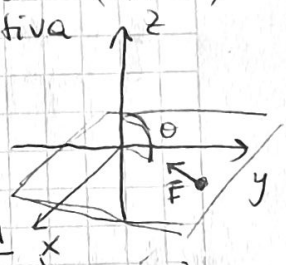
Guía 3

5. $V(r) = -\frac{K}{r^4} \Rightarrow \bar{F} = -\frac{\partial V(r)}{\partial r}\hat{r} = -\left(\frac{4K}{r^5}\right)\hat{r}$ Fuerza atractiva $(K>0)$



$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{K}{r^4}$$

φ) $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = \frac{d}{dt}\left(mr^2\dot{\varphi}\right) = 0 \Rightarrow \boxed{mr^2\dot{\varphi} = \ell \equiv cte}$

(ℓ momento ang.)

$$\Rightarrow \dot{\varphi} = \frac{\ell}{mr^2}$$

r) $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r}$

$$m\ddot{r} = mr\dot{\varphi}^2 - \frac{4K}{r^5} = \cancel{m}r\frac{\ell^2}{m^2r^4} - \frac{4K}{r^5} = \frac{\ell^2}{mr^3} - \frac{4K}{r^5}$$

$$\Rightarrow m\ddot{r} = \underbrace{\frac{\ell^2}{mr^3} - \frac{4K}{r^5}}_{-\frac{\partial V_{ef}(r)}{\partial r}} \Rightarrow V_{ef}(r) = -\int\left(\frac{\ell^2}{mr^3} - \frac{4K}{r^5}\right)dr =$$

$$= \frac{\ell^2}{2mr^2} - \frac{K}{r^4}$$

t) $\frac{\partial \mathcal{L}}{\partial t} = -\frac{dh}{dt} = 0$ En este caso $h = T + V = E$ porque $T(\lambda\dot{r}, \lambda\dot{\varphi}) = \lambda^2 T(\dot{r}, \dot{\varphi})$ y además $V \neq V(\dot{r}, \dot{\varphi})$

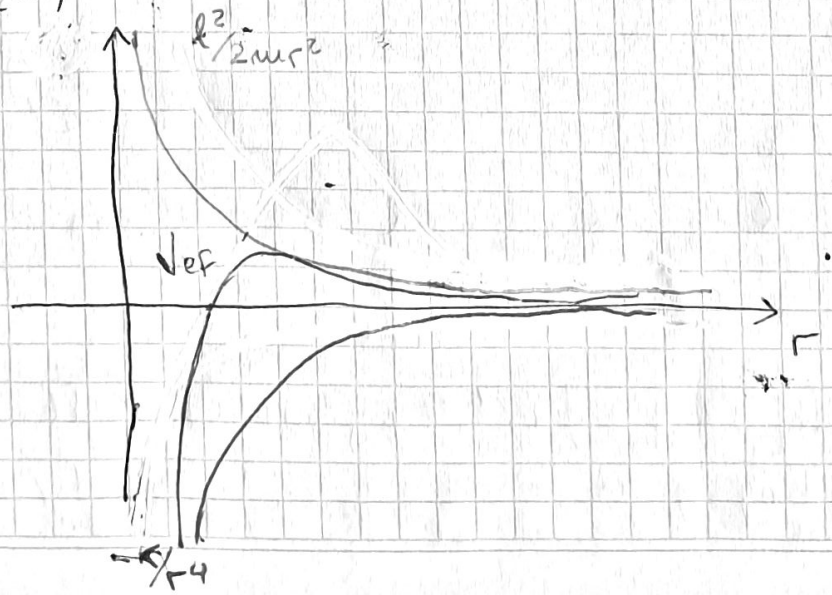
$$h = \underbrace{\frac{m}{2}\dot{r}^2}_{T(\dot{r})} + \underbrace{\frac{\ell^2}{2mr^2} - \frac{K}{r^4}}_{V_{ef}(r)} = E \equiv cte$$

