Last topic: Integral Equation Methods

Applicable to sound propagation (Helmholtz equation, wave equation), Electromagnetism (Maxwell equations), Laplace equation and its applications (Electrostatics, Poisson solver for incompressible flow, and many others).

We will focus on the Laplace equation case and point out similarities to other contexts.

We present the main ideas concerning integral equations, which are common to all of these problems, in the context of the Laplace equation.

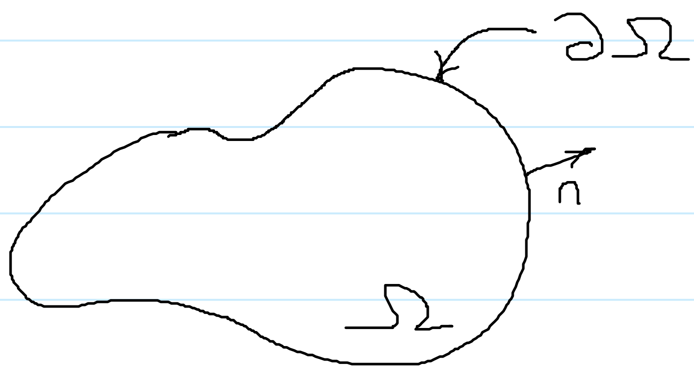
Before considering integral equations, a primer on theoretical aspects related to the Laplace equation is presented.

Laplace operator*

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} \qquad (u = u(x_1, x_2, \cdots, x_d)).$$

$\Delta u = 0$ in Ω : Laplace's equation

$\Delta u = h$ in Ω : Poisson equation.



$\Omega \subseteq \mathbb{R}^d$
open set

In either case boundary conditions must be assigned for

*Here we follow Fritz John's "Partial Differential Equations".

$x = (x_1, x_2, \cdots, x_d) \in \partial\Omega$, such as e.g

Dirichlet boundary conditions

$$u(x) = f(x) \qquad \text{for } x \in \partial\Omega,$$

Neumann boundary conditions

$$\frac{\partial u}{\partial n}(x) = g(x) \qquad \text{for } x \in \partial\Omega,$$

etc. Together, the equation and one of these boundary conditions make up a boundary value problem.

Green's identities

We consider Δ as an operator which acts on functions $u \in C^2(\Omega) \cap C(\overline{\Omega})$. For functions u and $v \in C^2(\overline{\Omega})$ we have

(i) $$\int_{\Omega} v \Delta u \, dx = -\int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, dS$$

(ii) $$\int_{\Omega} (v \Delta u - u \Delta v) \, dx = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) dS$$

(iii) $(v = 1)$

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, dS$$

(iv) $(v = u)$

$$\int_\Omega (\nabla u)^2 \, dx + \int_\Omega u \, \Delta u \, dx =$$

$$= \int_{\partial\Omega} u \cdot \frac{\partial u}{\partial n} \, dS$$

<u>Proof</u>: These identities follow from simple applications of the divergence theorem for vector fields

$$F = (F_1, F_2, \cdots, F_d) \in C^1(\Omega) \cap C(\bar{\Omega}),$$

namely

$$\int_\Omega \nabla \cdot F \, dx = \int_{\partial\Omega} F \cdot n \, dS.$$

Notes:

(a) Point (iii) shows that for the Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n}(x) = g(x) & \text{for } x \in \partial\Omega \end{cases}$$

to have a solution u it is necessary that

$$\int_{\partial\Omega} g \, dS = 0 .$$

(b) Using point (iv) it follow easily that a Dirichlet problem cannot admit more than one solution. Similarly, point (iv) shows that the solution of the Neumann problem is unique up to an additive constant.

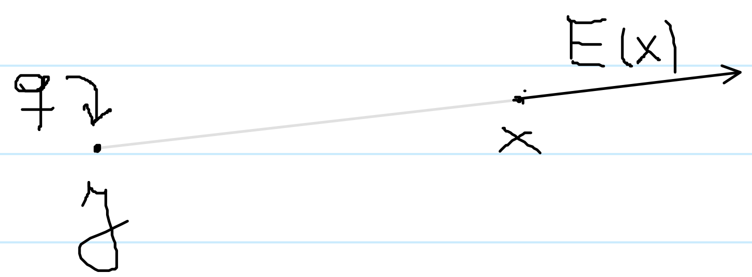
Motivating background concerning Electrostatics

Physical observation: a "point charge" (e.g., an electron) generates an "Electric field" $E = E(x)$ in space.

