2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2} - \lambda\right)^2$$

Puntos de equilibrio $\rightarrow$ $\left.\frac{\partial V}{\partial q_i}\right|_{q_0} = 0 \quad \forall i$

Hay que pedir que

$$\frac{\partial V}{\partial \varphi_1} = 0 \quad \& \quad \frac{\partial V}{\partial \varphi_2} = 0 .$$

Como $V(\varphi_1,\varphi_2) = V(\varphi_2-\varphi_1) \Rightarrow \frac{\partial V}{\partial \varphi_2} = -\frac{\partial V}{\partial \varphi_1}$

Alcanza con derivar una vez!

$$\frac{\partial V}{\partial \varphi_2} = \frac{k\left(\sqrt{2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2} - \lambda\right)\left(2a^2\sin(\varphi_2-\varphi_1) + \frac{\lambda^2}{2\pi^2}(\varphi_2-\varphi_1)\right)}{\sqrt{2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2}}$$

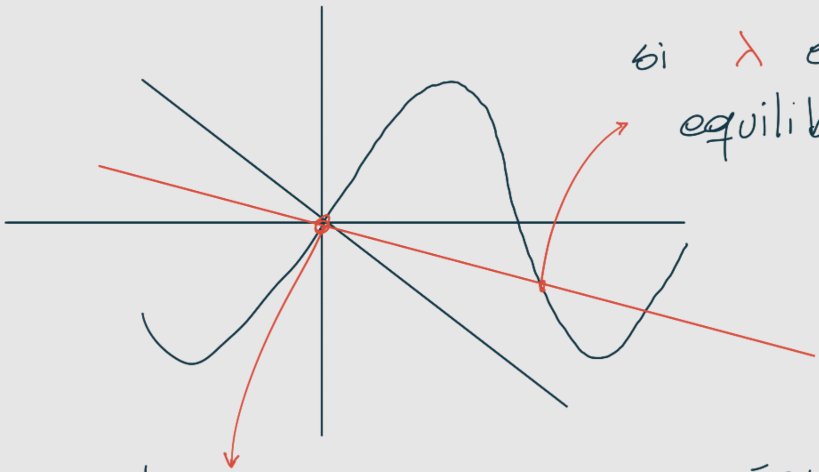
$$\frac{\partial V}{\partial \varphi_1} = 0 \quad \text{sii} \quad \sqrt{2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2} - \lambda = 0 \qquad (A)$$

$$\text{ó} \quad 2a^2\sin(\varphi_2-\varphi_1) + \frac{\lambda^2}{2\pi^2}(\varphi_2-\varphi_1) = 0 \qquad (B)$$

$$(A) = 0 \rightarrow 2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2 = \lambda^2$$

si $\boxed{\varphi_2-\varphi_1 = 2\pi}$ se cumple! (es lo que pide el ej)

$$(B) = 0 \quad \sin(\varphi_2-\varphi_1) = -\frac{\lambda^2}{4\pi^2 a^2}(\varphi_2-\varphi_1)$$



si λ es chico hay otro equilibrio.

El punto $\varphi_2-\varphi_1 = 0$ es engañoso pues se anula el denominador

$$\hookrightarrow \sqrt{2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2}$$

$$\rightarrow \lim_{\Delta\varphi \to 0} \frac{2a^2\sin(\Delta\varphi) + \frac{\lambda^2}{2\pi^2}\Delta\varphi}{\sqrt{2a^2(1-\cos(\Delta\varphi)) + \frac{\lambda^2}{4\pi^2}\Delta\varphi^2}} \rightarrow 2R$$

por lo que no es un mínimo.

"$\frac{0}{0}$"

Peq. Osc

Tomo $\varphi_1 = 0 + \eta_1$ & $\varphi_2 = 2\pi + \eta_2$

$$\rightarrow T = \frac{mR^2}{2}(\dot\eta_1^2 + \dot\eta_2^2) \Rightarrow \boxed{\mathbb{T} = mR^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

La matriz $\mathbb{V}$ la armamos derivando V.

