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Ashley H. Carter

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A class of inverse problems in physics

Ashley H. Carter
Physics Department, Drew University, Madison, New Jersey 07940

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If physics students are exposed at all to inverse problems, it is usually in the context of specialized treatments of quantum scattering processes. Inverse problems are, however, important in a wide variety of applications, such as gravitational lensing, seismological exploration, and underwater acoustic tomography. Students can be introduced to inverse problems in undergraduate courses in mechanics, optics, and quantum mechanics. This paper presents some simple examples whose solution in each case involves Abel's integral equation, illustrating the ubiquity of this mathematical relation in diverse areas of physics.

I. INTRODUCTION

In one form or another, inverse problems have been a subject of study for a very long time. Such problems involve the inversion of data to obtain information about forces or characteristics of the physical medium. They are exemplified in a classic paper by Mark Kac entitled, ''Can one hear the shape of a drum?''[1] Specifically, what can be inferred about the boundary configuration of a membrane from a knowledge of the eigenvalues of the normal modes of vibration?

Inverse methods are powerful mathematical tools, currently employed in several areas of physics. For example, in astrophysics, gravitational lensing is an inverse technique that is used to detect the presence of dark matter in the universe.[2] The light from a distant star or galaxy is bent by gravity as it passes near an invisible body closer to the observer. Multiple images are formed from which the mass of the unseen object can be estimated. In seismological exploration, travel times of elastic waves produced by pulsed sources are recorded to determine the depth, thickness, and composition of geological layers of the earth.[3] As part of a study of global warming, oceanographers are measuring the long-term changes in underwater sound speeds associated with temperature variations.[4]

As an example of an inverse problem, consider the one-dimensional motion of a particle subject to a force that is a function of the x coordinate alone:

$$m\frac{d^2x}{dt^2}=F(x).$$

A single integration of this equation gives the law of conservation of energy,

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2+V(x)=E, \tag{1}$$

in which $V(x)$ is the potential energy and E the total energy. Since the kinetic energy is non-negative, the particle must be confined to regions of space where $V(x)\leqslant E$. In the example of Fig. 1, the motion can only occur in the regions $x_1<x<x_2$ or $x>x_3$. The points x_1, x_2, and x_3, at which $V(x)=E$, are the *turning points* of the motion, where the velocity is zero.

If the particle is initially in the region between the turning points x_1 and x_2, the subsequent motion is bounded and oscillatory. The period of the motion is found by solving for dx/dt in Eq. (1) and integrating with respect to x. The result is

$$\tau(E)=(2m)^{1/2}\int_{x_1}^{x_2}\frac{dx}{[E-V(x)]^{1/2}}. \tag{2}$$

In the direct problem, we assume that the potential energy function is known, in which case Eq. (2) can in principle be used to obtain the period. Our concern here, however, is in the inverse problem: Given the period as a function of the total energy, find the functional form of the potential energy. This problem is discussed by Landau and Lifshitz.[5] It involves the solution of Abel's integral equation, which was investigated more than a century ago.[6]

We are interested in showing how similar problems arise in several different areas of physics. One may ask, for example: Can the force of interaction between colliding particles be determined from a knowledge of the angular dependence of the impact parameter? Can the variation of the index of refraction of an optical medium be found from the ray paths? Or, in quantum mechanics, can the form of the potential energy function be deduced from a knowledge of the energy level spacings?

Section II gives the Landau and Lifshitz solution of the inverse oscillatory problem. Section III illustrates the inverse problem in classical scattering theory. Sections IV and V present problems in geometrical optics and quantum mechanics and show some interesting special cases.

II. THE INVERSE OSCILLATOR PROBLEM

Suppose that from Eq. (2) we wish to find $V(x)$, assuming that $\tau(E)$ is known. In the integrand we consider the coordinate x to be a function of V, replacing dx by $(dx/dV)dV$. We shall restrict ourselves to even functions $V(x)=V(-x)$, symmetrical about the V axis. Then Eq. (2) becomes

$$\tau(E)=2(2m)^{1/2}\int_0^E\frac{(dx/dV)dV}{(E-V)^{1/2}}. \tag{3}$$

For convenience, we have taken the origin at the point of minimum potential energy, which we assume to be zero; that is, $V(0)=0$.

