PROBLEMA 1

Se toman 10 mediciones de una variable aleatoria con distribución gaussiana. Calcule la eficiencia de $s^2 = \sum_{i=1}^{m} \frac{(x_i - \bar{x})^2}{m-1}$ como estimador de σ^2 cuando μ es conocido.

$$N(\mu, \sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$

$$\mathcal{L}(x|\mu,\sigma^2) = \prod_{i=1}^{m} (2\pi\sigma^2)^{-1/2} \exp\left[-\frac{1}{2}\frac{(x_i-\mu)^2}{\sigma^2}\right]$$

$$\ln \mathcal{L}(x|\mu,\sigma^2) = -\frac{1}{2}\sum \ln(2\pi\sigma^2) - \frac{1}{2}\sum \frac{(x_i-\mu)^2}{\sigma^2}$$

$$\frac{\partial \ln \mathcal{L}}{\partial \sigma^2} = -\frac{1}{2}\sum \frac{2\pi}{2\pi\sigma^2} + \frac{1}{2}\sum_{i=1}^{m}\frac{(x_i-\mu)^2}{(\sigma^2)^2}$$

$$\frac{\partial \ln \mathcal{L}}{\partial \sigma^2} = -\frac{1}{2}\sum_{i=1}^{m} \frac{1}{\sigma^2} + \frac{1}{2}\left(\frac{1}{\sigma^2}\right)^2 \cdot \sum_{i=1}^{m}(x_i-\mu)^2$$

$$\frac{\partial \ln \mathcal{L}}{\partial \sigma^2} = -\frac{m}{2\sigma}\left(\sigma^2 - \sum_{i=1}^{m}\frac{(x_i-\mu)^2}{m}\right)$$

El estimador de mínima varianza es $t = \sum_{i=1}^{m}\frac{(x_i-\mu)^2}{m}$

y su varianza es

$$Var(t) = \frac{\left|\frac{\partial h}{\partial \theta} + \frac{\partial b}{\partial \theta}\right|}{A(\theta)} = \frac{2\sigma^4}{m}$$

Mientras que la de s^2 es:

$$Var(s^2) = \left(\frac{1}{m-1}\right)^2 \sum_{i=1}^{m} Var(x_i - \bar{x})^2 = \frac{1}{(m-1)^2} Var\left[x_i^2 - 2\bar{x}x_i + \bar{x}^2\right]$$

$$Var(s^2) = \left(\frac{\sigma}{m-1}\right)^2 \sum_{i=1}^{m} Var\left[\left(\frac{x_i-\bar{x}}{\sigma}\right)^2\right] = \frac{\sigma^4}{(m-1)^2} Var\left[\sum_{i=1}^{m}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2\right]$$

$$Var(s^2) = \frac{\sigma^4}{(m-1)^2} 2(m-1) = \frac{\sigma^4}{m-1} \Rightarrow \mathcal{E} = \frac{m-1}{m}$$

Si $m = 10 \Rightarrow \mathcal{E} = 90\%$

Problema 2. Muchos tréboles muy pocos con cuatro hojas

La v.a K: # de tréboles de cuatro hojas por m^2 ~ Poisson.

Los intervalos están ordenados de menor a mayor por lo tanto, se los puede asociar fácilmente con su K_{obs}

K	Intervalo	Frecuencia	Incluye $\mu = 3$
0	$[0; 2.3]$	5%	No
1	$[0; 3.9]$	15%	Si
2	$[0; 5.3]$	22.4%	Si
$K \geq 3$	$[0; >6]$	57.6%	Si

1) Como solo el primer intervalo no incluye el 3 y todos los otros si, la cobertura es 95%.

2) Usando que para $K=0$ el $\mu_{UP} = 2.3$

$$\sum_{K=0}^{K_{obs}} P(K|\mu_{UP}) = 1-CL \Rightarrow P(K=0|2.3) = 1-CL$$

$$e^{-2.3} = 1-CL \Rightarrow CL = 0.90$$

Problema 3 $\quad S_1 = K_1 - K_0 \qquad S_2 = K_2 - K_0$

$$\frac{\partial S_1}{\partial K_1} = 1 \qquad \frac{\partial S_1}{\partial K_0} = -1 \qquad \frac{\partial S_1}{\partial K_2} = 0$$

$$\frac{\partial S_2}{\partial K_1} = 0 \qquad \frac{\partial S_2}{\partial K_0} = -1 \qquad \frac{\partial S_2}{\partial K_2} = 1$$

$\sigma_{S_1}^2 = \sigma_{K_1}^2 + \sigma_{K_0}^2$ y $\sigma_{S_2}^2 = \sigma_{K_2}^2 + \sigma_{K_0}^2$ (error de la resta)

$$Cov(S_1, S_2) = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial S_1}{\partial K_i}\frac{\partial S_2}{\partial K_j} \underbrace{Cov(K_i, K_j)}_{\delta_{ij}\ Var(K_i)}$$

$$Cov(S_1, S_2) = \sum_{i=1}^{3} \left(\frac{\partial S_1}{\partial K_i}\right)\left(\frac{\partial S_2}{\partial K_i}\right) \sigma_{K_i}^2$$

$$Cov(S_1, S_2) = \frac{\partial S_1}{\partial K_1}\cancel{\frac{\partial S_2}{\partial K_1}}\sigma_{K_1}^2 + \cancel{\frac{\partial S_1}{\partial K_2}}\frac{\partial S_2}{\partial K_2}\sigma_{K_2}^2 + \frac{\partial S_1}{\partial K_0}\frac{\partial S_2}{\partial K_0}\sigma_{K_0}^2$$

$$Cov(S_1, S_2) = (-1)\cdot(-1)\cdot K_0$$

$S_1 = 100 - 50 \Rightarrow S_1 = 50$

$S_2 = 135 - 50 \Rightarrow S_1 = 85$

$$\mathbb{V} = \begin{pmatrix} Cov(S_1, S_1) & Cov(S_1, S_2) \\ Cov(S_2, S_1) & Cov(S_2, S_2) \end{pmatrix}$$

