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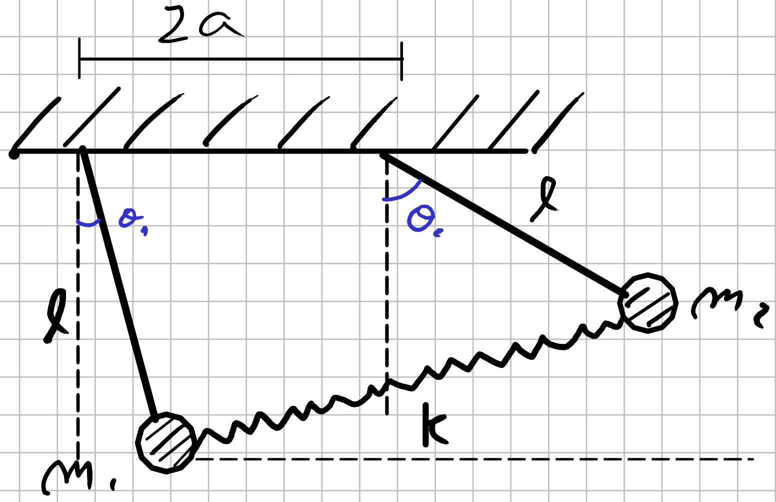


Coordenadas generalizadas:

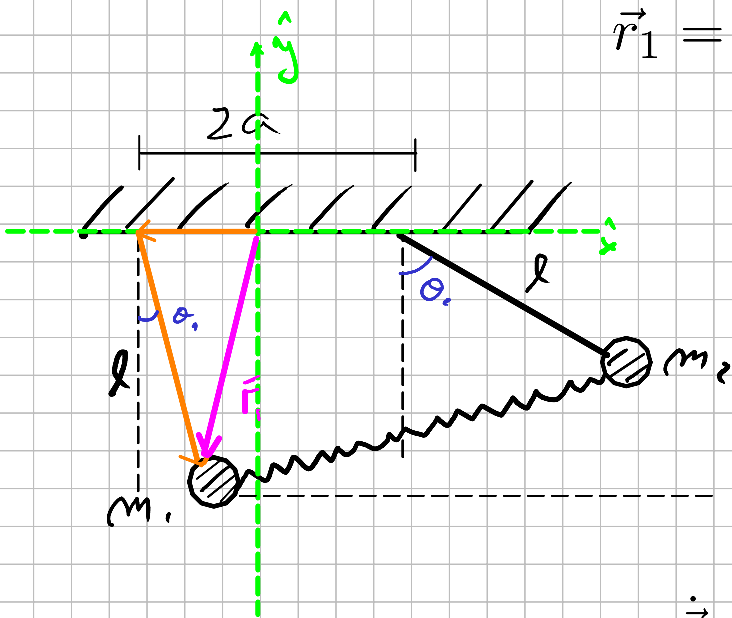
$$\{\theta_1, \theta_2\}$$

$$\vec{r}_1 = \vec{r}_1(\theta_1)$$
$$\vec{r}_2 = \vec{r}_2(\theta_2)$$

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}_1}^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}_2}^{\,2} - U(\vec{r}_1, \vec{r}_2)$$

Posicion particula 1:

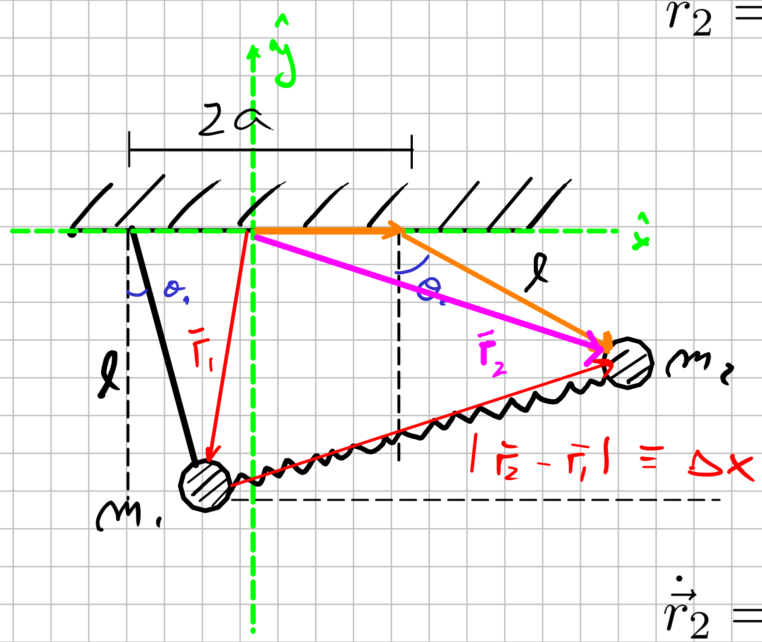
$$\vec{r}_1 = -a\hat{x} + l\sin\theta_1 \hat{x} - l\cos\theta_1 \hat{y}$$



$$\dot{\vec{r}_1} = l\dot{\theta}_1 \cos\theta_1 \hat{x} + l\dot{\theta}_1 \sin\theta_1 \hat{y}$$
$$\dot{\vec{r}_1}^{\,2} = l^2 \dot{\theta}_1^2$$

Posicion particula 2:

$$\vec{r}_2 = a\hat{x} + l\sin\theta_2\hat{x} - l\cos\theta_2\hat{y}$$



$$\dot{\vec{r}}_2 = l\dot{\theta}_2\cos\theta_2\hat{x} + l\dot{\theta}_2\sin\theta_2\hat{y}$$

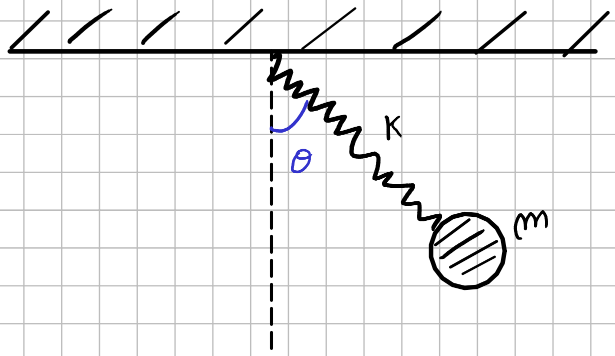
$$\dot{\vec{r}}_2^{\,2} = l^2\dot{\theta}_2^2$$

Escribimos el potencial:

$$U = -m_1 gl\cos\theta_1 - m_2 gl\cos\theta_2 + \frac{k}{2}|\vec{r}_1 - \vec{r}_2|^2$$

$$|\vec{r}_1{}^2 - \vec{r}_2{}^2|^2 = (2a + l(\sin\theta_1 - \sin\theta_2))^2 + l^2(\cos\theta_1 - \cos\theta_2)^2$$

$$\boxed{\begin{aligned}\mathcal{L} = &\frac{1}{2}l^2(m_1\dot{\theta}_1^2 + m_2\dot{\theta}_2^2) + m_1 g\cos\theta_1 + m_2 g\cos\theta_2 \\ &-\frac{kl^2}{2}[2l\cos(\theta_1 - \theta_2) + 4a(\sin\theta_1 - \sin\theta_2)] + ctes\end{aligned}}$$



$$\vec{r} = r\sin\theta\hat{x} - r\cos\theta\hat{y}$$

$$\dot{\vec{r}} = (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{x} - (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{y}$$

$$\dot{\vec{r}}^{\,2} = \dot{r}^2 + r^2\dot{\theta}^2$$

Escribimos el potencial:

$$U = mg(-r\cos\theta) + \frac{k}{2}|\vec{r}|^2 = -mgr\cos\theta + \frac{k}{2}r^2$$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{k}{2}r^2}$$

Para el ultimo:

15 $$\mathcal{L} = \frac{1}{2} m\dot{\vec r}^{\,2} + \frac{q}{c}\,\dot{\vec r}\cdot\vec A(t,\vec x) - q\,\phi(t,\vec x)$$

