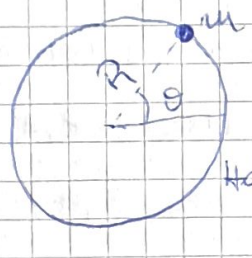
Vínculos / ligaduras

★ Caso 1: partícula libre

No hay ligaduras

Hay 3 grados de libertad: hay que especificar 3 variables (x, y, z) para dar la posición en un dado instante t.

★ Caso 2:

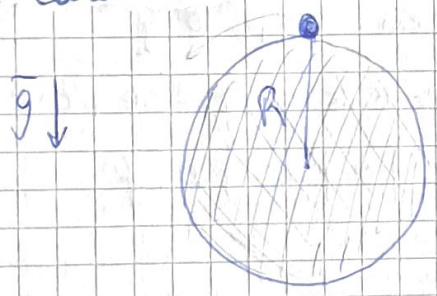


Hay 2 condiciones de vínculo:

Holónomas { 1) $z = 0$ (mov. bidimensional)
2) $r = R \equiv cte$

N° de grados de libertad: $3 - 2 = 1$ (me basta con θ)

★ Caso 3:



Vínculo: $r \geq R \rightarrow$ No holónomo

Nos interesan los vínculos holónomos: $f(\bar{r}_1, \bar{r}_2, \dots, \bar{r}_N, t) = 0$

En este caso,

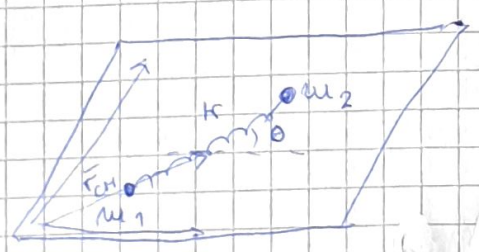
Si tenemos N partículas y k vínculos:

$m = N°\ g.l = 3.N - k = N°$ coord. generalizadas $(q_1, q_2, \dots, q_m)$

$$\bar{r}_i = \bar{r}_i(q_1, q_2, \dots, q_m, t) \quad i = 1, \dots, N$$

Guía 1

2. a)

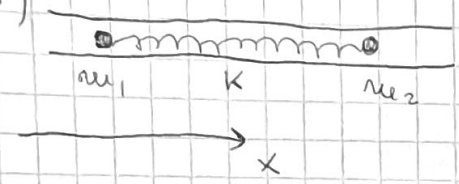


Vínculos: $z_1 = z_2 = 0$

$g.l = 3.2 - 2 = 4$

coord. generalizadas: $\{\bar{r}_{CM}, d_{12}, \theta\}$

(b)



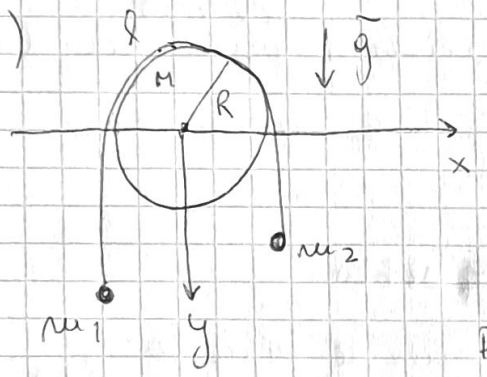
Vínculos $z_1 = z_2 = y_1 = y_2 = 0$

$g.l = 3.2 - 4 = 2$ Ej. coords gen: $\{x_1, x_2\}$

Otras coords. generalizadas: $\{x_{CM}, d_{12}\}$

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad d_{12} = |x_2 - x_1|$$

(c)



En total hay: 6 coords. para m_1 y m_2 + 3 coords. para el CM de la polea + 1 ángulo de giro alrededor de $\hat{z}$ = $\boxed{10}$

