Corchetes de Poisson

Sean $f = f(q_1, \dots, q_N, p_1, \dots, p_N, t)$ y $g = g(q_1, \dots, q_N, p_1, \dots, p_N, t)$ se define al corchete de Poisson de f y g como:

$$[f, g]_{q,p} := \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \cdot \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \qquad \{\cdot, \cdot\}$$

Corchetes fundamentales: $[q_i, q_j]_{q,p} = \sum_{k=1}^{N} \left(\frac{\partial q_i}{\partial q_k} \cdot \underbrace{\frac{\partial q_j}{\partial p_k}}_{0} - \underbrace{\frac{\partial q_i}{\partial p_k}}_{0} \cdot \frac{\partial q_j}{\partial q_k} \right) = 0$

$[p_i, p_j]_{q,p} = 0$

$$[q_i, p_j]_{q,p} = \sum_{k=1}^{N} \left(\frac{\partial q_i}{\partial q_k} \cdot \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \cdot \frac{\partial p_j}{\partial q_k} \right) = \delta_{ij}$$

Ecuaciones de Hamilton:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = [q_i, H]_{q,p} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} = [p_i, H]_{q,p}$$

Para una $f = f(q_1, \dots, q_N, p_1, \dots, p_N, t)$ arbitraria:

$$\dot{f} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \cdot \underbrace{\dot{q}_i}_{\frac{\partial H}{\partial p_i}} + \frac{\partial f}{\partial p_i} \cdot \underbrace{\dot{p}_i}_{-\frac{\partial H}{\partial q_i}} \right) + \frac{\partial f}{\partial t} = [f, H]_{q,p} + \frac{\partial f}{\partial t} \qquad \text{(ej. 12.(a))}$$

Si f no depende explícitamente de t: $\frac{\partial f}{\partial t} = 0$

$\Rightarrow \dot{f} = [f, H]_{q,p}$

Si además $[f, H]_{q,p} = 0 \Rightarrow f \equiv cte$ es cte de movimiento

<u>Resultado importante</u>: Si $(Q_i, P_i)_{i=1,\dots,N}$ son variables canónicas $\Rightarrow [\cdot, \cdot]_{q,p} = [\cdot, \cdot]_{Q,P}$

(los corchetes no dependen de la elección de variables canónicas conjugadas).

En particular: $[Q_i, Q_j] = [P_i, P_j] = 0$, $[Q_i, P_j] = \delta_{ij}$

Entonces, decimos que la transformación $(q_i, p_i) \to (Q_i, P_i)$ es canónica si $[Q_i, Q_j]_{q,p} = [P_i, P_j]_{q,p} = 0$ y $[Q_i, P_j]_{q,p} = \delta_{ij}$

Vamos a probar esto. Primero, vamos a necesitar demostrar algunas propiedades de las TC:

8. (a) $(q_i, p_i) \xrightarrow[TC]{} (Q_i, P_i)$

$$Q_i = Q_i(q_1, \dots, q_N, p_1, \dots p_N) \longrightarrow \dot{Q}_i = \sum_{j=1}^{N} \left(\frac{\partial Q_i}{\partial q_j} \cdot \dot{q}_j + \frac{\partial Q_i}{\partial p_j} \cdot \dot{p}_j \right) =$$

$$= \sum_{j=1}^{N} \frac{\partial Q_i}{\partial q_j} \cdot \frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j} \cdot \frac{\partial H}{\partial q_j} \Big) \quad (1)$$

Por otro lado, $\dot{Q}_i = \frac{\partial K}{\partial P_i} = \frac{\partial H}{\partial P_i}(q(Q_j, P_j), p(Q_j, P_j)) =$

si $F_i \neq F_i(t)$

$$= \sum_{k=1}^{N} \left(\frac{\partial H}{\partial q_k} \cdot \frac{\partial q_k}{\partial P_i} + \frac{\partial H}{\partial p_k} \cdot \frac{\partial p_k}{\partial P_i} \right) \quad (2)$$

Comparando (1) y (2): $\frac{\partial q_j}{\partial P_i} = -\frac{\partial Q_i}{\partial p_j}$ y $\frac{\partial p_j}{\partial P_i} = \frac{\partial Q_i}{\partial q_j}$