Coordenadas generalizadas: $\vec r = (x,y,z)$

$$\bar\nabla_{\dot r} = \left(\frac{\partial}{\partial \dot x}, \frac{\partial}{\partial \dot y}, \frac{\partial}{\partial \dot z}\right)$$

$$\frac{\partial\mathcal L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial\mathcal L}{\partial \dot x_i}\right) = 0 \longrightarrow \bar\nabla \mathcal L - \frac{d}{dt}\bar\nabla_{\dot r}\mathcal L = 0$$

$$\sum_i^3 \dot r_i \bar\nabla A_i$$

•) $\bar\nabla \mathcal L = \frac{q}{c}\,\underbrace{\bar\nabla(\dot{\bar r}\cdot\bar A)} - q\,\bar\nabla\phi$ $\qquad \frac{\partial\dot q}{\partial q} = 0$

$\sum r_i A_i = \dot r^i A_i$

$$\bar\nabla(\dot r^i A_i) = \bar\nabla \cancel{\dot r^i A_i} + \dot r^i\,\bar\nabla A_i = \dot r^i\,\bar\nabla A_i$$

$$= \sum_i \dot r^i(\partial_x A_i, \partial_y A_i, \partial_z A_i)$$

$$= (\dot x\partial_x A_x + \dot y \partial_x A_y + \dot z\partial_x A_z,\ \dot x\partial_y A_x + \dot y\partial_y A_y + \dot z\partial_y A_z,\ \dot x\partial_z A_x + \dot y\partial_z A_y + \dot z\partial_z A_z)$$

$$= (\dot x\partial_x A_x,\ \dot y\partial_y A_y,\ \dot z\partial_z A_z) +$$
$$+ (\dot y\partial_x A_y + \dot z\partial_x A_z,\ \dot x\partial_y A_x + \dot z\partial_y A_z,\ \dot x\partial_z A_x + \dot y\partial_z A_y)$$

Queremos llegar a $(\dot{\bar r}\cdot\bar\nabla)\cdot\bar A$

Ok: $(\dot x\partial_x A_x,\ \dot y\partial_y A_y,\ \dot z\partial_z A_z) = (\dot x\partial_x A_x + \underline{\dot y\partial_y A_x} + \underline{\dot z\partial_z A_x},$

$\dot x\partial_x A_y + \dot y\partial_y A_y + \dot z\partial_z A_y,\ \dot x\partial_x A_z + \dot y\partial_y A_z + \dot z\partial_z A_z)$

$- (\underline{\dot y\partial_y A_x} + \underline{\dot z\partial_z A_z},\ \dot x\partial_x A_y + \dot z\partial_z A_y,\ \dot x\partial_x A_z + \dot y\partial_y A_z) =$

$$= \underbrace{(\dot{\bar{r}}\cdot\bar{\nabla})}_{(\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z)}\cdot\bar{A} - (\dot{y}\partial_y A_x + \dot{z}\partial_z A_z,\ \dot{x}\partial_x A_y + \dot{z}\partial_z A_y,\ \dot{x}\partial_x A_z + \dot{y}\partial_y A_z)$$

$$(\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z)(A_x, A_y, A_z)$$

Por otro lado,

$$\bar{\nabla}\times\bar{A} = (\partial_y A_z - \partial_z A_y)\hat{x} + (\underline{\partial_z A_x - \partial_x A_z})\hat{y} + (\underline{\partial_x A_y - \partial_y A_x})\hat{z}$$

$$\dot{\bar{r}} = \dot{x}\hat{x} + \underline{\dot{y}\hat{y}} + \underline{\dot{z}\hat{z}}$$

$$\dot{\bar{r}}\times(\bar{\nabla}\times\bar{A}) = [\dot{y}\partial_x A_y - \dot{y}\partial_y A_x - \dot{z}\partial_z A_x + \dot{z}\partial_x A_z]\hat{x} +$$

$$+[\dot{z}\partial_y A_z - \dot{z}\partial_z A_y - \dot{x}\partial_x A_y + \dot{x}\partial_y A_x]\hat{y} +$$