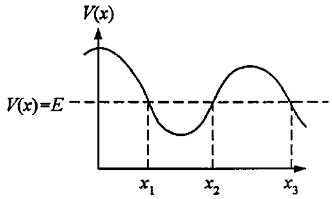

Fig. 1. A potential function with three turning points. Particle motion can occur in the regions $x_1 < x < x_2$ or $x > x_3$.

To solve this integral equation, multiply both sides by $(s-E)^{-1/2}$ and integrate with respect to E over the range $0 \leqslant E \leqslant s$. Here s is simply a parameter. The result is

$$\int_0^s \frac{\tau(E)dE}{(s-E)^{1/2}} = 2(2m)^{1/2} \int_0^s \frac{dE}{(s-E)^{1/2}} \int_0^E \frac{(dx/dV)dV}{(E-V)^{1/2}}$$

$$= 2(2m)^{1/2} \int_0^s \int_0^E \frac{(dx/dV)dVdE}{(s-E)^{1/2}(E-V)^{1/2}}. \quad (4)$$

Next we change the order of integration. From Fig. 2, we see that the double integral in Eq. (4) is over the triangle in the first quadrant above the line $V=E$ and extending from the origin to the horizontal line $E=s$. The V integral ranges from zero to the line $V=E$, indicated by the horizontal strip of width dE in the figure. Then the E integral sums over the horizontal strips from $E=0$ to $E=s$, covering the whole triangle.

Let us integrate with respect to E first. The E integral extends from the line $E=V$ to $E=s$, indicated by the vertical strip of width dV. The V integral sums over the vertical strips from $V=0$ to $V=s$. Making this change, we obtain

$$\int_0^s \frac{\tau(E)dE}{(s-E)^{1/2}} = 2(2m)^{1/2} \int_0^s \left(\frac{dx}{dV}\right) dV$$

$$\times \int_V^s \frac{dE}{(s-E)^{1/2}(E-V)^{1/2}}. \quad (5)$$

In the second integral on the right-hand side of Eq. (5), we introduce the new variable $w=(s-E)/(s-V)$. This integral is then found to be a beta function whose value is π:

$$\int_0^s \frac{dE}{(s-E)^{1/2}(E-V)^{1/2}} = \int_0^1 \frac{dw}{w^{1/2}(1-w)^{1/2}}$$

$$= B(\tfrac{1}{2}, \tfrac{1}{2}) = \pi.$$

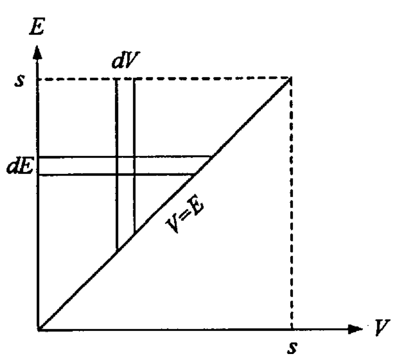

Fig. 2. Region of integration for Eq. (4).

The integration over V is trivial. When we write V in place of s, we obtain the final result

$$x(V) = \frac{1}{2\pi(2m)^{1/2}} \int_0^V \frac{\tau(E)dE}{(V-E)^{1/2}}. \quad (6)$$

As an example, suppose it is found that $\tau(E) \propto E^{-1/4}$. The integral in Eq. (6) in this instance is

$$x(V) \propto \int_0^V \frac{dE}{E^{1/4}(V-E)^{1/2}} = V^{1/4} \int_0^1 w^{-1/4}(1-w)^{-1/2} dw$$

$$= V^{1/4} B(\tfrac{3}{4}, \tfrac{1}{4}).$$

Thus $x \propto V^{1/4}$, or $V \propto |x|^4$, taking into account the symmetry requirement. This is the case of the classical anharmonic oscillator.

