because Planck outlived Hertz by 53 years!

[38]In his *Scientific Autobiography* Planck refers to Kundt as: "...the temperamental director of the Physics Institute, universally liked for his genuinely kind human feelings." [Max Planck (Ref. 1), on p. 25].

[39]On this see Heilbron (Ref. 33), especially chapter 4, pp. 149–203.

[40]Max Planck, "Mein Besuch bei Adolf Hitler," Phys. Bl. 3, 143 (1947). In this very brief report, Planck writes that he was still able to repeat *verbatim* Hitler's words to him.

A simple, conservative understanding of many time-driven systems

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Examination of two alternative models of a bead sliding on a rotating circular wire reveals the meaning of a conserved Hamiltonian where it is not the total energy. Analogies to other such dual treatments are discussed.

I. INTRODUCTION

When first exposed to Lagrangian mechanics many students experience a sense of euphoria which translates into an ability to forge ahead with the formulation of mechanics problems which, just previously, had seemed frought with almost insurmountable difficulty. In the typical course the emergence of the Lagrangian from the instructor's bag of tricks is preceded by an $\mathbf{F}=m\mathbf{a}$ approach to "toughies," such as the double pendulum and a cylinder rolling on a cylinder rolling on a plane, which seriously tax students' geometric abilities.

Once past the honeymoon of the first set of hand-picked problems chosen to advertise the method, the student realizes that Lagrangian mechanics also has its nonintuitive aspects, now perhaps made more so by being buried under one more level of formalism. Finding forces of constraint via Lagrange multipliers, and the various caveats associated with time-dependent Lagrangians, velocity-dependent forces and cases in which $L \neq T-V$ were not designed by the Madison Avenue of Physics. (All of these terms and more are explained and exemplified in standard graduate mechanics texts such as those by Goldstein[1] and Corben and Stehle)[2]. Nevertheless, the use of a scalar function rather than vectors and (implicit) antisymmetric tensors such as angular momentum is universally accepted as worth almost any price. In the pedagogy of the subject it is therefore useful to have simple examples which showcase problems that arise in more complex "realistic" cases.

We wish to point out a new use, in this spirit, for a well-known example: a bead (i.e., point mass m) sliding on a smooth, vertical, circular wire rotating with angular velocity ω in a uniform gravitational field g. This delightful problem is an example of a case in which, although the transformation to generalized (in this case rotating) coordinates is time-dependent, the Lagrangian is not time-dependent, and hence the Hamiltonian is a constant of the motion. It is used to study linear stability around steady motion. It even contains a parametric bifurcation or "crossover" between two linearly stable solutions, allowing a simple example of the treatment of nonlinear stability. The details of this problem appear in many books on mechanics, notably the impressive one by Arnold,[3] and also the undergraduate text by Baierlein,[4] which also offers an $\mathbf{F}=m\mathbf{a}$ approach, and we will not repeat them here. One may now ask, "If the Hamiltonian is a constant of the motion, but not the total energy, what is it? After all, it does seem like an energy." The answer, given most completely by Landau and Lifshitz,[5] is that it is the energy in the rotating reference frame. Here we will construct a model with one more degree of freedom for the same physical situation for which the Hamiltonian is the total energy and compare the two models.

II. THE USUAL TREATMENT

The Lagrangian for the bead sliding on a circular wire is given by

$$L=(ma^2/2)(\dot{\theta}^2+\omega^2 \sin^2 \theta)+mga \cos \theta,$$

where m is the mass of the bead, a is the radius of the wire, ω is the angular velocity of the wire, and (constant) gravity (g) is in the $-z$ direction (see Fig. 1). Note that there is an implicit time dependence because the bead is moving with an azimuthal angular velocity $\dot{\phi}=\omega$, otherwise expressed as the time-dependent constraint $\phi=\omega t$ in a system with two degrees of freedom. For simplicity we de-dimensionalize this expression by measuring energy in units of mga and time in units of $(a/g)^{1/2}$: Let $L=mga\mathscr{L}$, $t=(a/g)^{1/2}\tau$. We make the new identification $\dot{x}\equiv dx/d\tau$ and also define $\omega=(g/a)^{1/2}\Omega$. We find

$$\mathscr{L}=(1/2)(\dot{\theta}^2+\Omega^2 \sin^2 \theta)+\cos \theta.$$

Using the usual definition, the conserved Hamiltonian is

$$\mathscr{H}_\omega=(1/2)(\dot{\theta}^2-\Omega^2 \sin^2 \theta)-\cos \theta.$$

Here the subscript on $\mathscr{H}$ identifies this expression as related to the constant frequency formulation. We have not made use of the obvious identification $p_\theta\equiv\dot{\theta}$ because there will be no further use of Hamiltonian formalism. The naive expectation would surely be that what appears to be the kinetic energy would consist of positive terms. We will return to this point after examining an alternative treatment.

