The proof of (103) then follows easily from (31) by "induction from $(h-1)$ and h to $(h+1)$."

[31] It was my good friend Dr. Mary O'Brien of Lady Margaret Hall and the Department of Theoretical Physics, Oxford, who first called my attention to the existence of localized modes. See L. P. Howland, Amer. J. Phys. **33,** 269 (1965), for a description of a localized mode observable in a macroscopic experiment.

[32] A basic theorem of elementary number theory is used here; since it is equivalent to the uniqueness of the prime factorization of integers, any one having wide practical experience with integers ought to be willing to accept its validity without proof.

[33] This matter is discussed in some detail by Dean (Ref. 1), pp. 112–113. The "uncoupled oscillators" are of course the normal coordinates (11) [or (91)]; their frequencies are in a simple way related to the coefficients in the diagonalized Hamiltonian.

[34] Relevance of the oscillations of linear chains to molecular vibrations is discussed in various papers reprinted in *Mathematical Physics in One Dimension,* edited by E. H. Lieb and D. C. Mattis (Academic, New York, 1966), Chap. 2.

[35] Dr. O'Brien (Ref. 31) first made this observation.

[36] R. Weinstock, Amer. J. Phys. **38,** 1289 (1970).

[37] For an experimental arrangement to which the problem is applicable, see F. W. Sears, Amer. J. Phys. **37,** 645 (1969).

Conservation Laws for Gauge-Variant Lagrangians in Classical Mechanics

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When a physical system has some symmetry properties, it is described by equations of motion invariant under the corresponding transformation group. Its Lagrangian however need not be invariant and may be "gauge-variant," that is, vary by the addition of a total time derivative. A slightly generalized form of Noether's theorem nevertheless exists in such cases, still leading to conservation laws. The importance of considering such noninvariant Lagrangians and the associated conservation laws is illustrated by several examples: energy conservation, Galilean invariance, dynamical symmetries (harmonic oscillator and Kepler's problem), motion in a uniform electric field.

I. INTRODUCTION

Probably because of their utmost importance in modern physics, invariance principles and conservation laws nowadays hold a prominent place even in classical mechanics, and very rightly so. Most courses and textbooks contain a proof of Noether's theorem,[1] or at least implicitly use it to derive the great conservation laws from general invariance principles. However, some subtlety has to be exercised and, as examples will show below, an important point is missed in standard treatments. Indeed, when considering a physical system exhibiting some symmetry, one usually describes it by means of a Lagrangian invariant under the relevant transformation group. The invariance requirements are then shown to imply a conservation law. It is immediate to realize that this is a far too restricted framework. It suffices to think of an interacting system with time-translation invariance (for instance a single particle in a static external potential). As anybody knows, this implies a conservation law for the total energy $E=T+V$, where T and V are, respectively, the kinetic and potential energy. Accordingly, the Lagrangian $L=T-V$ *cannot* be invariant under time translations, except in trivial cases where kinetic and potential energies are separately conserved. In effect, energy conservation does not follow from a straightforward application of the standard Noether's theorem. Similar remarks can be made in the less elementary cases of the so-called "dynamical symmetries", exhibited by the Kepler's problem or the isotropic harmonic oscillator. The Hamiltonians of such systems in fact are invariant under specific invariance groups, which, in contrast to the rotation group for a spherically symmetric system for instance, do not leave separately invariant the kinetic and potential energies, but only their sum. As above, their difference, i.e., the Lagrangian, cannot be invariant, and the associated conservation laws cannot be derived from the standard Noether's theorem.

This apparent paradox is resolved easily by remembering that a symmetry of a physical system means invariance of its equations of motion, but not necessarily of its Lagrangian! Indeed, two Lagrangians are equivalent, that is lead to the same equations of motion, if they differ by a total time derivative; they lead to expressions for the action differing only by terms depending on the end-points and not on the path of integration. The variational problems thus are identical and so are the equations of motion. Taking into account the possibility of such a variance for Lagrangians describing invariant systems is essential here. Noether's theorem has an immediate generalization and a conservation law holds as well as for a strictly invariant Lagrangian. Of course, there is nothing original here in the statement and/or proof of this theorem; I only wish to stress its importance in rather elementary situations.