Pero hay vínculos: $z_1 = z_2 = z_p = 0$

(considerando el CM de la polea fijo): $x_{CM} = y_{CM} = 0$

$x_1 = -R$ y $x_2 = R$

•) $y_1 + \pi R + y_2 = l \Rightarrow g.l = 10 - 8 = 2$

Coords. gen. $\{y_1, \theta\}$

Si la soga no desliza: $\dot{y}_1 = R\dot{\theta} \Rightarrow dy_1 = R\,d\theta \Rightarrow y_1 = R\theta + cte$

$\Rightarrow$ 1 coord. gen $\{\theta\}$ ó $\{y_1\}$

NOTA

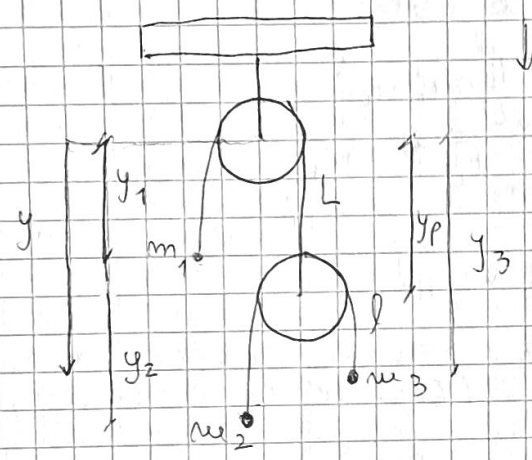
4. (b). Principio de D'Alembert:

$$\sum_{i=1}^{N}\left(\bar{F}_i^{(a)} - \dot{\bar{p}}_i\right)\cdot\delta\bar{r}_i = 0$$

Las fuerzas de vínculos no hacen trabajo virtual

$$\delta W_{virtual} = \sum_i \bar{F}_i^{(a)}\,\delta\bar{r}_i \qquad \delta W_{vínculo} = \sum_i \bar{f}_i\cdot\delta\bar{r}_i = 0$$

Desplazamiento virtual (compatible con los vínculos a tiempo t fijo)



Vínculos: $z_1 = z_2 = z_3 = 0$

$x_1 = cte$, $x_2 = cte$, $x_3 = cte$

$$y_1 + R_1\pi + y_p = L \rightarrow y_p = L - y_1 - R_1\pi$$

$$y_2 - y_p + R_2\pi + y_3 - y_p = l$$

$$\Rightarrow y_2 + y_3 - 2y_p = l - R_2\pi$$

$$\Rightarrow y_2 + y_3 - 2L + 2y_1 + 2R_1\pi = l - R_2\pi$$

$$y_2 + y_3 + 2y_1 = l - R_2\pi - 2R_1\pi + 2L$$

$\Rightarrow 3\cdot 3 - 7 = 2$ coords. generalizadas: ej.: $\{y_1, y_2\}$

Usemos el ppio de D'Alembert:

$$\bar{F}_1\cdot\delta\bar{r}_1 = m_1 g\,\hat{y}\cdot\delta y_1\,\hat{y} = m_1 g\,\delta y_1$$

$$\bar{F}_2\cdot\delta\bar{r}_2 = m_2 g\,\delta y_2$$

$$\bar{F}_3\cdot\delta\bar{r}_3 = m_3 g\,\delta y_3$$

$$\bar{p}_1 = m_1\dot{y}_1\,\hat{y} \Rightarrow \dot{\bar{p}}_1\,\delta\bar{r}_1 = m_1\ddot{y}_1\,\delta y_1$$

$$\dot{\bar{p}}_2\,\delta\bar{r}_2 = m_2\ddot{y}_2\,\delta y_2$$

$$\dot{\bar{p}}_3\,\delta\bar{r}_3 = m_3\ddot{y}_3\,\delta y_3$$

Usemos que $y_3 = \alpha - 2y_1 - y_2$ con $\alpha = 2L + l - 2R_1\pi - R_2\pi$

$$\ddot{y}_3 = -2\ddot{y}_1 - \ddot{y}_2 \qquad \delta y_3 = -2\delta y_1 - \delta y_2$$

$$\Rightarrow (m_1 g - m_1\ddot{y}_1)\,\delta y_1 + (m_2 g - m_2\ddot{y}_2)\,\delta y_2 +$$
$$+ (m_3 g + m_3\,2\ddot{y}_1 + m_3\ddot{y}_2)\cdot(-2\delta y_1 - \delta y_2) = 0$$