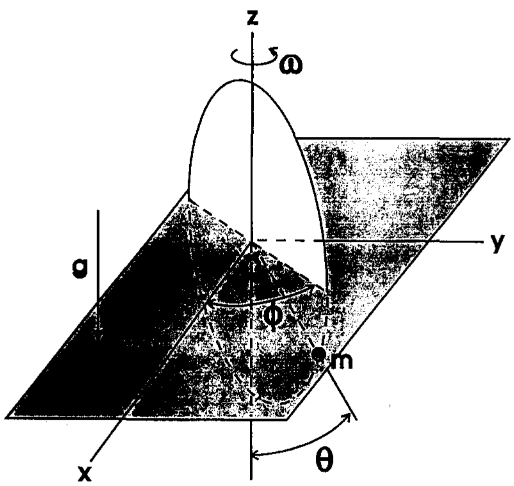

Fig. 1. The bead of mass m sliding on a smooth circular wire of radius a rotating about the vertical (z) axis, which is a diameter of the wire. The horizontal (x,y) plane is shaded. The plane of the wire is unshaded where it is above the (x,y) plane. All significant lines below the horizontal plane are dashed. Spherical coordinates (θ,ϕ) are used to describe the position of the bead. In the usual treatment, $\phi=\omega t$.

III. THE ALTERNATIVE TREATMENT

In our "strictly conservative" model we let the wire have a mass M, with $M \gg m$. The Lagrangian for this model is given by

$$L=(1/2)I\dot{\phi}^2+(ma^2/2)(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)+mga\cos\theta,$$

where $I=Ma^2/2$ is the moment of inertia of the wire. Introducing the same changes in notation as in the previous model, with the additional definition, $\mathscr{M}\equiv M/2m$, which is a measure of the ratio of the inertias of the wire and the bead, we find the following Lagrangian:

$$\mathscr{L}=(1/2)[\dot{\theta}^2+(\mathscr{M}+\sin^2\theta)\dot{\phi}^2]+\cos\theta.$$

This model has two degrees of freedom, but use of conservation of angular momentum about the axis of rotation, which is the physical way of saying that ϕ is a cyclic variable, and is expressed dimensionlessly as $l=(\mathscr{M}+\sin^2\theta)\dot{\phi}$, results in the following Hamiltonian:

$$\mathscr{H}_c=(1/2)[\dot{\theta}^2+l^2/(\mathscr{M}+\sin^2\theta)]-\cos\theta,$$

where $\mathscr{H}_c$ refers to the conservative formulation. By naive intuition this formulation does all of the "right things." In particular the kinetic energy consists of positive terms. This is to be expected, because the coordinate system used is not rotating.

IV. COMPARISON OF THE TWO FORMULATIONS

Of course the two formulations are not entirely equivalent, but they become ever more so as $\mathscr{M}$ becomes ever larger. To see this we expand the second term in the kinetic energy in powers of $\mathscr{M}^{-1}$ with the result

$$\mathscr{H}_c=(1/2)[\dot{\theta}^2+l^2/\mathscr{M}-(l/\mathscr{M})^2\sin^2\theta+\text{h.o.t.}]$$
$$-\cos\theta.$$

Our expression for angular momentum makes it clear that $l/\mathscr{M}$ is the angular velocity of the wire, which maintains a constant value in the limit that $\mathscr{M}$ becomes very large.[6] We identify $l/\mathscr{M}\equiv\Omega$, the angular velocity in the first formulation, and drop all terms which go to zero as inverse powers of $\mathscr{M}$. The result is

$$\mathscr{H}_c=l\Omega/2+(1/2)(\dot{\theta}^2-\Omega^2\sin^2\theta)-\cos\theta.$$

The final identification $\mathscr{H}_\omega=\mathscr{H}_c-l\Omega/2$ establishes the equivalence of the two formulations. The difference between the two Hamiltonians is seen to be just the *enormous* kinetic energy of rotation of the wire, a suitably intuitive result. The occurrence of the negative term in $\mathscr{H}_c$ plays the same role here as the term $-(1/2)\Omega^2\sin^2\theta$ plays in $\mathscr{H}_\omega$, in accord with the treatment of Ref. 5. It is also interesting to note that no qualitative changes occur in the motion even if the mass of the wire is not very large!

V. DISCUSSION

There are many similar cases in which a dynamical constraint can be replaced to a very good approximation by an extra degree of freedom along with an extra conservation law. "Very good approximation" usually means that some property of the extra degree of freedom is exceptional in some way, just as the mass of the wire is very large in our calculation, so that the backreaction on the extra degree of freedom is not apparent. In most of these cases the added conservation law is conservation of energy, particularly when "the usual treatment" is time-dependent, whereas in our problem the added conservation law is conservation of angular momentum, because the Hamiltonian is conserved in both alternative models. Some examples are:

(1) The behavior of a dielectric moving in and out of a capacitor whose plates are maintained at constant potential difference by a generator which can be modeled as an extra degree of freedom with a very large charge.[7]

(2) A corresponding magnetic problem in which a small magnet rotates in the field of a large electromagnet. A Lagrangian treatment of this system treating the source of the magnetic field as an extra degree of freedom was carried out long ago by Broer.[8] (3) The semiclassical theory of the interaction of radiation with an atom, as treated in every first year graduate text on quantum mechanics, involves a similar approximation. The radiation field is thought of as a single field mode which can interact resonantly with the atom. It has many quanta so that it can be treated as a classical field with minimum disturbance to itself.

