Noether's theorem in discrete classical mechanics

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produce (when the square modulus is perfomed) a factor proportional to t, as in Eq. (15); the major contribution (the only one to be considered if $t \to \infty$) comes from the integrals containing only the nontransformable function $\eta(t)$, while the transformable one, $g(t)$, describing the details of the perturbation rise but not the essential feature (to go to zero as $t \to -\infty$ and to 1 as $t \to \infty$), turns out to be unessential.

As a concluding remark, I would like to observe that the subject treated in this brief article is rarely exposed in a satisfactory way in textbooks and (by experience) in university courses (in spite of its importance); I think therefore that some words can be usefully spent in order to give an explanation whose aim is essentially a didactic one.

[1]L. I. Shiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955), 2nd ed., Chap. VIII.

[2]E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970), 2nd ed., Chap. 18.

[3]A. Messiah, *Mécanique Quantique* (Dunod, Paris, 1964), Chap. XVII.

[4]L. Landau and E. Lifchitz, *Mécanique Quantique* (M. I. R, Moscow, 1966), Chap. VI.

[5]J. J. Sakurai, *Advanced Quantum Mechanics* (Addison–Wesley, Reading, MA, 1967), Chap. 2.

[6]W. Heitler, *The Quantum Theory of Radiation* (Dover, New York, 1984), Chap. IV, Sec. 14.

[7]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), Chap. 6.

[8]L. I. Shiff, *Quantum Mechanics* (McGraw-Hill, Auckland, 1968),3rd ed., Chap. VIII.

[9]Reference 6, Chap. II, Sec. 8.

[10]I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. 1, Chap I.

[11]M. J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions* (Cambridge U.P., Cambridge, 1964), Chap. 3.

[12]Observe that, following, e.g., the approach of Ref. 9, Eq. (5) can be generalized at once as $\zeta(x) = \lim_{t \to \infty} [1 - q(x)\exp(ixt)]/x$, where $q(x)$ is a regular, continuous function with $q(0) = 1$; in our case, $q(x) = \exp(-ixt_0)$.

[13]M.Weissbluth, *Atoms and Molecules* (Academic, New York, 1978), Chaps. 9 (Sec. 4) and 14 (Sec. 4).

[14]E. J. Robinson, Phys. Rev. A **33**, 1461 (1986).

Noether's theorem in discrete classical mechanics

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A derivation of Noether's theorem in discrete classical mechanics from the invariance properties of the action is given. This derivation closely parallels the one given in classical field theory and emphasizes the fact that the involved symmetry transformations act on paths rather than on space and time points. Several illustrative examples are also presented.

I. INTRODUCTION

The relation between symmetry and conservation laws plays a major role in physics and, particularly, in classical mechanics. Noether's theorem is the most frequently used method in classical field theory for presenting this subject in a unified way. The concept of action is well suited to this purpose and, in fact, several textbooks[1] and articles[2] present Noether's theorem, derived with several degrees of sophistication from the invariance properties of the action.

However, many texts on classical mechanics that discuss the subject of motion equations for discrete systems from the principle of least action analyze the conservation laws through the examination of the transformation properties of the Lagrangian. We believe that the students could get a more unified view of these topics by presenting the Noether's theorem in a way parallel to that used in field theory.

We present a simple derivation of Noether's theorem along these lines paying special attention to the physical nature of the transformations of time and coordinates involved in the statement of the theorem. Finally, we give several examples on its use in particular situations

II. THE ACTION

Given a physical system having s degrees of freedom, let us collectively represent by $q(=q_1,q_2,...,q_s)$ and $\dot{q}(=\dot{q}_1,\dot{q}_2,...,\dot{q}_s)$ the sets of its generalized coordinates and velocities, respectively. The action[3]

$$S = \int_{t_1}^{t_2} L(q,\dot{q},t)dt \tag{1}$$

contains all the relevant information about the system that is independent of its precise dynamical state at a given time. Through the Lagrangian L, the action is a functional of the paths $q(t)$. Notice that once a path is given, the velocities $\dot{q}(t)$ are immediately obtained.