The electric field manifests itself as it gives rise to a force F that it exerts on any other charge in space.

Both E and F are vector quantities.

Coulomb's law states that the vector $E(x)$ resulting from a charge of magnitude q located at a point y



has the intensity

universal constant

$$|E(x)| = k_e \frac{q}{|x-y|^2},$$

it is parallel to $x-y$, and it points in the same (resp. the opposite) direction as $(x-y)$ provided the charge q is positive (resp. negative).

In other words, we have

$$E = k_e\, q \,\frac{(x-y)}{|x-y|^3}. \qquad (0)$$

The corresponding force on a (positive or negative) charge $\tilde{q}$ is given by

$$F = \tilde{q}\, E = k_e\, q \cdot \tilde{q}\, \frac{(x-y)}{|x-y|^3}.$$

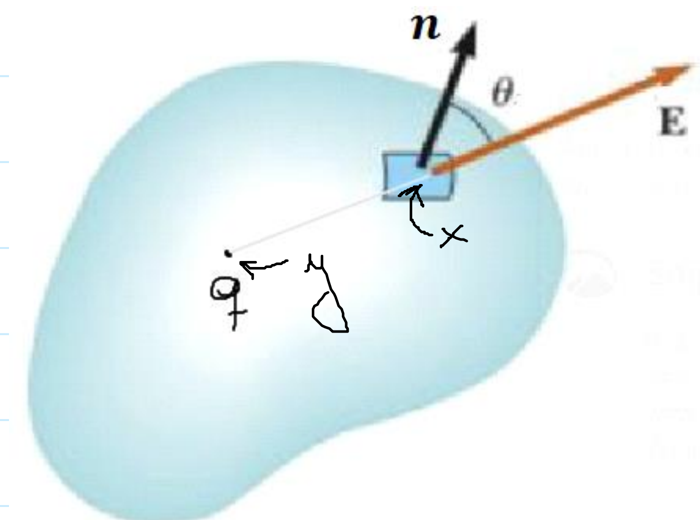
Note the exact analogy with the Newtonian gravitational force G produced by a point mass M on a second point mass m:

$$G = -\gamma \cdot M \cdot m\, \frac{(x-y)}{|x-y|^3}.$$

The constant k_e depends on the units used;

in what follows we use $k_e = 1$, for notational simplicity. We indeed have $k_e = 1$ if we use "electrostatic units".

Gauss' Law



q inside S

$r = |x - y|$

$$E \cdot n \, dS = q \underbrace{\frac{\cos(\theta)}{r^2} \, dS}_{d\Sigma}$$

$\leftarrow$ Element of area in the unit sphere

$$\rightsquigarrow \quad \int_S E \cdot n \, dS = \begin{cases} 4\pi q & \text{for } q \text{ within } S \\ 0 & \text{for } q \text{ outside } S \end{cases}$$

More "rigorous" proof later (using divergence thm.).

If the surface contains a number of charges

$q_1, q_2, q_3 \dots$ we obtain

$$\int_S E \cdot n \, dS = 4\pi \sum_{\substack{q_i \\ \text{inside } S}} q_i$$

For a charge density ρ in a volume W

we obtain

$$\int_S E \cdot n \, dS = 4\pi \int_V \rho(x) \, dx, \qquad (1)$$

where V is the part of W contained within S.

Applying the divergence theorem to the left-hand side of (1) we obtain (calling Ω the volume enclosed by S)

$$\int_{\Omega} \nabla \cdot E \, dx = \int_{\Omega} 4\pi\rho \, dx, \quad \text{or}$$

$$\int_{\Omega} (\nabla \cdot E - 4\pi\rho) \, dx = 0.$$

But, since this is valid for any domain Ω, it follows that

$$\nabla \cdot E = 4\pi\rho \qquad (2)$$

(differential form of Gauss' law).