(4) In the same spirit an analog pair of treatments exists for the study of the kind of simple dynamical system that a physicist is likely to consider: A periodically driven nonlinear system with one degree of freedom can be modeled with accuracy by a conservative system with a heavy mass oscillator replacing the driver. In effect one replaces an equal time map by a Poincaré section. These treatments are similar in the sense that they both are capable of deterministic chaos. Jargon has it that both of these have "one-and-a-half degrees of freedom."[9] However, chaotic systems are sufficiently sensitive so that details of the attractor and basin of attraction structures may show small qualitative changes in a comparison of the two models.

The reader unfamiliar with these examples is encouraged to look up the references given and consider each one from the point of view of our present treatment.

It is useful to point out the mechanical origin and the similarities of a set of current problems of this diversity when teaching "passé" classical mechanics.

[1]H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison–Wesley, Reading, MA, 1980).
[2]H. C. Corben and P. Stehle, *Classical Mechanics* (Wiley, New York, 1960; 2nd edition, reprinted by Dover, New York, 1994).
[3]V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1978, or 1989), pp. 87 and 88.
[4]R. Baierlein, *Newtonian Dynamics* (McGraw–Hill, New York, 1983), pp. 131–136.
[5]L. D. Landau and E. M. Lifshitz, *Mechanics, Volume 1 of Course of Theoretical Physics*, 3rd ed. (Pergamon, Oxford, 1976).
[6]Although paradoxically this expression may appear to depend on m because $\mathscr{M}$ depends on m, one must remember that the dimensionless angular momentum l also depends upon m, and the dependences cancel. This is an example of the loss of intuition that may occur when using dimensionless units, partially negating the simplicity they afford.
[7]J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory*, 3rd ed. (Addison–Wesley, Reading, MA, 1979), pp. 125–128.
[8]L. J. F. Broer, "On the statistical mechanics in a magnetic field," Physica **XII**, 49–60 (1946).
[9]G. M. Zaslavsky, R. Z. Sagdeev, D. A. Usikov, and A. A. Chernikov, *Weak Chaos and Quasiregular Patterns* (Cambridge University, Cambridge, 1991); in particular, see Chap. 5.

The circular disk parallel plate capacitor

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One of the more familiar systems in electrostatics is the parallel plate capacitor (PPC). While this system has received considerable attention in the close plate approximation, little is known about the exact solution for arbitrary plate separations. Although the solution was first given, in cylindrical coordinates by Sneddon, it was part of a more general treatise on mixed boundary value problems and appears to be unknown to much of the physics community. We present here a dedicated derivation of the solution to the boundary value problem for parallel disks, in cylindrical coordinates. The resulting expressions for potential and capacitance are in closed form, but depend on a function $f(t)$ which is determined from an integral equation of the Fredholm type, known as Love's equation. By adopting an orthogonal series approach to the solution of Love's equation, we have calculated the capacitance for a number of plate separation to plate radius ratios. A quantitative measure of the close plate approximation is then presented by comparing these values to those one would obtain using the elementary capacitance equation $\epsilon_0 A/d$.

I. INTRODUCTION

In beginning studies of electrostatics, the parallel plate capacitor (PPC) receives as much attention as any topic covered. With one simplifying condition, "plate separation small compared to plate area," the system is reduced to elementary status and becomes an ideal vehicle for exploring a variety of concepts: symmetry, Gauss's law, field energy, and of course capacitance, to name a few. One topic which does not get much attention, however, is the capacitance of a PPC with arbitrary, but finite, plate separation and area. Although textbook authors will qualitatively discuss "edge effects," we have not seen any attempt to quantify the "small separation to plate area" phrase nor have we seen any reference to the existence of exact solutions. In fact this system was solved (for circular parallel disks) by two rather different approaches. The first solution was obtained by Nicholson,[1] using oblate spheroidal coordinates and later embellished by Love.[2] A second solution was given by Snedden,[3] using cylindrical coordinates.

The Nicholson/Love approach is somewhat complicated because it uses dual oblate coordinate systems. The solution requires Legendre functions of both types and the symmetry is a bit forced as the circular plates are represented by flat spheroids. The Snedden approach has its own complications arising from the finite size of the circular plates. In cylindrical coordinates the plates are not closed coordinate surfaces, as in the oblate approach, and mixed boundary conditions are required. The real difficulty with Snedden's solution, however, is that solving the PPC is not the main focus of his work. Sneddon's primary interest is in solving dual integral equations. His formalism, developed for maximum generality, involves a maze of integral operators and abstract notation, which is then applied to the special case of the PPC.

Finally, we note that both approaches described above arrive at the same result. The capacitance and also the potential are derivable from a function $f(t)$, which is determined (using numerical techniques) from an integral equation of the Fredholm type, known as Love's equation. Much of the recent interest in this problem begins with Love's equation.[4]

Our objectives in this paper are first to solve the PPC in cylindrical coordinates, the most natural coordinate system for this system of circular parallel plates. This will assure that our approach is equivalent to Snedden's. Unlike Sned-