The equations of motion can be obtained from a prescription called the principle of least action or, more properly, the principle of stationary action. According to it, the actual motion of the system is such that the variation of the action is zero for fixed end points at t_1 and t_2.

The equations of motion are the well-known Euler–Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1,2,...,s . \tag{2}$$

The principle of least action is formulated independently of the choice of coordinates. So, we can easily change the description of the system. A change in our choice of coordinates is generally referred to as a passive transformation. The system keeps on being represented by a Lagrangian, even though its form will be generally different. Consequently, the form of the specific motion equations in the new coordinates changes with respect to the former ones. Nevertheless, there are very special transformations that can also be considered in the active sense, i.e., they can be actually performed on the system. These transformations are the most interesting in physics. Among them, those transformations that leave the action invariant up to an additive constant are related to conserved quantities, and are the only ones concerning us here.

III. NOETHER'S THEOREM IN DISCRETE CLASSICAL MECHANICS

A. Noetherian symmetry transformations

We consider a given path $q_i = q_i(t), i = 1,2,...,s$, and we transform it according to[4]

$$q_i' = q_i + \epsilon K_i(q,\dot{q},t) , \tag{3a}$$

$$t' = t + \epsilon\theta(q,\dot{q},t) , \tag{3b}$$

where ϵ is an infinitesimal parameter.

If we interpret this transformation in the active sense, each path $\gamma(q(t),t)$ is mapped onto $\gamma'(q'(t'),t')$. For one-dimensional systems, the situation is sketched in Fig. 1. On the path γ we choose two points A and B corresponding to times t_1 and t_2, respectively. These points transform into A′ and B′ corresponding to times

$$t_1' = t_1 + \epsilon\theta(q^A,\dot{q}^A,t_1) , \tag{4a}$$

$$t_2' = t_2 + \epsilon\theta(q^B,\dot{q}^B,t_2) . \tag{4b}$$

A sufficient condition for the transformations (3a) and (3b) to be a symmetry transformation of the physical system can be stated in terms of the action. We will require the action to have the same values, up to a first-order term in ϵ, along γ between A and B and along γ' between A′ and B′, i.e.,

$$\int_{t_1,\gamma}^{t_2} L(q,\dot{q},t)dt = \int_{t_1',\gamma'}^{t_2'} L(q',\dot{q}',t')dt' + O(\epsilon^2) , \tag{5}$$

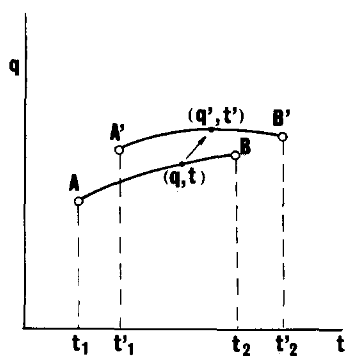

Fig. 1. Effect of a space and time transformation on the path for a one-dimensional system.

where

$$\dot{q}_i = \frac{dq_i}{dt}, \quad \dot{q}_i' = \frac{dq_i'}{dt'} . \tag{6}$$

From condition (5) it follows that if γ is a real path of the system, so is γ', because the actions in both sides of this equation, which are equal for any related paths γ and γ', attain the same minimum. Since the minimum condition is not affected by an additive constant to the action, we can substitute the condition (5) by a less restrictive one. Provided that we only consider first-order terms in ϵ, the new condition is

$$\int_{t_1,\gamma}^{t_2} L(q,\dot{q},t)dt = \int_{t_1',\gamma'}^{t_2'} L(q',\dot{q}',t')dt' + \epsilon A + O(\epsilon^2) , \tag{7}$$

where A is given by

$$A = f[q'(t_2'),t_2'] - f[q'(t_1'),t_1'] = \int_{t_1'}^{t_2'} \left(\frac{df(q',t')}{dt'}\right) dt' . \tag{8}$$