As mentioned in Sec. I, the oscillatory problem with solution Eq. (6) illustrates the generalized Abel equation,

$$f(s) = \int_a^s \frac{\phi(\xi)d\xi}{(s-\xi)^a} \quad (0<\alpha<1), \quad (7)$$

where $f(s)$ is a known function and $\phi(\xi)$ is to be determined. Upon inverting Eq. (7), we obtain

$$\phi(\xi) = \frac{\sin \pi\alpha}{\pi} \frac{d}{d\xi} \int_a^\xi \frac{f(s)ds}{(\xi-s)^{1-\alpha}}. \quad (8)$$

The factor $(\sin \pi\alpha)/\pi$ is just the reciprocal of the beta function $B(1-\alpha, \alpha)$. In the special case $\alpha=1/2$, Eq. (7) is known as Abel's integral equation.[7]

The conditions that Eq. (7) have the continuous solution Eq. (8) in the interval $a \leqslant s \leqslant b$ are given by Bôcher:[8] (1) $f(s)$ must be continuous in the interval; (2) $f(a)=0$; and (3), $\int_a^\xi f(s)(\xi-s)^{\alpha-1} ds$ must have a continuous derivative in the interval, except that $f(s)$ may possess a finite number of finite discontinuities in the interval.

The generalized Abel equation, Eq. (7), can also be solved by applying the Laplace convolution theorem.[9] The Abel equation appears in a wide variety of inverse problems, examples of which are given below.

III. CLASSICAL SCATTERING

Inverse methods are used extensively in scattering problems, in which the objective is to determine the force of interaction between the incident and scattered particles from the differential cross section or some related quantity. It is not our purpose to discuss the general analysis but rather to give an example encountered in classical mechanics—that of Rutherford scattering. The general problem has been discussed by Keller, Kay, and Shmoys.[10] The analysis presented here differs somewhat from theirs. The aim is to find the scattering potential, $V(r)$, as a function of the polar coordinate r.

In Rutherford scattering, an incident particle of mass m, charge q_1, and center-of-mass kinetic energy E is scattered by a heavy stationary particle of charge q_2 (see Fig. 3). Suppose that from analysis of experiments the impact parameter, b, is found to be

$$b = \frac{q_1 q_2}{8\pi\epsilon_0 E} \cot\left(\frac{\theta}{2}\right), \quad (9)$$

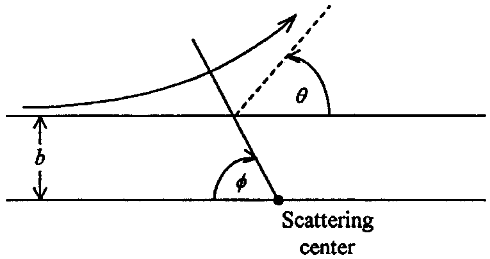

Fig. 3. The classical scattering problem showing the impact parameter b and the scattering angle θ.

where θ is the scattering angle.[11] The impact parameter is related to the angular momentum, L, and the energy, E, by

$$L = mvb = (2mE)^{1/2} b. \tag{10}$$

The scattering angle is given by

$$\theta = \pi - 2\phi, \tag{11}$$

where

$$\phi = \frac{L}{(2m)^{1/2}} \int_0^{u_{\max}} \frac{du}{\left[(E-V) - \dfrac{L^2u^2}{2m}\right]^{1/2}}. \tag{12}$$

Equations (10)–(12) are independent of the details of $V(r)$. Here $u = r^{-1}$, where r is the polar coordinate; $u_{\max}$ corresponds to $r_{\min}$, the distance of closest approach of the incident particle to the scattering center.

Our goal is to find $V(r)$ by using integral equations, given Eq. (12) and the preceding equations. If we set $U \equiv V + L^2u^2(2m)^{-1}$, Eq. (12) becomes

$$\phi = \frac{L}{(2m)^{1/2}} \int_0^{u_{\max}} \frac{du}{(E-U)^{1/2}}. \tag{13}$$

We change variables by writing $du = (du/dU)dU$, and note that $U(u_{\max}) = E$. Thus

$$\phi = \frac{L}{(2m)^{1/2}} \int_0^{E} \frac{(du/dU)dU}{(E-U)^{1/2}}. \tag{14}$$

This is Abel's integral equation, which can be solved using Eqs. (7) and (8) to give the equation of the potential function in the form

$$u(U) = \frac{(2m)^{1/2}}{\pi L} \int_0^{U} \frac{\phi(E)dE}{(U-E)^{1/2}}. \tag{15}$$