But from (0) (with $k_e = 1$) it follows that

$$E(x) = \int_{\Omega} \frac{\rho(y)\cdot(x-y)}{|x-y|^3}\, dy = \qquad (3)$$

$$= -\nabla_x \underbrace{\int_{\Omega} \frac{\rho(y)}{|x-y|}}_{\Phi(x)}$$

can be checked using the relation

$$\nabla_x \frac{1}{|x-y|} = -\frac{(x-y)}{|x-y|^3}$$

From (2) and (3) it follows that, letting

$$\Phi(x) = \int_{\Omega} \frac{\rho(y)}{|x-y|}\, dy \qquad (4)$$

we have

$$\Delta_x \Phi = -4\pi\rho(x)$$

But, what is

$$\Delta_x \frac{1}{|x-y|}$$

It is easy to check that

$$\Delta_x \frac{1}{|x-y|} = 0 \quad \text{for all } x \neq y.$$

Conclusion: we cannot pass the Laplacian under the integral in (4)! Or, if we do, we must use

$$\Delta_x \frac{1}{|x-y|} = -4\pi\, \delta(x-y) \qquad (5)$$

We will not rely on this somewhat unjustified discussion. But equation (5) is meaningful and correct, in the formalism of distributions.

We will use the "Fundamental solution"

$$N(x,y) = -\frac{1}{4\pi} \frac{1}{|x-y|}$$

(also called the "Green function") as a powerful tool for the solution of the Laplace equation in general settings (not only in electrostatics!)

Fundamental Solution in n Dimensions

Green's Formula

Switch notation from last class

$$d \to n \quad ; \quad n \to \nu$$

Last class we obtained the "Fundamental solution"

$$N(x,y) = -\frac{1}{4\pi}\frac{1}{|x-y|}$$

in 3-dimensional space, which, via "linear combination" of the form

$$\Phi(x) = \int_{\Omega} \frac{\rho(y)}{|x-y|}\,dy$$

yields a solution Φ of the Poisson equation

$$\Delta_x \Phi = -4\pi\rho(x) \quad \text{in } \Omega.$$

Note that this expression does not account for boundary conditions. We will use the fundamental solution to account boundary conditions as well (by means of "Green's formula"), but, before that, we seek a fundamental solution in general n-dimensional space.

To do this we rely on the radial dependence we expect in the fundamental solution $N = N(x,y)$ we seek, and we thus seek a function N of the form

$$N(x,y) = \Psi(|x-y|),$$

or, letting $r = |x-y|$,

$$N(x,y) = \psi(r).$$

The fundamental solution N should satisfy the Laplace equation for $x \neq y$ $(r \neq 0)$, which should give us an equation for the function ψ.

Letting $y=0$, for notational simplicity, we have

$$\frac{\partial}{\partial x_i} \psi(r) = \psi'(r) \, \frac{x_i}{r}$$

$$\frac{\partial^2}{\partial x_i^2} \psi(r) = \psi''(r) \, \frac{x_i^2}{r^2} + \psi'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right).$$

Thus

$$\begin{aligned} \Delta_x N(x,y) &= \psi''(r) + \frac{n}{r} \psi'(r) - \frac{1}{r} \psi'(r) \\ &= \psi''(r) + \frac{n-1}{r} \psi'(r) = 0. \end{aligned}$$

Using ODE separation of variables we obtain

$$\frac{\psi''}{\psi'} = \frac{1-n}{r}$$

$$\rightsquigarrow \quad \log \psi' = (1-n)\log(r) + \text{const.}$$

$$\rightsquigarrow \quad \psi' = C\, r^{1-n} \qquad (C \text{ constant}),$$

and, thus

$$\psi(r) = \begin{cases} C\,\dfrac{r^{2-n}}{2-n} & n>2 \\ C\log(r) & n=2 \end{cases} \quad + \underset{(\text{take } = 0)}{\text{const.}}$$

Note that for $n=3$ and taking $C = \frac{1}{4\pi}$, this expression reproduces the fundamental solution considered previously.