The function f can depend neither on $\dot{q}'$ nor on higher derivatives because A must remain constant under a variation of the path with fixed $q'(t_1')$ and $q'(t_2')$. In fact, the new path would differ from the previous one in the generalized velocities $\dot{q}'$ at t_1' and t_2'. By substituting A, given by Eq. (8) in Eq. (7), we have

$$\int_{t_1,\gamma}^{t_2} L(q,\dot{q},t)dt = \int_{t_1',\gamma'}^{t_2'} \left(L(q',\dot{q}',t') + \epsilon \frac{df(q',t')}{dt'} \right) dt' + O(\epsilon^2) . \tag{9}$$

Any transformation of the type given by Eqs. (3a) and (3b) such that (9) holds true for any time interval (t_1,t_2), and over any path γ and for some function f, is called a Noetherian symmetry transformation of the physical system.

B. Noetherian conservation laws

If the Eqs. (3a) and (3b) represent a Noetherian transformation, the quantity

$$\sum_i p_i K_i - E\theta + f, \tag{10}$$

where

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{11}$$

and[5]

$$E = \sum_i p_i \dot{q}_i - L , \tag{12}$$

is conserved, i.e., is a constant along real paths.

In order to prove this theorem we change, in the right-hand side of Eq. (9), the time variable t' into t and, accordingly, the limits t_1' and t_2' in the integral into t_1 and t_2, respectively. In this way both sides of Eq. (9) can be compared. The result is

$$\int_{t_1}^{t_2} L(q,\dot{q},t)dt$$
$$= \int_{t_1}^{t_2} \left(L(q',\dot{q}',t') + \epsilon \frac{df(q',t')}{dt'} \right) \left(\frac{dt'}{dt} \right) dt . \quad (13)$$

Now, we transform the primed variables in Eq. (13) according to (3a) and (3b) and, consequently,

$$\dot{q}_i' = \dot{q}_i + \epsilon(\dot{K}_i - \dot{\theta}\dot{q}_i) , \quad (14)$$

$$\frac{dt'}{dt} = 1 + \epsilon\dot{\theta} , \quad (15)$$

and obtain, up to first-order terms in ϵ,

$$\int_{t_1}^{t_2} L(q,\dot{q},t)dt$$
$$= \int_{t_1}^{t_2} \left[L(q,\dot{q},t) + \epsilon \left(\sum_i K_i \frac{\partial L}{\partial q_i} + \sum_i (\dot{K}_i - \dot{\theta}\dot{q}_i) \frac{\partial L}{\partial \dot{q}_i} + \theta \frac{\partial L}{\partial t} + \dot{\theta} L + \dot{f} \right) \right] dt . \quad (16)$$

Both integrals in (16) are extended over real paths in which motion equations hold. Therefore, if p_i is the conjugate canonical momentum to q_i,

$$p_i = \frac{\partial L}{\partial \dot{q}_i} , \quad (17)$$

its total time derivative is given by

$$\dot{p}_i = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} , \quad (18)$$

where we have used the motion equations. Then, substituting the definition (17) and the result (18) in (16), expressing $\partial L / \partial t$ in terms of $dL/dt, \partial L / \partial q, \partial L / \partial \dot{q}$, and using the arbitrariness of ϵ, Eq. (16) becomes

$$\int_{t_1}^{t_2} \left(\sum_i (\dot{p}_i K_i + p_i \dot{K}_i - p_i \dot{q}_i \dot{\theta}) - \dot{E}\theta + \dot{\theta} L + \dot{f} \right) dt = 0 . \quad (19)$$

Now, t_1 and t_2 are completely arbitrary; thus the integrand in (19) identically vanishes:

$$\sum_i (\dot{p}_i K_i + p_i \dot{K}_i - p_i \dot{q}_i \dot{\theta}) - \dot{E}\theta + \dot{\theta} L + \dot{f} = 0 , \quad (20)$$

which can be rewritten as

$$\frac{d}{dt} \left(\sum_i p_i K_i - \theta E + f \right) = 0 , \quad (21)$$

which is the result we wanted to show.