2. GENERALIZED NOETHER'S THEOREM

We consider a mechanical system with N degrees of freedom, described by generalized coordinates $q=\{q_1, q_2, \cdots, q_N\}$ and characterized by its Lagrangian $L(q, \dot{q}, t)$, with customary notations. Suppose that under some infinitesimal transformation of the coordinates, possibly velocity dependent

$$\delta q=\epsilon f(q, \dot{q}, t), \tag{1}$$

the Lagrangian varies by the total time derivative of a function of the coordinates

$$\delta L=\epsilon(d/dt)\Lambda(q, t). \tag{2}$$

It will be convenient to call such a transformation property a gauge-variation of the Lagrangian (see Sec. 6). The variation δL of the Lagrangian may be expressed in term of the coordinate variation (1) in the customary way

$$\delta L=(\partial L/\partial q)\partial q+(\partial L/\partial\dot{q})\delta\dot{q}, \tag{3}$$

which, using the Lagrange's equation of motion

$$(d/dt)(\partial L/\partial\dot{q})=\partial L/\partial q, \tag{4}$$

may be rewritten

$$\delta L=(d/dt)(\partial L/\partial\dot{q})\delta q+(\partial L/\partial\dot{q})(d/dt)(\delta q). \tag{5}$$

Finally,

$$\delta L=\epsilon(d/dt)[(\partial L/\partial\dot{q})f]. \tag{6}$$

Comparing with the hypothesis (2) on the gauge variance of the Lagrangian, we obtain a conservation law[1]

$$dF/dt=0 \tag{7}$$

for the quantity

$$F\equiv f(\partial L/\partial\dot{q})-\Lambda=\text{const.} \tag{8}$$

This differs from the standard result by the appearance of the second term Λ on the rhs, the physical importance of which we are going to illustrate in a few cases.

3. EXAMPLE 1: ENERGY CONSERVATION

If the Lagrangian L does not depend explicitly on the time t,

$$\partial L/\partial t=0, \tag{9}$$

then its variation under an infinitesimal time translation δt

$$\delta L=(dL/dt)\delta t \tag{10}$$

only follows from the variation of the coordinates

$$\delta q=\dot{q}\delta t. \tag{11}$$

The conditions of the theorem are fulfilled, with $f=\dot{q}$ and $\Lambda=L$, yielding a conservation law for

$$E=\dot{q}(\partial L/\partial\dot{q})-L, \tag{12}$$

the well-known expression for the energy. In the conventional treatments, conservation of energy appears as the result of a seemingly rather *ad hoc* manipulation of the Lagrangian which often baffles the student. Of course this manipulation is but a proof of the generalized Noether's theorem in the particular case considered. Still, it is pedagogically valuable to stay within a general framework, large enough to embody all the conservation laws of interest.

4. EXAMPLE 2: GALILEAN INVARIANCE

A collection of free particles with masses m_i $(i=1, 2\cdots n)$ has a Lagrangian

$$L=\sum_{i=1}^{n} \tfrac{1}{2}m_i\dot{\mathbf{q}}_i^2. \tag{13}$$

Under an infinitesimal Galilean transformation with velocity $\delta\mathbf{v}$, the coordinates vary according to

$$\delta\mathbf{q}_i=\delta\mathbf{v}\ t, \tag{14}$$

and the velocities obey the customary Galilean transformation law

$$\delta\dot{\mathbf{q}}_i=\delta\mathbf{v}. \tag{15}$$

The Lagrangian variation is

$$\delta L=\sum m_i\dot{\mathbf{q}}_i\cdot\delta\mathbf{v}=\delta\mathbf{v}\cdot(d/dt)(\sum m_i\mathbf{q}_i). \tag{16}$$

The generalized Noether's theorem then yields the conservation law for the "Galilean momentum"

$$\mathbf{G}=(\sum m_i\dot{\mathbf{q}}_i)t-(\sum m_i\mathbf{q}_i)=\text{const.} \tag{17}$$

Combined with the conservation law for the momentum

$$\mathbf{P}=\sum m_i\dot{\mathbf{q}}_i, \tag{18}$$

and the definition of the center of mass with position

$$\mathbf{R}=M^{-1}\sum m_i\mathbf{q}_i, \tag{19}$$

where M is the total mass

$$M=\sum m_i, \tag{20}$$

the conservation law (17) is seen to imply uniform motion of the center of mass

$$\mathbf{R}=M^{-1}(\mathbf{P}t-\mathbf{G}). \tag{21}$$

It may be recalled here that, provided the possibility of a gauge-variant Lagrangian is taken into account, Galilean invariance suffices to determine the form (13) for the Lagrangian of free particles. More precisely, all possible Galilean gauge variant Lagrangians are equivalent to the usual one (13), as may be shown by group theoretical arguments.[2] Let us illustrate this remark, for a single free particle, by considering the Lagrangian

$$L=\tfrac{1}{2}m(\mathbf{q}-\dot{\mathbf{q}}t)^2/t^2. \tag{22}$$

This seemingly exotic expression may be rewritten

$$L=\tfrac{1}{2}m\dot{\mathbf{q}}^2-(d/dt)[\tfrac{1}{2}m(\mathbf{q}^2/t)], \tag{23}$$