In general, letting ω_n denote the surface area of the unit sphere in $\mathbb{R}^n$, we select constants as follows:

$$\Psi(r) = \begin{cases} \dfrac{r^{2-n}}{(2-n)\omega_n} & n > 2 \\[2ex] \dfrac{1}{2\pi} \log(r) & n = 2, \end{cases}$$

(The rationale concerning this selection will emerge shortly.)

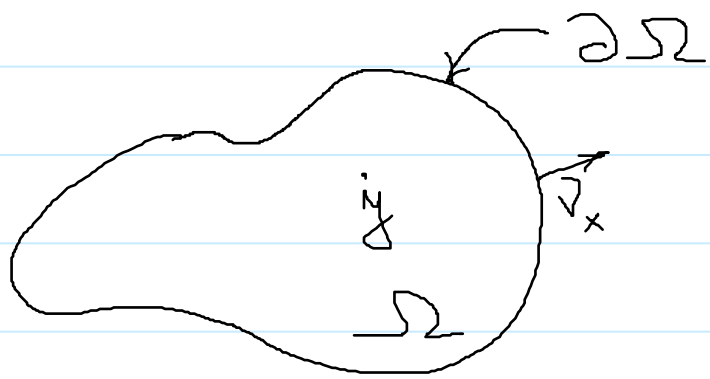
Green's Formula

We wish to use the Green identity (ii) (p. 237)

$$\int_{\Omega} (v \Delta u - u \Delta v)\, dx = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial \nu_x} - u \frac{\partial v}{\partial \nu_x} \right) dS_x$$

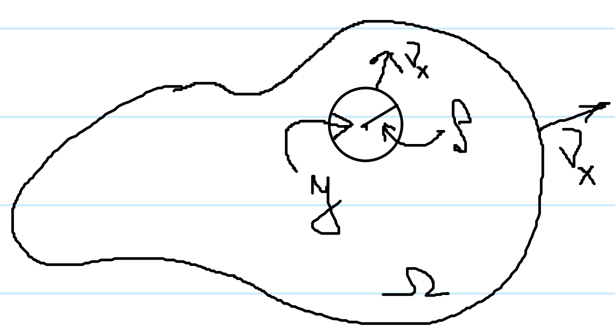
with $v(x) = N(x,y)$ $(y \in \Omega$ fixed$)$,

and for $u \in C^2(\overline{\Omega})$.



But this is not possible, since the Green identity requires both u and v to be smooth within Ω.

To bypass this difficulty we apply the Green identity in the domain $\Omega \setminus B_\rho(y)$.



Since $\Delta_x v(x) = 0$ for $x \neq y$ we obtain

$$\int_{\Omega \setminus B_\rho(y)} v \Delta u \, dx = \int_{\partial(\Omega \setminus B_\rho(y))} \left(v \frac{\partial u}{\partial \nu_x} - u \frac{\partial v}{\partial \nu_x} \right) dS_x =$$

$$= \int_{\partial \Omega} \left(N(x,y) \frac{\partial u}{\partial \nu_x} - u \frac{\partial N}{\partial \nu_x}(x,y) \right) dS_x - \qquad (1)$$

$$- \int_{\partial B_\rho(y)} \left(N(x,y) \frac{\partial u}{\partial \nu_x} - u \frac{\partial N}{\partial \nu_x}(x,y) \right) dS_x .$$

We wish to take limit as $\rho \to 0$ in (1).

The integral over $\partial B_\rho(y)$ equals

$$\int_{\partial B_\rho(y)} N(x,y) \frac{\partial u}{\partial \nu_x} dS_x - \int_{\partial B_\rho(y)} u \frac{\partial N}{\partial \nu_x}(x,y) \, dS_x . \qquad (2)$$

The first of the integrals in (2) tends to zero as $\rho \to 0$, since for $x \in \partial B_\rho(y)$ we have

$$N(x,y) = \begin{cases} \mathcal{O}(\rho^{2-n}) & n > 2 \\ \mathcal{O}(\log(\rho)) & n = 2, \end{cases} \qquad (3)$$

as well as

Element of area in the unit sphere

$$dS_x = \rho^{n-1} d\Sigma . \qquad (4)$$

To deal with the second integral in (2) we note that for $x \in \partial B_\rho(y)$

$$\frac{\partial N}{\partial \nu_x}(x,y) = \frac{(x-y)\cdot \nu_x}{\omega_n \rho^n} = \frac{1}{\omega_n \rho^{n-1}} .$$

It follows that the second integral in (2) tends to $u(y)$ as $\rho \to 0$.