This result is easily generalized for infinitesimal transformations that depend on several independent parameters $\epsilon_{(\alpha)}$ $(\alpha = 1,2,...,r)$

$$q_i' = q_i + \sum_\alpha \epsilon_{(\alpha)} K_i^{(\alpha)}(q,t) , \quad (22a)$$

$$t' = t + \sum_\alpha \epsilon_{(\alpha)} \theta_{(\alpha)}(q,t) . \quad (22b)$$

If this transformation, when acting on any real path of the system, is such that the equality

$$\int_{t_1}^{t_2} L(q,\dot{q},t)dt$$
$$= \int_{t_1'}^{t_2'} \left(L(q',\dot{q}',t') + \sum_\alpha \epsilon_{(\alpha)} f_{(\alpha)}(q',t') \right) dt' + O(\epsilon^2) \quad (23)$$

holds, the system exhibits r conservation laws:

$$\frac{d}{dt} \left(\sum_i p_i K_i^{(\alpha)} - E\theta_{(\alpha)} + f_{(\alpha)} \right) = 0 \quad (\alpha = 1,2,...,r) , \quad (24)$$

where

$$E = \sum_i p_i \dot{q}_i - L . \quad (25)$$

This is the content of Noether's theorem.[6]

Whenever the symmetry transformations are such that time is transformed identically or, at most, suffers a uniform translation

$$t' = t + \epsilon , \quad (26)$$

condition (13) is equivalent to

$$L(q,\dot{q},t) = L(q',\dot{q}',t') + \epsilon \frac{df(q',t')}{dt'} . \quad (27)$$

This is the symmetry condition given in most textbooks, and it expresses the (generalized) invariance of the Lagrangian.

IV. EXAMPLES

Noether's theorem allows us to associate a conserved quantity to every symmetry transformation on the physical system.

A. Isolated systems

Classical isolated systems are symmetrical under the ten-parameter inhomogeneous Galilei group.[7] The corresponding ten operations in this group are three space translations, one time translation, three space rotations, and three boosts. The associated ten conserved quantities are linear momentum **P**, energy E, angular momentum **J**, and the vector

$$\mathbf{M} = \sum_i m_i \mathbf{r}_i - \mathbf{P}t .$$

The subject concerning these conserved quantities and their relations to space and time symmetries is well covered in many textbooks, but without making use of Noether's theorem, which is, however, explicitly used, for instance, in the article by Havas and Stachel.[2] In fact, Galilean symmetry for isolated systems may be analyzed from the Lagrangian only.

B. Particle system in a time-dependent uniform external field

We assume that the internal potential energy of the system U depends on the distances between particles only. Therefore, it is translationally invariant. The Lagrangian is

$$L = \sum_i m_i \dot{\mathbf{r}}_i^2/2 - U + \sum_i \mathbf{F}_i(t)\cdot\mathbf{r}_i , \quad (28)$$

which, under the three-parameter transformation

$$\mathbf{r}_i = \mathbf{r}_i + \boldsymbol{\epsilon}\,, \tag{29}$$

becomes L', such that

$$L = L' - \sum_i \mathbf{F}_i(t)\cdot\boldsymbol{\epsilon}\,. \tag{30}$$

Comparing this result with (27) and using the general form (24) for the conservation laws, we obtain

$$\frac{d}{dt}\left[\sum_i\left(\mathbf{p}_i - \int \mathbf{F}_i(t)dt\right)\right] = 0\,. \tag{31}$$

The fact that (29) is a symmetric transformation of the space coordinates shows that the space is homogeneous in a certain sense, but it can be easily shown that it is not isotropic.