The function $\phi(E)$ is found by combining Eqs. (9)–(11):

$$\phi(E) = \frac{\pi}{2} - \cot^{-1}\left(\frac{LE^{1/2}}{\alpha}\right), \tag{16}$$

where

$$\alpha \equiv (2m)^{1/2}\left(\frac{q_1q_2}{8\pi\epsilon_0}\right).$$

Substituting Eq. (16) in Eq. (15) gives

$$\left[\frac{\pi L}{(2m)^{1/2}}\right] u = \int_0^U \frac{\phi(E)dE}{(U-E)^{1/2}} = I = I_1 - I_2, \tag{17}$$

where

$$I_1 = \frac{\pi}{2} \int_0^U \frac{dE}{(U-E)^{1/2}} \tag{18}$$

and

$$I_2 = \int_0^U \frac{\cot^{-1}(LE^{1/2}/\alpha)dE}{(U-E)^{1/2}}. \tag{19}$$

Integrating I_2 by parts, we get

$$I_2 = \pi U^{1/2} - \frac{L}{\alpha} \int_0^U \frac{(U-E)^{1/2}dE}{E^{1/2}(1+L^2E/\alpha^2)}.$$

Setting $y^2 = E/U$ and $d^2 = \alpha^2/L^2U$, we obtain

$$I = \frac{2\alpha}{L} \int_0^1 \frac{(1-y^2)^{1/2}dy}{d^2+y^2} = \frac{2\alpha}{L} \cdot \frac{\pi}{2}\left[\frac{(d^2+1)^{1/2}}{d} - 1\right]. \tag{20}$$

Then Eq. (17) gives

$$\left[\frac{\pi L}{(2m)^{1/2}}\right] u = \frac{\pi\alpha}{L}\left[\left(1 + \frac{L^2U}{\alpha}\right)^{1/2} - 1\right]. \tag{21}$$

Solving for U, we obtain

$$U = \frac{2\alpha u}{(2m)^{1/2}} + \frac{L^2u^2}{2m} = V(u) + \frac{L^2u^2}{2m}.$$

Recalling that $u = 1/r$, we have, finally,

$$V(r) = \frac{2\alpha}{(2m)^{1/2}r} = \frac{q_1q_2}{4\pi\epsilon_0 r}, \tag{22}$$

as expected for Rutherford scattering.

In some applications, the differential scattering cross section $D(\theta)$ may be more readily determined than the impact parameter $b(\theta)$. The function $\phi(E)$ in that case can be found from the relation

$$L^2 = 2mE \int_0^{\pi-2\phi} D(\theta)\sin\theta\, d\theta, \tag{23}$$

assuming that the integral can be evaluated as a function of ϕ.

IV. GEOMETRICAL OPTICS

An inverse problem in ray optics is the following: Given the trajectories of light (or sound) rays, find the variation of the refractive index of the medium n, with respect to a position coordinate. Consider a ray path in the x–y plane in a medium in which n varies continuously in the y direction only. Let θ be the ray angle at an arbitrary point (x,y), measured with respect to the x axis, and θ_0 be its value at the origin (see Fig. 4). We take $n(0) = 1$. Snell's law, valid for any point in the medium, is then

$$n(y)\cos\theta = \cos\theta_0. \tag{24}$$

Since $dx/dy = \cot\theta$, it follows that

$$x = \int \frac{\cos\theta_0\, dy}{[n^2(y) - \cos^2\theta_0]^{1/2}}. \tag{25}$$

It is convenient to introduce dimensionless variables $x' = ax$ and $y' = ay$ and write

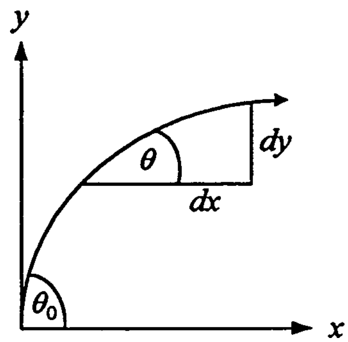

Fig. 4. Geometry of a ray path for which $n=n(y)$.