$$\mathbb{V} = \left.\begin{pmatrix} \frac{\partial^2 V}{\partial\varphi_1\partial\varphi_1} & \frac{\partial^2 V}{\partial\varphi_1\partial\varphi_2} \\ \frac{\partial^2 V}{\partial\varphi_2\partial\varphi_1} & \frac{\partial^2 V}{\partial\varphi_2\partial\varphi_2} \end{pmatrix}\right|_{\substack{\varphi_1=0\\ \varphi_2=2\pi}} ; \quad \text{pero como } V = V_{(\varphi_2-\varphi_1)}$$

$$\Rightarrow \frac{\partial^2 V}{\partial\varphi_1^2} = \frac{\partial^2 V}{\partial\varphi_2^2} = -\frac{\partial^2 V}{\partial\varphi_1\partial\varphi_2} = -\frac{\partial^2 V}{\partial\varphi_2\partial\varphi_1}$$

$$\rightarrow \boxed{\mathbb{V} = \underbrace{\left.\frac{\partial^2 V}{\partial\varphi_2^2}\right|_{\substack{\varphi_1=0\\ \varphi_2=2\pi}}}_{\equiv \gamma} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} = \begin{pmatrix} \gamma & -\gamma \\ -\gamma & \gamma \end{pmatrix}$$

Alcanza, de nuevo, con sólo una derivada

$$\frac{\partial^2 V}{\partial\varphi_2^2} = k\left[\frac{\left(a^2\sin(\varphi_2-\varphi_1) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)\right)^2}{2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2} + \right.$$

$$- \frac{\left(\sqrt{2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2} - \lambda\right)\left(a^2\sin(\varphi_2-\varphi_1) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)\right)^2}{\left(2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2\right)^{3/2}}$$

$$\left. + \frac{\left(\sqrt{2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2} - \lambda\right)}{\sqrt{2a^2(1-\cos(\varphi_2-\varphi_1)) + \frac{\lambda^2}{4\pi^2}(\varphi_2-\varphi_1)^2}}\left(2a^2\cos(\varphi_2-\varphi_1) + \frac{\lambda^2}{4\pi^2}\right)\right]$$

Al evaluar en $\Delta\varphi = 2\pi$ sólo queda el 1° término

$$\rightarrow \frac{\partial^2 V}{\partial\varphi_2^2} = \frac{k\lambda^2}{4\pi^2} \rightarrow \boxed{\mathbb{V} = \frac{k\lambda^2}{4\pi^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}$$

$$\rightarrow \boxed{\mathcal{L} = \frac{\dot{\bar{\eta}}^T\,\mathbb{T}\,\dot{\bar{\eta}}}{2} - \frac{\bar{\eta}^T\,\mathbb{V}\,\bar{\eta}}{2}}$$

Proponemos $\vec{\eta} = \bar{A} e^{i\omega t} \rightarrow \ddot{\vec{\eta}} = -\omega^2 \vec{\eta}$

$$\rightarrow \boxed{(\mathbb{V} - \omega^2 \mathbb{T})\vec{A} = 0}$$ Pedimos $\det(\mathbb{V} - \omega^2 \mathbb{T}) = 0$

$$\det \begin{vmatrix} \frac{k\lambda^2}{4\pi^2} - mR^2\omega^2 & -\frac{k\lambda^2}{4\pi^2} \\ -\frac{k\lambda^2}{4\pi^2} & \frac{k\lambda^2}{4\pi^2} - mR^2\omega^2 \end{vmatrix} = 0$$

$$\rightarrow \left(\frac{k\lambda^2}{4\pi^2} - mR^2\omega^2\right)^2 - \left(\frac{k\lambda^2}{4\pi^2}\right)^2 = 0 \rightarrow \frac{k\lambda^2}{4\pi^2} - mR^2\omega^2 = \pm \frac{k\lambda^2}{4\pi^2}$$

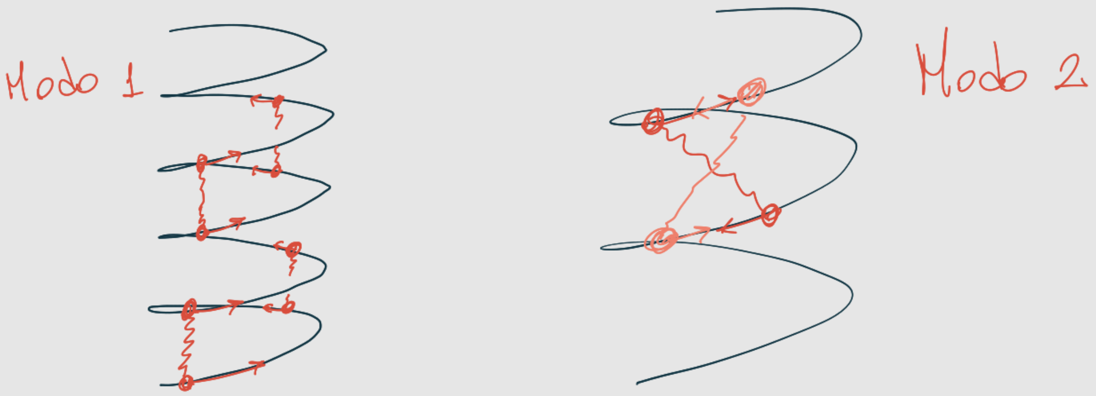
$$\rightarrow \boxed{\omega_1^2 = 0} \rightarrow \mathbb{V}\bar{A}_1 = 0 \Rightarrow \bar{A}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