C. A particle in a plane-wavelike external field

The Lagrangian for the particle is given by

$$L = m\mathbf{v}^2/2 - U(\mathbf{r} - \mathbf{u}t)\,, \tag{32}$$

where $\mathbf{u}$ is the wave propagation velocity. The one-parameter transformation,

$$\mathbf{r}' = \mathbf{r} + \mathbf{u}\epsilon\,, \tag{33a}$$

$$t' = t + \epsilon\,, \tag{33b}$$

leaves invariant the argument in U and therefore it is a symmetry transformation. The corresponding conservation law is

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{p} - E) = 0$$

or

$$\frac{dE}{dt} = \mathbf{u}\cdot\frac{d\mathbf{p}}{dt}\,, \tag{34}$$

where, according to (12),

$$E = T + U\,. \tag{35}$$

For a wave of finite extension, a time integration of (34) between the instants t_1 and t_2, preceding and following, respectively, the passage of the wave by the position of the particle, yields

$$\mathbf{u}\cdot(\mathbf{p}_2 - \mathbf{p}_1) = T_2 - T_1\,. \tag{36}$$

D. Particle in a potential field, homogeneous function of r[8]

In this case,

$$U(\alpha\mathbf{r}) = \alpha^n U(\mathbf{r})\,,$$

where α is arbitrary and n is the degree of homogeneity of U. We will show that, for a particular n, the action is invariant under a linear transformation on $\mathbf{r}$ and t,

$$\mathbf{r}' = \alpha\mathbf{r}\,, \tag{37a}$$

$$t' = \beta t\,, \tag{37b}$$

partially suggested by the assumed homogeneity of U. Since our previous analysis on action invariance was performed by using only infinitesimal transformations, we rewrite Eqs. (37a) and (37b) in infinitesimal form, by putting

$$\alpha = 1 + \epsilon, \quad \beta = 1 + \gamma\epsilon \tag{38}$$

and obtain

$$\mathbf{r}' = \mathbf{r} + \epsilon\mathbf{r}\,, \tag{39a}$$

$$t' = t + \gamma\epsilon t\,, \tag{39b}$$

where γ is to be specified later. The transformed action is

$$S' = \int_{t'_1}^{t'_2}\left(\frac{m\mathbf{v}'^2}{2} - U(\mathbf{r}')\right)dt'$$
$$= \int_{t_1}^{t_2}\left(\frac{m(1 + 2\epsilon - 2\gamma\epsilon)v^2}{2} - (1 + n\epsilon)U(\mathbf{r})\right)(1 + \gamma\epsilon)dt\,. \tag{40}$$

If the action is to be invariant, i.e., $\delta S = 0$, then

$$1 + 2\epsilon - 2\gamma\epsilon + \gamma\epsilon = 1$$

and

$$1 + n\epsilon + \gamma\epsilon = 1\,.$$

Hence,

$$\gamma = 2 \quad\text{and}\quad n = -2\,.$$

In this case, there is a conserved law obtained from Eq. (21):

$$\left(\frac{d}{dt}\right)(\mathbf{p}\cdot\mathbf{r} - 2Et) = 0\,.$$

[1]See, for example, H. Goldstein, *Classical Mechanics* (Addison–Wesley, Reading, MA, 1980), 2nd ed., pp. 588–596; A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Macmillan, New York, 1964).

[2]One of the most referenced papers is E. L. Hill, Rev. Mod. Phys. **23**, 253 (1951). Also useful, but shorter, are P. Havas and J. Stachel, Phys. Rev. **185**, 1636 (1969); T. Boyer, Am. J. Phys. **34**, 475 (1966).

[3]We consider here only systems that have an action with the general form given by Eq. (1). In particular, dissipative systems will not be covered.

[4]For the sake of simplicity we consider a one-parameter transformation. The extension to general situations is presented below. As far as we are interested in analyzing the invariances of the action, the presence of higher-order derivatives of q in K_i and θ is of no use since the Lagrangians do not contain them either. On the other hand, since we transform paths on which q are known functions of t, K_i and θ could be considered as properly chosen functions of t.

[5]The quantity E, when conserved, is the energy of the system.

[6]Actually, there are two Noether's theorems. The one not dealt with here refers to the invariance of S with respect to transformation groups that depend on arbitrary functions instead of on arbitrary parameters.

[7]Of course, in a relativistic theory, the pertinent space-time symmetry group is the ten-parameter Poincaré group.

[8]This example is an adaptation of a problem in G. L. Kotkin and V. G. Serbo, *Collection of Problems in Classical Mechanics* (Pergamon, Oxford, 1971).