$$x'=\int\frac{\cos\theta_0\,dy'}{[n^2(y')-\cos^2\theta]^{1/2}}. \quad (26)$$

Consider optical media in which the ray with angle θ_0 at the origin has a turning point ($\theta=0$) at $y'=\eta$, where η is given by $n(\eta)=\cos\theta_0$, according to Eq. (24). If we define the ''half-range'' of such a ray by the relation

$$x'_{y=\eta}\equiv\rho(\theta_0), \quad (27)$$

then Eq. (26) yields

$$\rho(\theta_0)=\int_0^\eta\frac{\cos\theta_0\,dy'}{[n^2(y')-\cos^2\theta_0]^{1/2}}. \quad (28)$$

With the substitutions $n^2(y')=\xi$, $\cos^2\theta_0=s$, and $dy'=(dy'/d\xi)d\xi$, this equation becomes

$$\rho(\theta_0)=\pm\frac{1}{i}\int_1^s\frac{s^{1/2}(dy'/d\xi)d\xi}{(s-\xi)^{1/2}}. \quad (29)$$

If, further, we set $\rho(\theta_0)/s^{1/2}=f(s)$, and $\pm(1/i)(dy'/d\xi)=\phi(\xi)$, Eq. (29) assumes the form of Eq. (7) with $a=1$, $\alpha=\frac{1}{2}$. The solution, following Eq. (8), is

$$y'=\frac{1}{\pi}\int_1^\xi\frac{\rho(\theta_0)ds}{s^{1/2}(s-\xi)^{1/2}}. \quad (30)$$

Note that $\rho(\theta_0)$ is a function of s through the relation $\cos^2\theta_0=s$. Equation (30) is the result we seek. We solve for $y'(\xi)=y'(n^2)$, thence for $n(y')$.

The variation of n with y can also be found from the travel time along the ray to the turning point using the formula

$$\frac{dy'}{d\xi}=\frac{1}{\pi\xi}\frac{d}{d\xi}\int_1^\xi\frac{t'_\rho ds}{(s-\xi)^{1/2}}.$$

Here t' is the dimensionless time ac_0t, where c_0 is the optical velocity at the origin, and t'_ρ is its value at the half-range point $\rho(\theta_0)$. Like $\rho(\theta_0)$, t'_ρ is a function of s, since $s\equiv\cos^2\theta_0$.

An example is provided by parabolic ray paths, satisfying

$$x'=2\cos\theta_0[\sin\theta_0\pm(\sin^2\theta_0-y')^{1/2}]. \quad (31)$$

For this case, the half-range is

$$\rho(\theta_0)=2\cos\theta_0\sin\theta_0=2s^{1/2}(1-s)^{1/2}.$$

Then Eq. (30) gives

$$y'=\frac{2}{\pi}\int_1^\xi\frac{(1-s)^{1/2}ds}{(s-\xi)^{1/2}}=1-\xi=1-n^2(y').$$

Thus

$$n^2(y')=1-y',\ \text{or}\ n^2(y)=1-ay. \quad (32)$$

This example furnishes a nice illustration of the well-known analogy between geometrical optics and classical mechanics. Parabolic rays correspond to parabolic particle trajectories in projectile motion, with $a\to2g/v_0^2$, where v_0 is the initial velocity and g the acceleration due to gravity. Therefore,

$$n^2(y)=1-ay\to1-\left(\frac{2g}{v_0^2}\right)y=1-\frac{mgy}{\frac{1}{2}mv_0^2}=1-\frac{V}{E}.$$

A kind of Snell's law for two-dimensional classical-mechanical motion is

$$\left(1-\frac{V}{E}\right)^{1/2}\cos\theta=\cos\theta_0.$$

The inverse formulation for geometrical optics allows us to address an especially intriguing question: Is there a refraction index profile such that all rays, regardless of initial ray angle, have the same half-range and hence return to the x' axis at the same point? This is the case for which $\rho(\theta_0)$ is a constant, independent of θ_0, and which we might as well take to be unity. For this example, Eq. (30) is

$$y'=\frac{1}{\pi}\int_1^\xi\frac{ds}{s^{1/2}(s-\xi)^{1/2}}. \quad (33)$$

Here it is convenient to let $u^2=s$. Then

$$y'=\frac{2}{\pi}\int_1^{\xi^{1/2}}\frac{du}{(u^2-\xi)^{1/2}}=\frac{2}{\pi}\cosh^{-1}\left(\frac{u}{\xi^{1/2}}\right)\Bigg|_1^{\xi^{1/2}}$$
$$=-\frac{2}{\pi}\cosh^{-1}(\xi^{-1/2}).$$

Taking account of the fact that the hyperbolic cosine is an even function, and recalling that $\xi^{1/2}=n(y')$, we obtain the solution

$$n(y')=\text{sech}\left(\frac{\pi y'}{2}\right), \quad (34)$$

for the required variation of the index of refraction.