Para $r \to 0$,

$V_{ef} \sim -\frac{K}{r^4} \to -\infty$

Para $r \to \infty$,

$V_{ef} \sim \frac{\ell^2}{2mr^2} \to 0$



NOTA

Órbita circular existe si $\exists\, r_c \,/\, \left.\frac{\partial V_{ef}}{\partial r}\right|_{r_c} = 0$

$$\left.\frac{\partial V_{ef}}{\partial r}\right|_{r_c} = \frac{-\ell^2}{m r_c^3} + \frac{4k}{r_c^5} = 0 \Rightarrow \frac{\ell^2}{m} r_c^2 = 4k$$

$$\Rightarrow \boxed{r_c = \frac{2}{\ell}\sqrt{km}}$$

Como $r_c \equiv cte \Rightarrow \dot{r} = 0 \Rightarrow E_c = V_{ef}(r) = \frac{\ell^2}{2mr_c^2} - \frac{k}{r_c^4}$

$$E_c = V_{ef}(r_c) = \frac{\ell^2 \ell^2}{2m\, 4km} - \frac{\cancel{k}\,\ell^4}{16\, k^{\cancel{2}} m^2} = \boxed{+\frac{\ell^4}{16\, km^2} = E_c}$$

$$\boxed{T = \frac{2\pi}{\dot{\varphi}} = \frac{2\pi m r_c^2}{\ell}}$$

Estabilidad de la órbita circular:

Es estable si $\left.\frac{\partial^2 V_{ef}}{\partial r^2}\right|_{r_c} > 0$

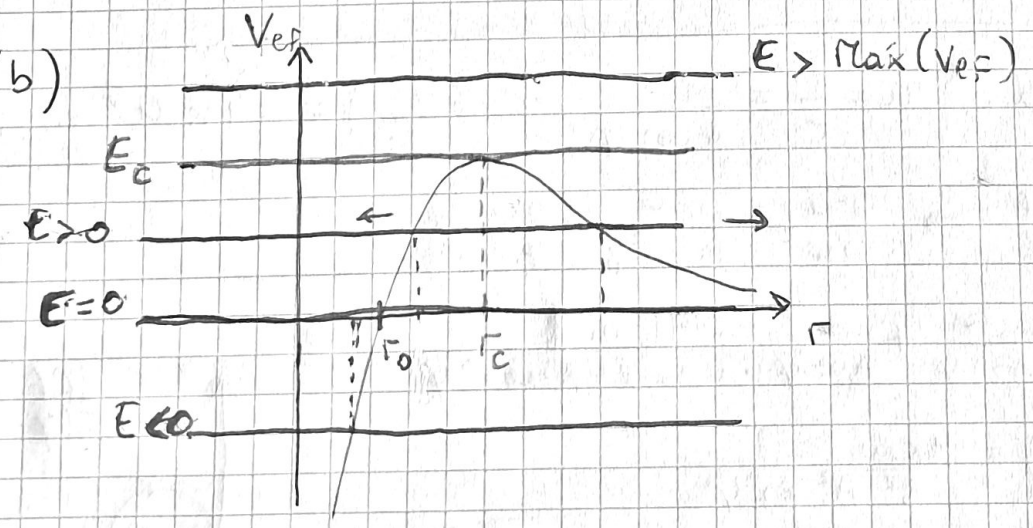
Es inestable si $\left.\frac{\partial^2 V_{ef}}{\partial r^2}\right|_{r_c} < 0$

$$\left.\frac{\partial^2 V_{ef}}{\partial r^2}\right|_{r_c} = +\frac{3\ell^2}{m r_c^4} - \frac{20k}{r_c^6} = +\frac{3\ell^2}{m\frac{16}{\ell^4}k^2m^2} - \frac{20\cancel{k}}{\frac{64}{\ell^6}k^{\cancel{3}\,2}m^3} =$$

$$= +\frac{3}{16}\frac{\ell^6}{k^2m^3} - \frac{5}{16}\frac{\ell^6}{k^2m^3} = \frac{-\ell^6}{8k^2m^3} < 0$$

$$\Rightarrow \boxed{\text{Es inestable}}$$

(b)