Thus, taking limit as $\rho \to 0$ in (1) yields

$$u(y) = \int_{\Omega} N(x,y)\,\Delta u\,dx \; - \tag{5}$$

$$- \int_{\partial\Omega} \left(N(x,y)\frac{\partial u}{\partial \nu_x} - u\,\frac{\partial N}{\partial \nu_x}(x,y) \right) dS_x .$$

Thus, for a Poisson problem with $\Delta u = h$ the Green formula gives the solution u in terms of h and the boundary values of u and $\frac{\partial u}{\partial \nu_x}$.

For the Laplace problem $\Delta u = 0$, in particular we have

$$u(y) = - \int_{\partial\Omega} \left(N(x,y)\frac{\partial u}{\partial \nu_x} - u\,\frac{\partial N}{\partial \nu_x}(x,y) \right) dS_x \tag{6}$$

But, unfortunately, these boundary values cannot both be prescribed.

To bypass this difficulty we propose to use only one of the integrals: Letting $S = \partial\Omega$ we thus use one of the following expressions

1) $$u(x) = \int_S \frac{\partial N(x,y)}{\partial \nu_y} \Phi(y)\, d\sigma(y)$$

2) $$u(x) = \int_S N(x,y)\, \Phi(y)\, d\sigma(y)$$

(Note slight additional notational changes, $S = \partial\Omega$ and $x \leftrightarrow y$, now following Folland's "Introduction to Partial Differential Equations".)

Of course we may not expect $\Phi = u$ or

$\Phi = \frac{\partial u}{\partial \nu_y}$!

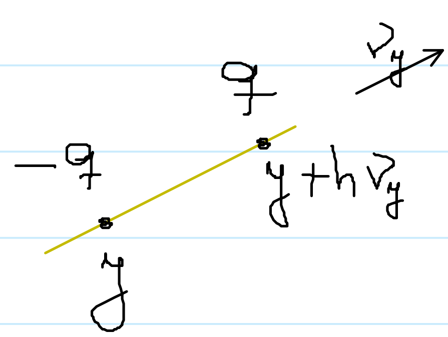
But we do see that for any function Φ defined on S the functions defined in 1) and 2) satisfy Laplace's equation for all $x \in \Omega$!

(Note that $\frac{\partial N}{\partial \nu_y}(x,y)$ is a harmonic function of x for $x \notin S$!)

Note: In the electrostatic context the integral 2) amounts to a surface distribution of charge, with density ϕ.

We can give a related interpretation t the integral 1).

To do this we consider pairs of charges in close proximity of each other



$$\frac{1}{h}\left(\frac{q}{|x-y-h\nu_y|} - \frac{q}{|x-y|}\right)$$

$$\xrightarrow[h \to 0]{} \quad q \frac{\partial}{\partial \nu_y} \frac{1}{|x-y|}$$

This limit is the field of a dipole.

The integral 1) can thus be viewed as a surface distribution of dipoles, or, in other words, a double layer of charges (monopoles). The integral 1) is thus called the "Double-layer potential". The integral 2) is analogously called the "Single-layer potential".

Both, the single- and double-layer potentials are solutions of the Laplace equation for $x \notin S$ (e.g., for $x \in \Omega$).

To solve a Dirichlet problem with boundary values f we use a single-layer (resp. double-layer) representation and seek ϕ such that, for all $x_0 \in S$,

$$\lim_{x \to x_0} \left(SLP[\phi] \right) = f(x_0)$$

(resp.

$$\lim_{x \to x_0} \left(DLP[\phi] \right) = f(x_0) \Big).$$

In the Neumann case, we would similarly seek ϕ such that the normal derivative of the SL or DL tends to $g(x_0)$ as $x \to x_0 \in S$.

These equalities provide equations for the density ϕ.

Once ϕ has been obtained the solution u is obtained by substitution into the layer potential used.