NOTA

Como son variaciones independientes

$$\begin{cases} m_1 g - m_1 \ddot{y}_1 - 2m_3 g - 4m_3 \ddot{y}_1 - 2m_3 \ddot{y}_2 = 0 & (1) \\ m_2 g - m_2 \ddot{y}_2 - m_3 g - m_3 2\ddot{y}_1 - m_3 \ddot{y}_2 = 0 & (2) \end{cases}$$

$$\left.\begin{array}{l} (1)\ \ddot{y}_1 (m_1 + 4m_3) + 2m_3 \ddot{y}_2 = g(m_1 - 2m_3) \\ (2)\ \ddot{y}_2 (m_2 + m_3) + 2m_3 \ddot{y}_1 = g(m_2 - m_3) \end{array}\right\} \text{obtengo } \ddot{y}_1 \text{ e } \ddot{y}_2$$

(c) Euler - Lagrange

A partir de D'Alembert: $\sum_{i=1}^{N} (\bar{F}_i^{(a)} - \dot{\bar{p}}_i) \cdot \delta \bar{r}_i = 0$

pasando a coords. generalizadas se obtiene:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad \text{con } T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$

$$j = 1, \dots, n \qquad Q_j = \sum_{i=1}^{N} \bar{F}_i^{(a)} \cdot \frac{\partial \bar{r}_i}{\partial q_j}$$

y además si $\bar{F}_i = -\bar{\nabla}_i V$ con $V \neq V(\dot{q})$

$$\Rightarrow \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j} \underbrace{(T-V)}_{L}\right] - \frac{\partial}{\partial q_j} \underbrace{(T-V)}_{L} = 0$$

son n ecuaciones (Ecs. de Euler-Lagrange)

En el problema:

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 + \frac{1}{2} m_3 \dot{y}_3^2$$

$$V = -m_1 g y_1 - m_2 g y_2 - m_3 g y_3$$

Nuestras coordenadas generalizadas son $\{y_1, y_2\}$ ($n = 2$ g.l)

Reemplazando $y_3 = \alpha - 2y_1 - y_2$, $\dot{y}_3 = -2\dot{y}_1 - \dot{y}_2$

$$L = T - V = \frac{1}{2}\left[m_1 \dot{y}_1^2 + m_2 \dot{y}_2^2 + m_3 (-2\dot{y}_1 - \dot{y}_2)^2\right] +$$

$$+ g\left[m_1 y_1 + m_2 y_2 + m_3 (\alpha - 2y_1 - y_2)\right]$$

Las 2 ecuaciones de E-L son:

$$y_1)\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) - \frac{\partial L}{\partial y_1} = 0$$

$$\frac{d}{dt}\left[m_1 \dot{y}_1 + 2m_3 (2\dot{y}_1 + \dot{y}_2)\right] - m_1 g + 2m_3 g = 0$$

NOTA

$$\Rightarrow \ddot{y}_1 (m_1 + 4m_3) + 2m_3 \ddot{y}_2 = g(m_1 - 2m_3) \quad (1)$$

$$y_2) \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_2}\right) - \frac{\partial L}{\partial y_2} = 0$$

$$\frac{d}{dt}\left[m_2 \dot{y}_2 + m_3 (2\dot{y}_1 + \dot{y}_2)\right] - m_2 g + m_3 g = 0$$

$$\ddot{y}_2 (m_2 + m_3) + \ddot{y}_1 \, 2m_3 = g(m_2 - m_3) \quad (2)$$

NOTA