K_0, K_1 y K_2 son v.a. con distribución poissoniana.
y por lo tanto S_1 y S_2 también $\Rightarrow \sigma_{S_1}^2 = S_1$ y $\sigma_{S_2}^2 = S_2$

$$\mathbb{V} = \begin{pmatrix} K_1 - K_0 & K_0 \\ K_0 & K_2 - K_0 \end{pmatrix}$$

2) $R = S_2/S_1 \qquad \dfrac{\partial R}{\partial S_2} = \dfrac{1}{S_1} \qquad \dfrac{\partial R}{\partial S_1} = -\dfrac{S_2}{S_1^2}$

$$\sigma_R^2 = Cov(R, R) = \sum_{i=1}^{2}\sum_{j=1}^{2}\left(\frac{\partial R}{\partial S_i}\right)\left(\frac{\partial R}{\partial S_j}\right) Cov(S_i, S_j)$$

$$\sigma_R^2 = \left(\frac{\partial R}{\partial S_1}\right)^2 \sigma_{S_1}^2 + \left(\frac{\partial R}{\partial S_2}\right)^2 \sigma_{S_2}^2 + 2\left(\frac{\partial R}{\partial S_1}\right)\left(\frac{\partial R}{\partial S_2}\right) Cov(S_1, S_2)$$

$$\sigma_R^2 = \left(\frac{S_2}{S_1}\right)^2 \frac{\sigma_{S_1}^2}{S_1^2} + \frac{\sigma_{S_2}^2}{S_1^2} + 2\cdot\left(-\frac{S_2}{S_1^2}\right)\cdot\frac{1}{S_1}\cdot K_0$$

$$\sigma_R^2 = R^2\left[\frac{1}{S_1} + \frac{1}{S_2} - \frac{2K_0}{S_1 S_2}\right] = R^2\left(\frac{S_1 + S_2 - 2K_0}{S_1 \cdot S_2}\right)$$

Con $K_0 = 50$, $K_1 = 100$ y $K_2 = 135$

$S_1 = 50$ y $S_2 = 85$ $\quad R = 1.7$

$$\sigma_R = R\sqrt{\frac{S_1+S_2-2.K_0}{S_1.S_2}} \Rightarrow \sigma_R = 0.154$$

$R = 1,70 \pm 0.15$ quedó a 2σ de $R=2$

Si olvidaba incluir la correlación $\sigma_R = 0.30$ y hubiese creído que era compatible.

Problema 4 F = # medio de fondo mensual

S = # medio de $\bar{\nu}$ mensual con $P = 50\%$

Se midieron k_1, k_0 y k_2 cuyas esperanzas son

$\langle k_0 \rangle = F$; $\langle k_1 \rangle = S + F$; $\langle k_2 \rangle = 2S + F$

y todas ellas son v.a. con distribución poissoniana

$$\Rightarrow \mathcal{L}(k_0, k_1, k_2 | S; F) = P(k_0 | \mu = F) \cdot P(k_1 | \mu = S+F) P(k_2 | \mu = 2S+F)$$

$$\mathcal{L}(k_0, k_1, k_2 | S, F) = \frac{e^{-F} F^{k_0}}{k_0!} \cdot \frac{e^{-(S+F)} (S+F)^{k_1}}{k_1!} \cdot \frac{e^{-(2S+F)} (2S+F)^{k_2}}{k_2!}$$

$$\begin{aligned} \ln \mathcal{L}(\underline{k} | S, F) = & -F + k_0 \ln(F) - \ln(k_0!) \\ & -(S+F) + k_1 \ln(S+F) - \ln(k_1!) \\ & -(2S+F) + k_2 \ln(2S+F) - \ln(k_2!) \end{aligned}$$

$$\frac{\partial \ln \mathcal{L}}{\partial F} = -1 + \frac{k_0}{F} - 1 + \frac{k_1}{S+F} - 1 + \frac{k_2}{2S+F}$$

$$\frac{\partial \ln \mathcal{L}}{\partial F} = \frac{k_0}{F} + \frac{k_1}{S+F} + \frac{k_2}{2S+F} - 3$$

$$\frac{\partial \ln \mathcal{L}}{\partial S} = -1 + \frac{k_1}{S+F} - 2 + \frac{k_2 \cdot 2}{2S+F}$$

$$\frac{\partial \ln \mathcal{L}}{\partial S} = \frac{k_1}{S+F} + \frac{2k_2}{2S+F} - 3$$

Las dos ecuaciones a resolver son:

$$\frac{\partial \ln \mathcal{L}}{\partial F} = 0 \Rightarrow \frac{k_0}{F} + \frac{k_1}{S+F} + \frac{k_2}{2S+F} = 3$$

$$\frac{\partial \ln \mathcal{L}}{\partial S} = 0 \Rightarrow \frac{2k_2}{2S+F} + \frac{k_1}{S+F} = 3$$