so that it only differs from the usual Lagrangian by a total time derivative and is equivalent to it. Now (22) is strictly invariant under a Galilean transformation (14, 15) and the conservation law (17) follows from the conventional Noether's theorem. However, under a space translation

$$\delta\mathbf{q}=\delta\mathbf{a}, \tag{24}$$

the Lagrangian variation is

$$\delta L=[(\mathbf{q}-\dot{\mathbf{q}}t)/t^2]\cdot\delta\mathbf{a}=\delta\mathbf{a}\cdot(d/dt)(-m\mathbf{q}/t), \tag{25}$$

so that the generalized Noether's theorem (8) is necessary here to prove conservation of the momentum

$$\mathbf{p}=-m(\mathbf{q}-\dot{\mathbf{q}}t)/t-(-m\mathbf{q}/t)=m\dot{\mathbf{q}}. \tag{26}$$

This admittedly rather artificial example is only here to point out at the necessity of keeping in mind the generalized form of Noether's theorem if one wants to make full use of the flexibility of a Lagrangian description by considering possible gauge variations.

5. EXAMPLE 3: DYNAMICAL SYMMETRIES

A. The Isotropic Harmonic Oscillator

Consider an N-dimensional isotropic oscillator, the Lagrangian of which may be written

$$L=\tfrac{1}{2}\sum_{i=1}^{N}\dot{q}_i^2-\tfrac{1}{2}\sum_{i=1}^{N}q_i^2, \qquad (27)$$

with a convenient choice of units. The equations of motion are

$$\ddot{q}_i+q_i=0 \qquad (i=1,2\cdots N). \qquad (28)$$

Under the infinitesimal variation of the coordinates

$$\delta q_j=\tfrac{1}{2}\epsilon(\dot{q}_k\delta_{jl}+\dot{q}_l\delta_{jk}) \qquad j=1,2\cdots N \;\; k,l \text{ fixed} \qquad (29)$$

the velocities, using (28), transform according to

$$\delta\dot{q}_j=-\tfrac{1}{2}\epsilon(q_k\delta_{jl}+q_l\delta_{jk}) \qquad j=1,2\cdots N \;\; k,l \text{ fixed}. \qquad (30)$$

The resulting variation of the Lagrangian,

$$\delta L=\sum\dot{q}_i\delta\dot{q}_i-\sum q_j\delta q_j, \qquad (31)$$

reads

$$\delta L=-\epsilon(\dot{q}_lq_k+q_l\dot{q}_k)=-\epsilon(d/dt)(q_lq_k) \qquad (32)$$

Applying our generalized Noether's theorem, we thus obtain the conservation laws

$$M_{kl}=\dot{q}_k\dot{q}_l+q_kq_l=\text{const} \qquad (k,l=1,2\cdots N). \qquad (33)$$

It is well known that these conservation laws correspond to invariance of the Hamiltonian under a unitary group.[3] We have shown here that this group does not leave the Lagrangian invariant, but causes it to undergo a gauge variation, which suffices to ensure invariance of the equations of motion and to yield the usual conservation laws.

B. The Kepler Problem

With suitable units, the Lagrangian reads

$$L=\tfrac{1}{2}\dot{\mathbf{q}}^2+q^{-1}, \qquad (34)$$

and the equations of motion are

$$\ddot{\mathbf{q}}+(\mathbf{q}/q^3)=0. \qquad (35)$$

Consider the following variation of the coordinates

$$\delta q_i=\epsilon(\dot{q}_iq_k-\tfrac{1}{2}q_i\dot{q}_k-\tfrac{1}{2}\mathbf{q}\cdot\dot{\mathbf{q}}\delta_{ik}) \qquad i=1,2,3 \;\; k \text{ fixed}, \qquad (36)$$

and the associated variation of the velocities

$$\delta\dot{q}_i=\tfrac{1}{2}\epsilon(\dot{q}_i\dot{q}_k-\dot{\mathbf{q}}^2\delta_{ik}-(q_iq_k/q^3)+(\delta_{ik}/q)] \qquad i=1,2,3 \;\; k \text{ fixed}. \qquad (37)$$

A simple calculation yields the variation of the Lagrangian

$$\delta L=\epsilon[(\dot{q}_k/q)-(\mathbf{q}\cdot\dot{\mathbf{q}}/q^3)q_k]=\epsilon(d/dt)(q_k/q). \qquad (38)$$

The theorem (8) thus applies here and furnishes a conservation law

$$A_k=\dot{\mathbf{q}}^2q_k-\mathbf{q}\cdot\dot{\mathbf{q}}\dot{q}_k-(q_k/q) \qquad (k=1,2,3). \qquad (39)$$

In other words, we have recovered the customary Lenz vector

$$\mathbf{A}\equiv\dot{\mathbf{q}}\times(\mathbf{q}\times\dot{\mathbf{q}})-(\mathbf{q}/q)=\text{const} \qquad (40)$$

associated with the hidden symmetry of the Hamiltonian under an orthogonal group in four dimensions,[3] and shown that this group is a gauge-variance group for the Lagrangian.