&

$$\boxed{\omega_2^2 = \frac{k\lambda^2}{2\pi^2 mR^2}} \rightarrow \frac{k\lambda^2}{4\pi^2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\bar{A}_2 = 0 \rightarrow \bar{A}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

La sol. general es entonces $\bar{\eta} = \mathrm{Re}\left(\sum_i c_i \bar{A}_i e^{i\omega_i t}\right)$

$$\boxed{\bar{\eta}(t) = (c_1 + c_1' t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cos(\omega_2 t + \phi)\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$



Coord. normales: Como $\omega_1 \neq \omega_2 \Rightarrow \bar{A}_1 \mathbb{T} \bar{A}_2 = 0$

def. $\vec{\xi} = A^T \mathbb{T} \bar{\eta} \rightarrow A^T \mathbb{T} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} mR^2 & 0 \\ 0 & mR^2 \end{pmatrix}$

$\Rightarrow \bar{\xi} = mR^2 \begin{pmatrix} \eta_1 + \eta_2 \\ \eta_1 - \eta_2 \end{pmatrix}; \; A^T \mathbb{T} = mR^2 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

es sólo una normalización, da igual si está o no

Sabemos que, a menos de una normalización

$\mathcal{L}(\bar{\xi}, \dot{\bar{\xi}}) = \frac{1}{2} \dot{\xi}_1^2 + \frac{1}{2}\left(\dot{\xi}_2^2 - \omega_2^2 \xi_2^2\right)$ y por lo tanto

$\omega_1 = 0$

$\xi_1(t) = a + bt$ & $\xi_2(t) = A \cos(\omega_2 t + \phi)$

Nota

$V(\varphi_1, \varphi_2)$ se podia aproximar a mano pero era más jugado. Queremos $V(\Delta\varphi)$ cuando $\Delta\varphi \simeq 2\pi$

$\rightarrow \; 2a^2 \underbrace{(1 - \cos\Delta\varphi)}_{\sim 0 \text{, es de orden cero}} + \lambda^2 \left(\frac{\Delta\varphi}{2\pi}\right)^2 \simeq \lambda^2 \left(\frac{\Delta\varphi}{2\pi}\right)^2$

este gana ya que no es chico

$$V(\Delta\varphi) \cong \frac{k}{2}\lambda^2 \left(\frac{\Delta\varphi}{2\pi} - 1\right)^2 = \frac{1}{2} \frac{k\lambda^2}{4\pi^2}(\Delta\varphi - 2\pi)^2 = \frac{1}{2}\frac{k\lambda^2}{4\pi^2}(\eta_2 - \eta_1)^2.$$

Si bien la cuenta sale, no era facil de ver y lo más directo era derivar.

MECÁNICA CLÁSICA 1er C. 2024

1er Parcial resuelto

1)

$\bar{A} = -A_0 \log\frac{r}{r_0}\hat{z}$

a) $U = -q\bar{v}\cdot\bar{A} = qA_0\dot{z}\log\frac{r}{r_0}$

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) - qA_0\dot{z}\log\frac{r}{r_0}$$

r) $\frac{\partial\mathcal{L}}{\partial r} = mr\dot{\varphi}^2 - qA_0\frac{\dot{z}}{r}$ $\qquad \frac{\partial\mathcal{L}}{\partial\dot{r}} = m\dot{r}$

$\rightarrow m\ddot{r} - mr\dot{\varphi}^2 + qA_0\frac{\dot{z}}{r} = 0$

φ) $\frac{\partial\mathcal{L}}{\partial\varphi} = 0$ $\qquad \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = mr^2\dot{\varphi}$ $\rightarrow \frac{d}{dt}(mr^2\dot{\varphi}) = 0$