$$+[\dot{x}\partial_z A_x - \dot{x}\partial_x A_z - \dot{y}\partial_y A_z + \dot{y}\partial_z A_y]\hat{z}$$

$$= (\dot{y}\partial_x A_y + \dot{z}\partial_x A_z,\ \dot{x}\partial_y A_x + \dot{z}\partial_y A_z,\ \dot{x}\partial_z A_x + \dot{y}\partial_z A_y)$$

$$-(\dot{y}\partial_y A_x + \dot{z}\partial_z A_x,\ \dot{z}\partial_z A_y + \dot{x}\partial_x A_y,\ \dot{x}\partial_x A_z + \dot{y}\partial_y A_z)$$

$$\Rightarrow (\dot{y}\partial_x A_y + \dot{z}\partial_x A_z,\ \dot{x}\partial_y A_x + \dot{z}\partial_y A_z,\ \dot{x}\partial_z A_x + \dot{y}\partial_z A_y) =$$

$$= \dot{\bar{r}}\times(\bar{\nabla}\times\bar{A}) + (\dot{y}\partial_y A_x + \dot{z}\partial_z A_x,\ \dot{z}\partial_z A_y + \dot{x}\partial_x A_y,\ \dot{x}\partial_x A_z + \dot{y}\partial_y A_z)$$

$$\bar{\nabla}(\dot{\bar{r}}\cdot\bar{A}) = (\dot{\bar{r}}\cdot\bar{\nabla})\cdot\bar{A} + \dot{\bar{r}}\times(\bar{\nabla}\times\bar{A})$$

$$\Rightarrow \bar{\nabla}\mathcal{L} = \frac{q}{c}\left[(\dot{\bar{r}}\cdot\bar{\nabla})\cdot\bar{A} + \dot{\bar{r}}\times(\bar{\nabla}\times\bar{A})\right] - q\,\bar{\nabla}\varphi$$

$$\bullet)\quad \bar\nabla_{\dot{\bar r}} \mathcal{L} = m.\dot{\bar r} + \frac{q}{c}\,\bar\nabla_{\dot{\bar r}}(\dot{\bar r}.\bar A) = m\dot{\bar r} + \frac{q}{c}\,\bar A$$

$$\frac{d}{dt}\left(\bar\nabla_{\dot{\bar r}}\mathcal{L}\right) = m\ddot{\bar r} + \frac{q}{c}.\frac{d}{dt}\bar A \qquad \left[\dot x_i\,\frac{\partial \bar A}{\partial x_c}\right] = (\dot x.\bar\nabla)$$

$$\underline{\text{ob}}: \ A = A(\bar x, t) \Rightarrow \frac{d}{dt}\bar A = \frac{\partial A}{\partial t} + \sum_i \underbrace{\frac{\partial \bar A}{\partial x_i}\,.\,\overbrace{\frac{dx_i}{dt}}^{=\dot x_i}}_{=(\dot x.\bar\nabla)\bar A}$$

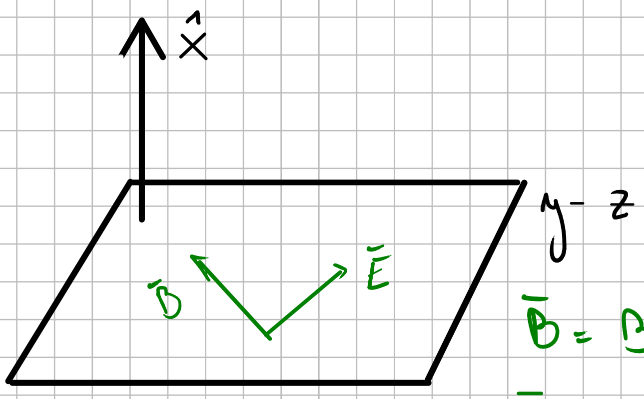
Juntando los dos terminos:

$$\Rightarrow m\ddot{\bar r} + \frac{\partial\bar A}{\partial t}.\frac{q}{c} + \frac{q}{c}\cancel{(\dot{\bar r}.\bar\nabla).\bar A} = \frac{q}{c}\cancel{(\dot{\bar r}.\bar\nabla)\bar A} + \dot{\bar r}\times(\bar\nabla\times\bar A) - q\bar\nabla\varphi$$

$$\Rightarrow m\ddot{\bar r} = q.\Big[\underbrace{-\nabla\varphi - \frac{\partial A}{\partial t}}_{=\bar E} + \frac{\dot{\bar r}}{c}\times\underbrace{(\bar\nabla\times\bar A)}_{=\bar B}\Big]$$

$$\Rightarrow \boxed{m.\ddot{\bar r} = q\left[\bar E + \frac{\dot{\bar r}}{c}\times\bar B\right]}$$

BONUS: Particula en campo E-M uniforme y constante



$$\bar B = B\hat z$$

$$\bar E = E_y\hat y + E_z\hat z$$

$$m\ddot x = \frac{q}{c}\dot y B_0$$

$$m\ddot y = qE_y - \frac{q}{c}\dot x B_0$$

$$m\ddot z = qE_z$$

$\hat{z}$: MRUV: $$\boxed{z(t) = \frac{qE_z}{2m}t^2 + v_{0z}t}$$

$\hat{x}$ - $\hat{y}$: $$m(\ddot{x} + i\,\ddot{y}) + i\,\frac{qB_0}{c}(\dot{x} + i\,\dot{y}) = i\,qE_y$$

$$\Omega = \dot{x} + i\dot{y} \Rightarrow \dot{\Omega} + i\,\frac{qB_0}{mc}\,\Omega = \frac{iq}{m}E_y$$

$$\Omega = A e^{i\omega t} + \frac{B_0}{E_y}c \quad , \quad \omega = \frac{qB_0}{mc}$$

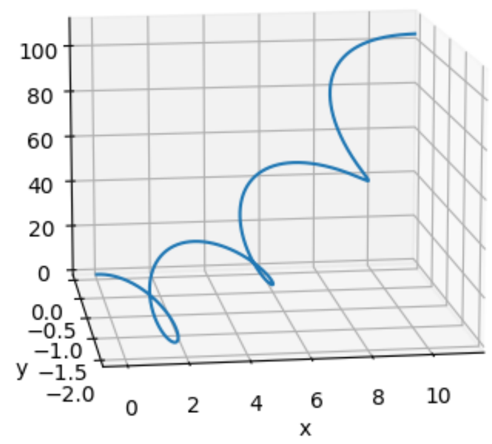
$$\Rightarrow \dot{x} = A\cos\omega t + \frac{B_0}{E_y}\cdot c \Rightarrow x(t) = \frac{A}{\omega}\sin\omega t + \frac{B_0}{E_y}ct + D_1$$

$$\dot{y} = -A\sin\omega t \Rightarrow y(t) = +\frac{A}{\omega}\cos\omega t + D_2$$

Elegimos $x(0) = y(0) = z(0) = 0$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= \frac{A}{w}\sin wt + \frac{E_y}{B_0}c\,t \\ y(t) &= \frac{A}{w}(\cos(wt) - 1) \end{aligned}}$$

$\omega \propto B_0$



Pueden encontrar mas info en el Landau II (teo clasica de campos), sec § 17

La propiedad de calculo vectorial que mostramos es la siguiente:

$$\bar{\nabla}(\bar{A}\cdot\bar{B}) = (\bar{A}\cdot\bar{\nabla})\bar{B} + (\bar{B}\cdot\bar{\nabla})\bar{A} + \bar{B}\times(\bar{\nabla}\times\bar{A}) + \bar{A}\times(\bar{\nabla}\times\bar{B})$$

donde usamos que

$$\bar{B} = \dot{\bar{r}} \Rightarrow \nabla\cdot\dot{\bar{r}} = \nabla\times\dot{\bar{r}} = 0 \quad \left(\frac{\partial \dot{q}}{\partial q} = 0\right)$$