Double-layer potentials

$$u(x) = \int_S \frac{\partial N}{\partial \nu_y}(x,y)\, \phi(y)\, d\sigma(y) \qquad \text{(DLP)}$$

Calling

$$K(x,y) = \frac{\partial N}{\partial \nu_y}(x,y) \quad \text{for } x \in S \text{ and } y \in S,$$

for $x \in S$ we let

$$T_K[\phi](x) = \int_S K(x,y)\, \phi(y)\, d\sigma(y).$$

We shall study the limit of DLP as x tends to the boundary, and its relation to $T_K[\phi]$

To do this we need the following fact:

assuming the surface S is smooth (C^2)

the kernel K is a kernel of order $n-2$ on S,

that is

$$|K(x,y)| \leq \frac{A}{|x-y|^{n-2}}$$

for all $x, y \in S$, where A denotes a

positive real constant.

Kernel Singularities and Jump Conditions

As mentioned last class, our discussion of integral equations is restricted to domains Ω with a smooth (C^2) boundary S. Under such assumption we consider the Double- and Single-Layer potentials

$$1) \quad u(x) = \int_S \frac{\partial N}{\partial \nu_y}(x,y)\, \phi(y)\, d\sigma(y)$$

$$2) \quad u(x) = \int_S N(x,y)\, \phi(y)\, d\sigma(y)$$

where

$$N(x,y) = \begin{cases} \dfrac{|x-y|^{2-n}}{(2-n)\omega^n} & n > 2 \\[2ex] \dfrac{1}{2\pi} \log|x-y| & n = 2, \end{cases}$$

and, by differentiation

$$\frac{\partial N(x,y)}{\partial \nu_y} = -\frac{(x-y)\cdot \nu_y}{\omega_n |x-y|^n} \qquad \left(\begin{array}{c}\text{Even for}\\ n=2!\end{array}\right)$$

For $x \in S$ and $y \in S$ we also write

$$K(x,y) = \frac{\partial N(x,y)}{\partial \nu_y}.$$

It is important to note that both the kernels

$M(x,y) = N(x,y)$ $(n \geq 3)$ and $M(x,y) = K(x,y)$ $(n \geq 2)$

are kernels of order $n-2$ on S, that

both satisfy

$$|M(x,y)| \leq \frac{A}{|x-y|^{n-2}}$$

for all $x, y \in S$, where A denotes a positive real constant.

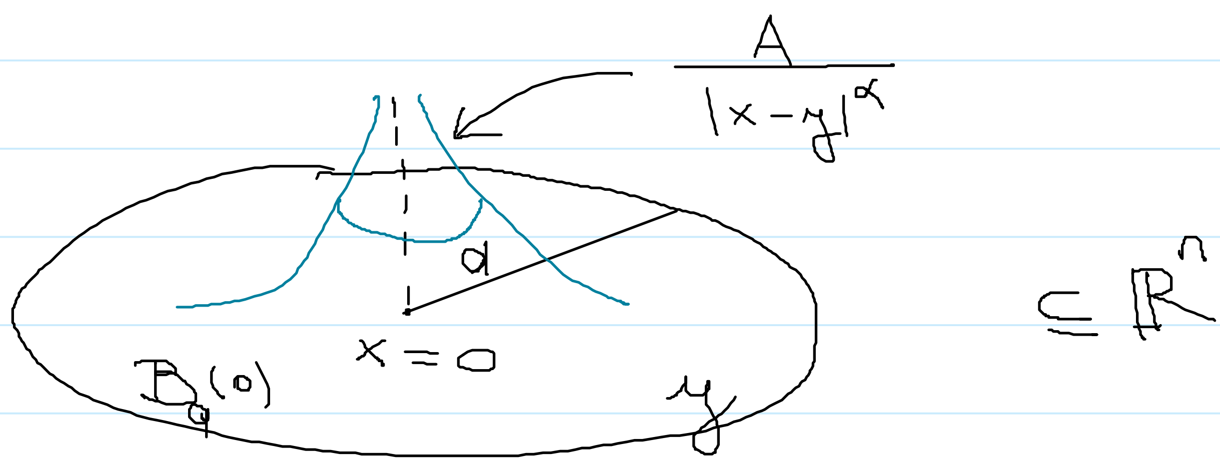
This is a straightforward fact for the kernel $M = N$, even for x, y not on S, and even if S is not smooth. And we will verify this for $M = K$ (for smooth S).