Usemos esto en:

$$[Q_i, P_j]_{q,p} = \sum_{k=1}^{N} \left(\underbrace{\frac{\partial Q_i}{\partial q_k}}_{\frac{\partial p_k}{\partial P_i}} \cdot \frac{\partial P_j}{\partial p_k} - \underbrace{\frac{\partial Q_i}{\partial p_k}}_{-\frac{\partial q_k}{\partial P_i}} \cdot \frac{\partial P_j}{\partial q_k} \right) = \frac{\partial P_j}{\partial P_i} = \delta_{ij}$$

Análogamente, ver: $[Q_i, Q_j]_{q,p} = [P_i, P_j]_{q,p} = 0$

9. Asumiendo que (Q, P) son variables canónicas, podemos demostrar que una transformación es canónica si se verifica: $[q_i, p_j]_{Q,P} = \delta_{ij}$, $[q_i, q_j]_{Q,P} = [p_i, p_j]_{Q,P} = 0$

$$[x, p_x] = \left[X \cos\lambda + P_y \frac{\sin\lambda}{m\omega}, -m\omega Y \sin\lambda, P_x \cos\lambda \right] =$$

Usamos props del ej. 10.

$$[\alpha f + \beta g, h] = \alpha [f, h] + \beta [g, h]$$

$$[f, g] = -[g, f]$$

$$[x, p_x] = -m\omega \sin\lambda \cos\lambda \overbrace{[x, y]}^{0} + \cos^2\lambda \overbrace{[x, p_x]}^{1} -$$

$$- \sin^2\lambda \underbrace{[p_y, y]}_{-1} + \cos\lambda \frac{\sin\lambda}{m\omega} \underbrace{[p_y, p_x]}_{0} = 1$$

$$[x, y] = \left[x\cos\lambda + p_y \frac{\sin\lambda}{m\omega},\ y\cos\lambda + \frac{p_x \sin\lambda}{m\omega}\right] =$$

$$= \cos^2\lambda \overbrace{[x, y]}^{0} + \cos\lambda \frac{\sin\lambda}{m\omega} \underbrace{[x, p_x]}_{1} + \sin\lambda \frac{\cos\lambda}{m\omega} \underbrace{[p_y, y]}_{-1} +$$

$$+ \frac{\sin^2\lambda}{m^2\omega^2} \underbrace{[p_y, p_x]}_{0} = 0$$

Transformaciones canónicas infinitesimales

Consideremos una TC: $F_2(\bar{q}, \bar{P}) = \underbrace{\sum_{i=1}^{N} q_i P_i}_{\text{transf. identidad}} + \epsilon\, G(\bar{q}, \bar{P})$ con $\epsilon \ll 1$

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j + \epsilon \frac{\partial G}{\partial q_j} \Rightarrow P_j = p_j - \epsilon \frac{\partial G}{\partial q_j} \Rightarrow \delta p_j = -\epsilon \frac{\partial G}{\partial q_j}$$

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \epsilon \frac{\partial G}{\partial P_j} = q_j + \epsilon \frac{\partial G}{\partial (p_j - \epsilon \frac{\partial G}{\partial q_j})} \simeq q_j + \underbrace{\epsilon \frac{\partial G}{\partial p_j}}$$

$$\Rightarrow \delta q_j = \epsilon \frac{\partial G}{\partial p_j}$$

$$\delta f = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \underbrace{\delta q_i}_{\epsilon \frac{\partial G}{\partial p_i}} + \frac{\partial f}{\partial p_i} \underbrace{\delta p_i}_{-\epsilon \frac{\partial G}{\partial q_i}} \right) = [f, G]\epsilon$$

Si $f = H \Rightarrow \delta H = \epsilon [H, G]$

Si G es una simetría de H: $[H, G] = 0$

$\Rightarrow G \equiv$ cte de movimiento (Teorema de Noether)

12. (b) $G = p_i \Rightarrow [H, G] = \sum_{j=1}^{N} \left(\frac{\partial H}{\partial q_j} \underbrace{\frac{\partial G}{\partial p_j}}_{\delta_{ij}} - \frac{\partial H}{\partial p_j} \underbrace{\frac{\partial G}{\partial q_j}}_{0} \right) = \frac{\partial H}{\partial q_i} = 0$ pues q_i es cíclico

$\Rightarrow G = p_i \equiv$ cte

14. $\mathcal{L} = \frac{m}{2}(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2) - V(\rho, z)$

$$p_\rho = \frac{\partial \mathcal{L}}{\partial \dot{\rho}} = m\dot{\rho}, \quad p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi}, \quad p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z}$$

$$H = T + V = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) + V(\rho, z)$$

