Desarrollo de $\ln \phi(t)$

$$\ln \phi(t) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k \ln \phi}{\mathrm{d}t^k}\Big|_0 \, t^k \;=\; \sum_{k=0}^{\infty} \frac{1}{k!} \, i^k M_k \; t^k$$

Desarrollo de $\ln\phi(t)$

$$\ln\phi(t) = \ln\phi\big|_0 + \frac{\mathrm{d}\ln\phi}{\mathrm{d}t}\Big|_0 t + \frac{1}{2}\,\frac{\mathrm{d}^2\ln\phi}{\mathrm{d}t^2}\Big|_0 t^2 + \frac{1}{3!}\,\frac{\mathrm{d}^3\ln\phi}{\mathrm{d}t^3}\Big|_0 t^3 + \mathcal{O}(t^4)$$

① $$\phi_X(t) = E\left(e^{itX}\right) \quad\Rightarrow\quad \phi\big|_0 = E\left(e^{i0X}\right) = 1 \quad\Rightarrow\quad \ln\phi\big|_0 = 0$$

② $$\frac{\mathrm{d}\ln\phi}{\mathrm{d}t} = \frac{1}{\phi}\,\frac{\mathrm{d}\phi}{\mathrm{d}t} \quad\Rightarrow\quad \frac{\mathrm{d}\ln\phi}{\mathrm{d}t}\Big|_0 = i\langle x\rangle = i\,\mu = i\,M_1$$

③ $$\frac{\mathrm{d}^2\ln\phi}{\mathrm{d}t^2} = \frac{1}{\phi}\,\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} - \frac{1}{\phi^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 \quad\Rightarrow\quad \frac{\mathrm{d}^2\ln\phi}{\mathrm{d}t^2}\Big|_0 = i^2\langle x^2\rangle - (i\langle x\rangle)^2 = -\sigma^2 = i^2\,M_2$$

④ $$\frac{\mathrm{d}^3\ln\phi}{\mathrm{d}t^3} = \frac{1}{\phi}\,\frac{\mathrm{d}^3\phi}{\mathrm{d}t^3} - \frac{3}{\phi^2}\,\frac{\mathrm{d}\phi}{\mathrm{d}t}\,\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + \frac{2}{\phi^3}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^3 \quad\Rightarrow\quad \frac{\mathrm{d}^3\ln\phi}{\mathrm{d}t^3}\Big|_0 = i^3\left[\langle x^3\rangle - 3\langle x\rangle\langle x^2\rangle + 2\langle x\rangle^3\right] = i^3\,M_3$$

$$\boxed{\ln\phi(t) = i\,\mu\,t - \frac{1}{2}\,\sigma^2\,t^2 - \frac{i}{6}\,M_3\,t^3 + \mathcal{O}(t^4)}$$

Desarrollo de $\ln \phi(t)$

$$\boxed{\ln \phi(t) = i\mu t - \frac{1}{2}\,\sigma^2\,t^2 - \frac{i}{6}\,M_3\,t^3 + \mathcal{O}(t^4)}$$

Para la gaussiana, la función característica era:

$$\phi(t) = e^{i\mu t - \frac{1}{2}\,t^2\sigma^2}$$

$$\boxed{\ln \phi(t) = i\mu t - \frac{1}{2}\,t^2\sigma^2}$$

Para la normal:

$$\boxed{\ln \phi(t) = -\frac{1}{2}\,t^2}$$

Teorema Central del Límite

Sean $\{X_i\}$ V.A. independientes con $E(X_i) = \mu_i$, $Var(X_i) = \sigma_i^2$ $\Longrightarrow$ $Z = \Sigma X_i$ tiene $E(Z) = \Sigma \mu_i$, $Var(Z) = \Sigma \sigma_i^2$

El TCL dice entonces que cuando $n \to \infty$: $Y = \frac{Z - E(Z)}{\sqrt{Var(Z)}} = \frac{\Sigma X_i - \Sigma \mu_i}{\sqrt{\Sigma\, \sigma_i^2}} = \frac{\Sigma X_i - \Sigma \mu_i}{\sqrt{n}\, \sigma} \sim N(0,1)$

$$\phi_Y(t) = E\left[e^{itY}\right] = E\left[e^{it\frac{\sum(X_i-\mu_i)}{\sqrt{n}\sigma}}\right] = E\left[e^{i\frac{t}{\sqrt{n}\sigma}\sum(X_i-\mu_i)}\right] = \phi_{\Sigma(X_i-\mu_i)}\left(\tfrac{t}{\sqrt{n}\sigma}\right) = \prod_{i=1}^{n} \phi_{(X_i-\mu_i)}\left(\tfrac{t}{\sqrt{n}\sigma}\right)$$

$$\ln \phi_Y(t) = \sum_{i=1}^{n} \ln \phi_{(X_i-\mu_i)}\left(\tfrac{t}{\sqrt{n}\sigma}\right)$$

$$= \sum_{i=1}^{n} \left[-\tfrac{1}{2}\, \sigma^2 \left(\tfrac{t}{\sqrt{n}\sigma}\right)^2 - \tfrac{i}{6}\, M_3 \left(\tfrac{t}{\sqrt{n}\sigma}\right)^3 + \mathcal{O}\left(\tfrac{t}{\sqrt{n}\sigma}\right)^4 \right]$$

$$= \sum_{i=1}^{n} \left[-\frac{t^2}{2n} - \frac{i}{6}\, t^3\, \frac{M_3}{\sigma^3}\, \frac{1}{n^{3/2}} + \mathcal{O}\left(\frac{t^4}{n^2\sigma^4}\right) \right]$$

$$= -\frac{1}{2}\, t^2 - \frac{i\, M_3}{6\, \sigma^3}\, \frac{1}{n^{1/2}}\, t^3 + \mathcal{O}\left(\frac{t^4}{n\sigma^4}\right) \qquad \Longrightarrow \qquad \boxed{\lim_{n\to\infty} \ln \phi_Y(t) = -\frac{1}{2}\, t^2}$$