But first we consider the significance of this property: such kernels M are integrable on S, (that is, the integral of $|M|$ on S is finite!) In fact, this is true for any kernel M of order α

$$|M(x,y)| \leq \frac{A}{|x-y|^{\alpha}}$$

for any $\alpha < n-1$.

This is easily checked. Considering a flat surface $S \subseteq \mathbb{R}^2$ and assuming $x = 0 \in S$, it suffices to integrate in a circular neighborhood around $y = 0$



We obtain for, e.g. $n=3$, $n-1=2$

Jacobian

$$\int_{B_a(0)} \frac{A}{|y|^\alpha}\, dy = \int_0^{2\pi} d\theta \int_0^a \frac{A}{r^\alpha}\, r\, dr$$

(For general n, the spherical coordinate Jacobian in $\mathbb{R}^{n-1}$ is given by

$$J_{n-1} = J(r, \theta, \phi_1, \phi_2, \ldots, \phi_{n-3}) = r^{n-2} \prod_{k=1}^{n-3} \sin^{n-2-k} \phi_k \quad .$$
)

For general n, then, the radial integral equals

$$\int_0^a \frac{A}{r^\alpha}\, r^{n-2}\, dr = A \int_0^a r^{n-\alpha-2}\, dr .$$

This quantity is finite iff

$$n-\alpha-2 > -1 ,$$

or, equivalently, iff

$$\alpha < n-1 ,$$

as claimed.

The case of a general $(n-1)$-dimensional "surface" (manifold) $S \subseteq \mathbb{R}^n$ (curve in $\mathbb{R}^2$, surface in $\mathbb{R}^3$, hypersurface in $\mathbb{R}^n$) can be treated similarly, by means of a suitable parametrization of S around $x=y$.

Thus, the significance is that kernels of order α with $\alpha < n-1$ are integrable. The kernel N for $n=2$, which was left out of the discussion, is also integrable, as it may be checked similarly. Thus, all of these kernels are integrable ($n \geq 2$)

Going back to the topic on page 268, we now show (but only in outline) that K is a kernel of order $n-2$. (Problem 1 in HW set VIII provide hints for a rigorous proof in the case $n=2$. The general case can be treated analogously.)

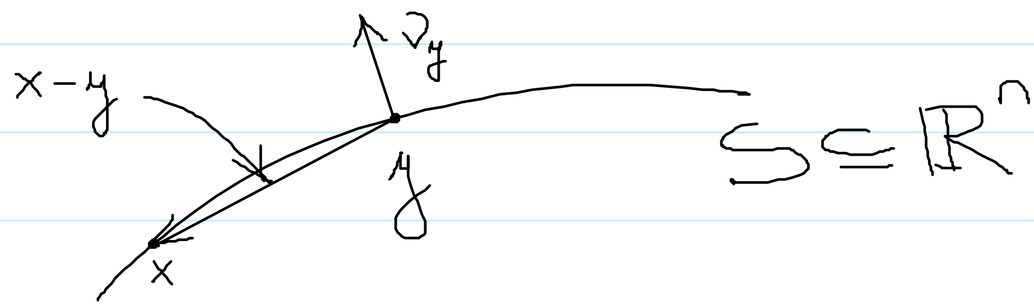
We have

$$K(x,y) = -\frac{(x-y)\cdot \nu_y}{\omega_n |x-y|^n}, \qquad (1)$$

from which it immediately follows that K is a kernel of order $n-1$. But we need more!

The crux of the matter is that the numerator vanishes not only to order one, but, indeed, to order two, as $x \to y$.