φ cíclica $\Rightarrow p_\varphi \equiv$ cte $= L_z$

Tomemos $G = p_\varphi \Rightarrow [H, G] = \frac{\partial H}{\partial \rho} \overbrace{\frac{\partial G}{\partial p_\rho}}^{0} - \frac{\partial H}{\partial p_\rho} \overbrace{\frac{\partial G}{\partial \rho}}^{0} + \frac{\partial H}{\partial \varphi} \overbrace{\frac{\partial G}{\partial p_\varphi}}^{1} -$

$- \frac{\partial H}{\partial p_\varphi} \overbrace{\frac{\partial G}{\partial \varphi}}^{0} + \frac{\partial H}{\partial z} \overbrace{\frac{\partial G}{\partial p_z}}^{0} - \frac{\partial H}{\partial p_z} \overbrace{\frac{\partial G}{\partial z}}^{0} = 0 \Rightarrow [H, p_\varphi] = \text{cte} \Rightarrow p_\varphi$ cte de mov.

Potencial central: $\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) - V(r)$

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mr^2\sin^2\theta\,\dot{\varphi} \Rightarrow \dot{\varphi} = \frac{p_\varphi}{mr^2\sin^2\theta}$$

$$\Rightarrow H = T + V = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) + V(r)$$

$$\bar{L} = r\hat{r} \times m\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\varphi}\hat{\varphi}\right) =$$

$$= r^2\dot{\theta}\hat{\varphi}m + r^2 m\sin\theta\,\dot{\varphi}(-\hat{\theta}) = p_\theta\hat{\varphi} - \frac{p_\varphi}{\sin\theta}\hat{\theta} =$$

$$= p_\theta(-\sin\varphi, \cos\varphi, 0) - \frac{p_\varphi}{\sin\theta}(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) =$$

$$= \left(-p_\theta\sin\varphi - p_\varphi\frac{\cos\theta}{\sin\theta}\cos\varphi,\ p_\theta\cos\varphi - p_\varphi\frac{\cos\theta}{\sin\theta}\sin\varphi,\ p_\varphi\right)$$

$$G = -p_\theta\sin\varphi - p_\varphi\frac{\cos\theta}{\sin\theta}\cos\varphi = L_x \qquad G = L_y$$

$$[H, G] = \frac{\partial H}{\partial r}\cdot\underbrace{\frac{\partial G}{\partial p_r}}_{0} - \frac{\partial H}{\partial p_r}\cdot\underbrace{\frac{\partial G}{\partial r}}_{0} + \frac{\partial H}{\partial \theta}\cdot\frac{\partial G}{\partial p_\theta} - \frac{\partial H}{\partial p_\theta}\cdot\frac{\partial G}{\partial \theta} + \overbrace{\frac{\partial H}{\partial \varphi}}^{0}\cdot\frac{\partial G}{\partial p_\varphi} - \cancel{}$$

$$-\frac{\partial H}{\partial p_\varphi}\cdot\frac{\partial G}{\partial \varphi} = \overset{-}{+}\cancel{\frac{p_\varphi^2}{mr^2\sin^3\theta}\cos\theta\cos\varphi}\sin\varphi + \frac{p_\theta}{mr^2}\left(\cancel{-1} - \frac{\cos^2\theta}{\sin^2\theta}\right)p_\varphi\sin\varphi\cos\varphi +$$

$$\overset{+}{-}\frac{p_\varphi}{mr^2\sin^2\theta}\cdot\left(p_\theta\cos\varphi - p_\varphi\cancel{\frac{\cos\theta}{\sin\theta}\sin\varphi}\right) =$$

$$= -\frac{p_\theta p_\varphi}{mr^2\sin^2\theta}\sin\varphi\cos\varphi + \frac{p_\varphi p_\theta}{mr^2\sin^2\theta}\sin\varphi\cos\varphi = 0 \checkmark \Rightarrow L_x = cte$$

Si $G = p_\varphi = L_z \Rightarrow [H, p_\varphi] = \underbrace{\frac{\partial H}{\partial \varphi}}_{\varphi\ \text{cíclico}}\cdot\frac{\partial G}{\partial p_\varphi} = 0 \checkmark \Rightarrow L_z = cte$

$$\Rightarrow \bar{L} = cte$$