$\rightarrow mr^2\ddot{\varphi} + 2mr\dot{r}\dot{\varphi} = 0$

z) $\frac{\partial\mathcal{L}}{\partial z} = 0$ $\qquad \frac{\partial\mathcal{L}}{\partial\dot{z}} = m\dot{z} - qA_0\log\frac{r}{r_0}$ $\rightarrow \frac{d}{dt}\left(m\dot{z} - qA_0\log\frac{r}{r_0}\right) = 0$

$\rightarrow m\ddot{z} - \frac{qA_0}{r}\dot{r} = 0$

b) $\delta r = \varepsilon_1$: $\mathcal{L}' = \frac{1}{2}m(\dot{r}^2 + (r+\varepsilon_1)^2\dot{\varphi}^2 + \dot{z}^2) - qA_0\dot{z}\log\frac{r+\varepsilon_1}{r_0}$

$\rightarrow \delta\mathcal{L} = mr\dot{\varphi}^2\varepsilon_1 - qA_0\frac{\dot{z}}{r}\varepsilon_1 + \mathcal{O}(\varepsilon_1^2)$

No es simetría (no es derivada total)

Demostración completa (no necesaria)

Dada una función $f(q,\dot{q},t)$, su derivada total siempre es de la forma

$$\frac{df}{dt} = \frac{\partial f(q,\dot{q},t)}{\partial q}\dot{q} + \frac{\partial f(q,\dot{q},t)}{\partial\dot{q}}\ddot{q} + \frac{\partial f(q,\dot{q},t)}{\partial t}$$

En particular, si $\ddot{q}$ no aparece, $\frac{\partial f}{\partial\dot{q}} = 0$ y toda derivada total es de la forma $\frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t}$

$\delta\varphi = \varepsilon_2$: $\quad \mathcal{L}' = \mathcal{L} \rightarrow \delta\mathcal{L} = 0 = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\dot\varphi}\cancel{\delta\dot\varphi}^{=0}$

$$= \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\varphi}\varepsilon_2\right)$$

$$\rightarrow \frac{\partial\mathcal{L}}{\partial\dot\varphi} = mr^2\dot\varphi = cte = \ell$$

$\delta z = \varepsilon_3$: $\quad \mathcal{L}' = \mathcal{L} \rightarrow \delta\mathcal{L} = 0 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot z}\varepsilon_3\right)$

$$\rightarrow \frac{\partial\mathcal{L}}{\partial\dot z} = m\dot z - qA_0\log\frac{r}{r_0} = cte = p_z$$

$\delta t = \varepsilon_4$: $\quad \mathcal{L}' = \mathcal{L} \rightarrow \frac{\partial\mathcal{L}}{\partial t} = -\frac{dh}{dt} = 0$

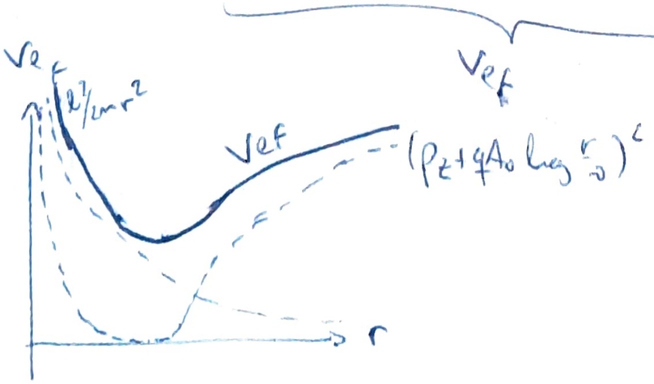
$$\rightarrow h = \dot r\frac{\partial\mathcal{L}}{\partial\dot r} + \dot\varphi\frac{\partial\mathcal{L}}{\partial\dot\varphi} + \dot z\frac{\partial\mathcal{L}}{\partial\dot z} = \frac{1}{2}m(\dot r^2 + r^2\dot\varphi^2 + \dot z^2) = cte$$

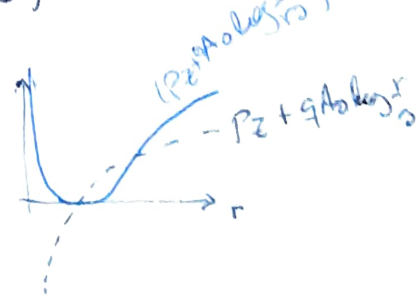
$\delta z = \varepsilon_5 t$, $\quad \delta\dot z = \varepsilon_5$

$$\delta\mathcal{L} = m\dot z\varepsilon_5 - qA_0\log\frac{r}{r_0}\varepsilon_5 \rightarrow \text{no es simetría}$$

c) En h reemplazo $\dot\varphi = \frac{\ell}{mr^2}$, $\quad \dot z = \frac{1}{m}\left(p_z + qA_0\log\frac{r}{r_0}\right)$

$$\rightarrow h = \frac{1}{2}m\dot r^2 + \underbrace{\frac{\ell^2}{2mr^2} + \frac{1}{2m}\left(p_z + qA_0\log\frac{r}{r_0}\right)^2}_{V_{ef}}$$