It turns out that the travel time in the horizontal direction is also the same for all rays. We can easily find an expression for the dimensionless travel time $t'=\alpha c_0t$ analogous to Eq. (26):

$$t'=\int\frac{n^2(y')dy'}{[n^2(y')-\cos^2\theta_0]^{1/2}}.$$

The travel time $\Delta t'$ to half-range is calculated by evaluating this integral between zero and η, where $n(\eta)=\cos\theta_0$. Using Eq. (34), we can show that $\Delta t'=1$, or $\Delta t=1/ac_0$, independent of the initial angle θ_0. Thus the hyperbolic secant profile gives rise to perfect focusing and zero time dispersion.

This very special case has been studied in connection with fiber optics. The theoretical optimum index of refraction profile for graded-index optical fibers that optimizes the meridional modes is $n(r)=n_0\,\text{sech}(ar)$, where r is the radial coordinate of a cylindrical fiber.[12]

V. QUANTUM MECHANICAL BOUND STATES

In quantum mechanics an inverse problem for bound states is as follows: Given a set of discrete energy eigenvalues, find the functional form of the potential energy, $V(x)$. The problem can be approached via the Wentzel–Kramers–Brillouin (WKB) method, also known as the semiclassical approximation. The approximation is valid when the potential is a slowly varying function of x, i.e., when

$$\left|\frac{dV(x)}{dx}\right| \ll \frac{1}{\hbar m}[2m(E-V(x))]^{3/2}.$$

Here m is the particle mass, $E-V$ is its kinetic energy, and $\hbar$ is Planck's constant divided by 2π.

The WKB quantization condition for a one-dimensional potential $V(x)$ is

$$\int_{x_1}^{x_2}[E-V(x)]^{1/2}\,dx=(n+\epsilon)\frac{\pi\hbar}{(2m)^{1/2}}, \quad n=0,1,2,..., \tag{35}$$

where ϵ is a fraction, and x_1 and x_2 are the classical turning points of the motion. If $V(x)$ is an even function $V(x)=V(-x)$, increasing monotonically for $x>0$, then $\epsilon=\frac{1}{2}$ and Eq. (35) can be written[13]

$$\Phi(E)\equiv\int_0^a[E-V(x)]^{1/2}dx=\left(n+\frac{1}{2}\right)\frac{\pi\hbar}{2(2m)^{1/2}}, \tag{36}$$

where a is given by $V(a)=E$. Differentiating Φ with respect to E, and changing the variable of integration from x to V, we get

$$\frac{d\Phi}{dE}=\frac{1}{2}\int_0^E\frac{(dx/dV)dV}{(E-V)^{1/2}}. \tag{37}$$

Here we tacitly assume that $V(0)=0$ by a suitable choice of energy scale. Equation (37) has the same form as Eq. (7), whose solution is Eq. (8). The latter yields the result

$$\frac{dx}{dV}=\frac{2}{\pi}\frac{d}{dV}\int_0^V\frac{(d\Phi/dE)dE}{(V-E)^{1/2}}.$$

Integration gives

$$x(V)=\frac{2}{\pi}\int_0^V\frac{(d\Phi/dE)dE}{(V-E)^{1/2}}. \tag{38}$$

Equation (36) is an implicit relationship between the discrete energy levels E_n and the corresponding quantum numbers $n=0,1,2...$. Thus the derivative $d\Phi/dE$ is more properly approximated by $\Delta\Phi/\Delta E$. We note, however, that the WKB approximation improves as n increases and $\Delta E/E$ gets smaller. In this limit the energy spectrum becomes quasi-continuous. Thus

$$\begin{aligned}\frac{d\Phi}{dE}&=\lim_{\Delta E\to 0}\left(\frac{\Delta\Phi}{\Delta E}\right)=\lim_{\Delta E\to 0}\left(\frac{\Delta\Phi}{\Delta n}\frac{\Delta n}{\Delta E}\right)\\&=\frac{\pi\hbar}{2(2m)^{1/2}}\lim_{\Delta E\to 0}\left(\frac{\Delta n}{\Delta E}\right)\\&=\frac{\pi\hbar}{2(2m)^{1/2}}\left(\frac{dE}{dn}\right)^{-1},\end{aligned} \tag{39}$$