Hallemos $r_0 / V_{ef}(r_0) = 0$

$$\Rightarrow \frac{l^2}{2m r_0^2} = \frac{K}{r_0^4} \Rightarrow \frac{l^2}{2m} r_0^2 = K$$

$$\boxed{r_0 = \sqrt{2}\sqrt{\frac{Km}{l}} = \frac{r_c}{\sqrt{2}}} \quad l \neq 0$$

(c) Probemos que la ecuación de movimiento para r en un potencial central equivale a la ec. de Binet:

$$\ddot{r} = \frac{l^2}{m^2 r^3} + \frac{F(r)}{m} \quad y \quad \dot{\varphi} = \frac{l}{m r^2}$$

Cambio de variable $r = \frac{1}{u} \Rightarrow \dot{\varphi} = \frac{l}{m} u^2$

$$\dot{r} = \frac{d}{dt}(r) = \frac{d}{dt}\left(\frac{1}{u}\right) = \dot{\varphi}\frac{\partial}{\partial\varphi}\left(\frac{1}{u}\right) = \frac{l}{m} u^2 \left(-\frac{1}{u^2}\right)\frac{\partial u}{\partial \varphi} =$$

$$= -\frac{l}{m}\frac{\partial u}{\partial \varphi}$$

$$\ddot{r} = \frac{d}{dt}\left(-\frac{l}{m}\frac{\partial u}{\partial \varphi}\right) = \dot{\varphi}\frac{\partial}{\partial\varphi}\left(-\frac{l}{m}\frac{\partial u}{\partial \varphi}\right) = \frac{l}{m} u^2\left(-\frac{l}{m}\frac{\partial^2 u}{\partial \varphi^2}\right) =$$

$$= -\frac{l^2}{m^2} u^2 \frac{\partial^2 u}{\partial \varphi^2}$$

$$\Rightarrow -\frac{l^2}{m^2} u^2 \frac{\partial^2 u}{\partial \varphi^2} = \frac{l^2}{m^2} u^3 + \frac{F}{m}\left(\frac{1}{u}\right)$$

$$\frac{\partial^2 u}{\partial \varphi^2} = -u - \frac{m}{l^2 u^2} F\left(\frac{1}{u}\right)$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial \varphi^2} + u + \frac{m}{l^2 u^2} F\left(\frac{1}{u}\right) = 0} \quad \text{Ec. de Binet}$$

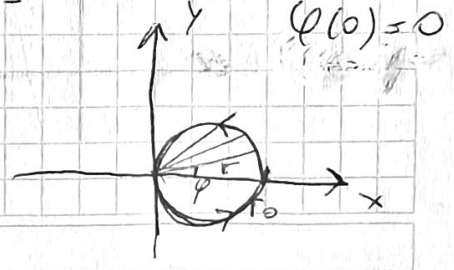
Reemplazando $u = \frac{1}{r} = \frac{1}{A\cos\varphi}$ y $F\left(\frac{1}{u}\right) = -4Ku^5$

$$\frac{\partial u}{\partial \varphi} = +\frac{\operatorname{sen}\varphi}{A\cos^2\varphi} \Rightarrow \frac{\partial^2 u}{\partial \varphi^2} = \frac{\cos\varphi}{A\cos^2\varphi} + \frac{2\operatorname{sen}^2\varphi}{A\cos^3\varphi}$$

$$\Rightarrow \frac{1}{A\cos\varphi} + \frac{2\operatorname{sen}^2\varphi}{A\cos^3\varphi} + \frac{1}{A\cos\varphi} - \frac{4Km}{l^2 A^3\cos^3\varphi} = 0$$

$$2 + 2\tan^2\varphi = \frac{4Km}{l^2 A^2\cos^2\varphi}$$

$$2A^3 = \frac{4Km}{l^2} \Rightarrow \boxed{A = \frac{\sqrt{2}\sqrt{Km}}{l} = r_0}$$



NOTA

$$\dot{r}(\varphi) = -A\dot{\varphi}\sin\varphi$$

$$\dot{r}(\varphi=0) = 0$$

$$\Rightarrow E \equiv cte = V_{ef}(\varphi=0) = \frac{\ell^2}{2m r_0^2} - \frac{k}{r_0^4} = 0 \Rightarrow \boxed{E \equiv 0}$$