6. MOTION IN AN ELECTROMAGNETIC FIELD

A particle with mass m and electric charge e in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ is described by a Lagrangian

$$L=\tfrac{1}{2}m\dot{\mathbf{q}}^2+e\dot{\mathbf{q}}\cdot\mathbf{A}-eV, \qquad (41)$$

where $\mathbf{A}$ and V are the vector and scalar potentials of the fields, such that

$$\mathbf{E}=-(\partial\mathbf{A}/\partial t)-\text{grad}V \qquad (42a)$$

and

$$\mathbf{B}=\text{rot}\mathbf{A}. \qquad (42b)$$

As is well known these potentials are not uniquely defined, since a gauge transformation

$$\mathbf{A} \to \mathbf{A} + \operatorname{grad}\Lambda \quad (43a)$$

$$V \to V - (\partial\Lambda/\partial t) \quad (43b)$$

leaves the fields unchanged. Indeed the Lagrangian (41) transforms into an equivalent one

$$L \to L + e(d/dt)\Lambda. \quad (44)$$

Accordingly, the properties of a Lagrangian describing a particle in an electromagnetic field with some symmetry are not straightforward. More precisely, the invariances of the fields (that is, of the real physical situations) are not usually reflected as invariances of the potentials. The corresponding Lagrangian will then be gauge variant rather than invariant and the associated conservation laws will be derived by using the generalized Noether's theorem.

As the simplest example, let us consider a uniform and constant electric field **E**. This is a time and space translationally invariant situation. However, potentials do not exist which would have the same invariance properties, since they would obviously give zero fields! A time invariant scalar potential $V = -\mathbf{E}\cdot\mathbf{q}$ may be chosen. The associated Lagrangian

$$L_1 = \tfrac{1}{2}m\dot{\mathbf{q}}^2 + e\mathbf{E}\cdot\mathbf{q}, \quad (45)$$

however, is not invariant under space translations since

$$\delta\mathbf{q} = \delta\mathbf{a}$$

yields

$$\delta L_1 = e\mathbf{E}\cdot\delta\mathbf{a} = (d/dt)(e\mathbf{E}t)\cdot\delta\mathbf{a}. \quad (47)$$

Our theorem accordingly implies conservation of

$$m\dot{\mathbf{q}} - e\mathbf{E}t = \text{const}, \quad (48)$$

showing the uniformly accelerated nature of the motion. Alternatively one could describe the same field by a uniform vector potential $\mathbf{A} = -e\mathbf{E}t$. The corresponding Lagrangian

$$L_2 = \tfrac{1}{2}m\dot{\mathbf{q}}^2 - e\mathbf{E}\cdot\dot{\mathbf{q}}t, \quad (49)$$

is obviously equivalent to the first one (45) since

$$L_2 - L_1 = -e\mathbf{E}\cdot(\dot{\mathbf{q}}t + \mathbf{q}) = -e(d/dt)(\mathbf{E}\cdot\mathbf{q}t). \quad (50)$$

However L_2 is now invariant under space translations and we immediately obtain the conservation law (48), which is but the conservation law for the momentum $\mathbf{p} = m\dot{\mathbf{q}} - e\mathbf{A}$ in the present gauge. It is now the conservation law for the energy which requires using the generalized Noether's theorem, since a variation

$$\delta\mathbf{q} = \dot{\mathbf{q}}\delta t, \quad (51)$$

corresponds to the Lagrangian varying by

$$\delta L_2 = [dL_2/dt) - (\partial L_2/\partial t)]\delta t, \quad (52)$$

or still

$$\delta L_2 = [(dL_2/dt) + e\mathbf{E}\cdot\dot{\mathbf{q}}]\delta t = (d/dt)(L_2 + e\mathbf{E}\cdot\mathbf{q})\delta t. \quad (53)$$

We then obtain the expected conservation law

$$\tfrac{1}{2}m\dot{\mathbf{q}}^2 - e\mathbf{E}\cdot\mathbf{q} = \text{const}. \quad (54)$$

Further examples, in particular the interesting case of a uniform constant magnetic field with gauge-variance under rotations, and more general considerations may be found in a paper by Tassie and Buchdahl.[4]

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[1] See, for example, E. L. Hill, Rev. Mod. Phys. **23,** 253 (1951); C. Palmieri and B. Vitale, Nuovo Cimento **66A,** 299 (1970) and additional references therein.

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