Table I. Functional form of potential energy for $E_n \propto n^k$ in the WKB approximation in quantum mechanics.

n dependence of E_n	$V(x)$	Physical case
$n^{2/3}$	$\propto\lvert x\rvert$	Ramp
n	$\propto x^2$	Harmonic oscillator
$n^{4/3}$	$\propto x^4$	Anharmonic oscillator
n^2	$\cdots$	Infinite square well

where we regard dE/dn as a function of E. Substituting this expression in Eq. (38), we arrive at

$$x(V)=\frac{\hbar}{(2m)^{1/2}}\int_0^V\frac{dE}{(dE/dn)(V-E)^{1/2}}. \tag{40}$$

The example of the harmonic oscillator illustrates the method. For this case, $E_n=(n+1/2)\hbar\omega$ and $dE/dn=\hbar\omega$, so that

$$x(V)=\frac{\hbar}{(2m)^{1/2}}\frac{1}{\hbar\omega}\int_0^V\frac{dE}{(V-E)^{1/2}}=\frac{2}{\omega}\left(\frac{V}{2m}\right)^{1/2}.$$

Squaring and rearranging terms, we obtain $V(x)=\frac{1}{2}m\omega^2x^2$. Thus, knowledge of the energy level *spacing* (the dependence on n) is sufficient to determine the functional form of $V(x)$, at least under the conditions $V(x)=V(-x)$ and $V(0)=0$, and within the limits of applicability of the WKB approximation.

To generalize the preceding example, suppose that E is proportional to n^k. Then $dE/dn\propto n^{k-1}\propto E^{1-1/k}$. Substituting this in Eq. (40), we obtain

$$\begin{aligned}x(V)&\propto\int_0^V E^{1/k-1}(V-E)^{-1/2}dE\\&=V^{(2-k)/2k}\int_0^1 z^{1/k-1}(1-z)^{-1/2}dz\\&=V^{(2-k)/2k}B\left(\frac{1}{k},\frac{1}{2}\right).\end{aligned} \tag{41}$$

As in the oscillator problem, $B(1/k,1/2)$ is the beta function, valid for positive k. Hence Eq. (41) gives

$$V(x)\propto x^{2k/(2-k)}. \tag{42}$$

Some cases of physical interest are given in Table I. Note that for $E_n\propto n^{-2}$ (not listed), Eq. (42) gives the correct "quasi-Coulomb" potential $V(x)\propto 1/x$. (Although the beta function is not defined for $k=-2$, the integral nevertheless exists.) The tabulation shows how the energy level spacing depends on the "rate of climb" of the potential function.

In applying the WKB quantization condition to the above cases, we assume that x varies from $-\infty$ to $+\infty$. For the analysis of a spherically symmetric potential in which the coordinate r varies from 0 to ∞, the quantization condition is

$$\Phi(E)\equiv\int_0^{r_0}[E-V(r)]dr=\left(n+\frac{3}{4}\right)\frac{\pi\hbar}{(2m)^{1/2}}, \quad n=0,1,2,..., \tag{43}$$

where the turning point r_0 is given by $V(r_0)=E$.[14] Equation (43) is valid for s states (zero angular momentum). The method of solution is the same as before, with r replacing x.

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PARKING PRIVILEGES AT BERKELEY

The monetary value of the prize was initially very substantial, corresponding to about fifteen times the yearly salary of a distinguished professor. In 1959 it amounted to $21,184 for each of us, and my net annual salary at the time was about $13,000. Of course, the prize also provides many less tangible advantages: invitations, prestige among one's colleagues, the chance to be on various committees, numerous opportunities to serve as an ornamental plant, and even some minor monetary advantages. At Berkeley, in recent years, one is even given a private parking place on campus!

Emilio Segrè, *A Mind Always in Motion—The Autobiography of Emilio Segrè* (University of California Press, Berkeley, 1993), pp. 272.