2)

a) $\bar{L} = \bar{r} \times \bar{p} \rightarrow \bar{r}$ siempre es $\perp$ a $\bar{L}$.

Si $\bar{L}$ es cte, es siempre $\perp$ a un plano

Algebraicamente, $\bar{r} \cdot \bar{L} = 0$ es la ecuación de un plano

b) $\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{\lambda}{4} r^4$

$l = mr^2\dot{\varphi} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$

$E = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{\lambda}{4} r^4 = \frac{1}{2} m\dot{r}^2 + \underbrace{\frac{l^2}{2mr^2} + \frac{\lambda}{4} r^4}_{V_{eff}}$

Órbitas circulares: $\left.\frac{\partial V_{eff}}{\partial r}\right|_{r_c} = 0$

$\frac{\partial V_{eff}}{\partial r} = -\frac{l^2}{mr^3} + \lambda r^3$

$\rightarrow -\frac{l^2}{mr_c^3} + \lambda r_c^3 = 0 \rightarrow r_c = \left(\frac{l^2}{m\lambda}\right)^{1/6}$

$\rightarrow l^2 = m\lambda r_c^6$

$E = V_{eff}(r_c)$

$= \frac{l^2}{2mr_c^2} + \frac{\lambda}{4} r_c^4 = \frac{1}{2mr_c^2} \cdot m\lambda r_c^6 + \frac{\lambda}{4} r_c^4 = \frac{3}{4} \lambda r_c^4$

$\rightarrow r_c = \left(\frac{4E}{3\lambda}\right)^{1/4}$

c) EdM radial: $\ddot{r} - \frac{l^2}{m^2 r^3} + \frac{\lambda}{m} r^3 = 0$

$r = r_c + \delta r \rightarrow \frac{1}{r^3} = \frac{1}{r_c^3 (1 + \frac{\delta r}{r_c})^3} \approx \frac{1}{r_c^3}\left(1 - \frac{3\delta r}{r_c}\right) = \frac{1}{r_c^3} - \frac{3}{r_c^4} \delta r$

$r^3 \approx r_c^3 + 3 r_c^2 \delta r$

$\delta\ddot{r} - \underbrace{\frac{l^2}{m^2 r_c^3} + \frac{\lambda r_c^3}{m}}_{=0} \underbrace{- \frac{3l^2}{m^2 r_c^4}}_{= \frac{3\lambda r_c^2}{m}} \delta r + \frac{3\lambda}{m} r_c^2 \delta r = 0$

$\delta\ddot{r} + \frac{6\lambda r_c^2}{m} \delta r = 0$

También puedo expandir la energía a 2do orden y/o escribir todo en términos de l en lugar de r_c (pero no las dos)

Defino $\omega_0^2 = \frac{\lambda r_c^2}{m}$, $\omega^2 = 6\omega_0^2$

$\Rightarrow \ddot{\delta r} + 6\omega_0^2 \, \delta r = 0 \quad \Rightarrow \delta r = \varepsilon \cos(\sqrt{6}\omega_0 t + \delta)$

$r(t) = r_c + \varepsilon \cos(\sqrt{6}\omega_0 t + \delta)$

Para φ:

$$\dot{\varphi} = \frac{l}{m r^2} = \frac{\sqrt{m\lambda}\, r_c^3}{m(r_c + \delta r)^2} \simeq \underbrace{\sqrt{\frac{\lambda}{m}}\, r_c}_{=\omega_0} - 2\sqrt{\frac{\lambda}{m}}\, \delta r$$

$$\dot{\varphi} = \omega_0 - 2\frac{\omega_0}{\varepsilon}\, \varepsilon \cos(\sqrt{6}\omega_0 t + \delta)$$

$$\varphi = \omega_0 t - \frac{2}{\sqrt{6}} \frac{\varepsilon}{r_c} \operatorname{sen}(\sqrt{6}\omega_0 t + \delta) + \varphi_0$$