To see this, consider the local geometry



We have

$$(x-y)\cdot\nu_y = \underbrace{|x-y|}\ \underbrace{\frac{(x-y)}{|x-y|}\cdot\nu_y}_{\substack{\text{tend to zero}\\ \text{as } x\to y}}$$

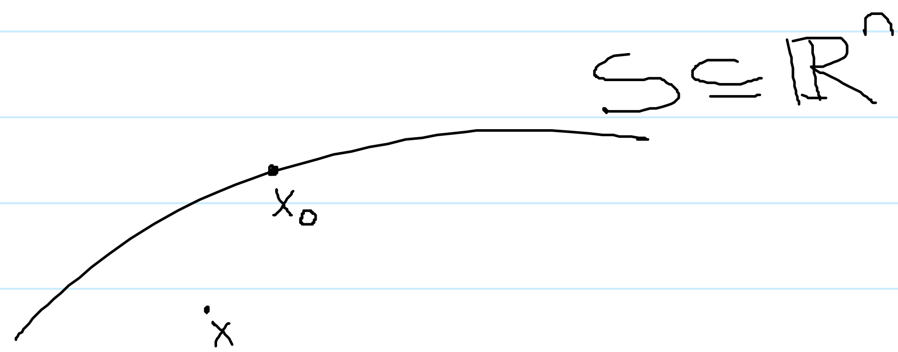
By Taylor expansion, each of these terms can be bounded by a constant times $|x-y|$.

Thus, the numerator in (1) is bounded by a constant times $|x-y|^2$, and, thus, the kernel (1) is bounded by $\dfrac{A}{|x-y|^{n-2}}$, as claimed.

Note that this bound only holds for $x, y \in S$.

If $x \notin S$ only the poorer bound $\dfrac{A}{|x-y|^{n-1}}$ holds.

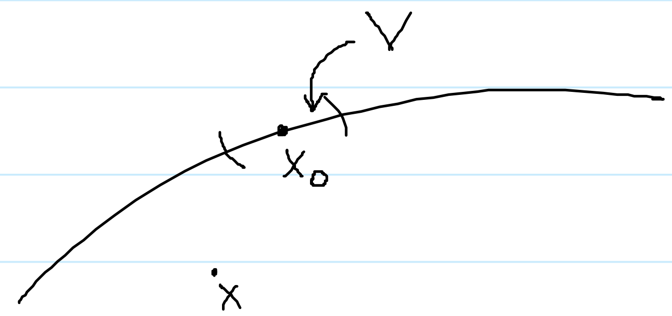
We may now consider the evaluation of the limits, which we can then use to obtain equations for ϕ.



The SLP limit is obtained easily: for $x_0 \in S$ we have

$$\lim_{x \to x_0} \int_S N(x,y)\, \phi(y)\, d\sigma(y) = \int_S N(x_0,y)\, \phi(y)\, d\sigma(y)$$

(Idea: since $N(x,y)$ is $\mathcal{O}(|x-y|^{n-2})$ both for $x \in S$ and $x \notin S$ we can remove a small neighborhood V around x_0 on the integration surface



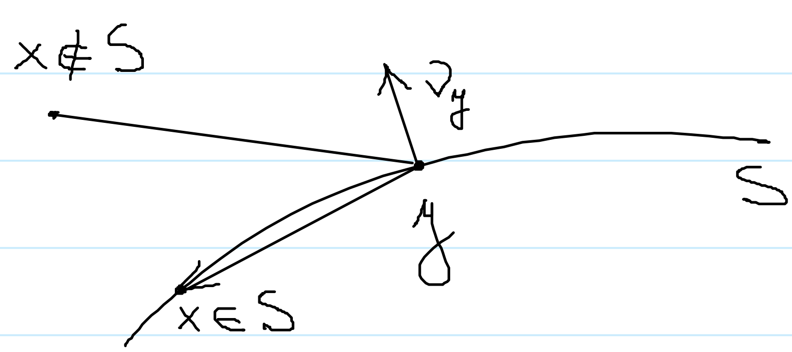
while introducing only a small error, which tends to zero as the neighborhood V shrinks to x_0.)

The remaining integral is clearly continuous.

Let us now consider the limit for the DLP

$$\lim_{x \to x_0} \int_S \frac{\partial N}{\partial \nu_y}(x,y)\, \phi(y)\, d\sigma(y).$$

We have shown that the DLP is $O(|x-y|^{n-2})$ for $x \in S$, but it is not $O(|x-y|^{n-2})$ for $x \notin S$.



Thus, the error introduced by removing a neigborhood is not small as V shrinks, for points $x \notin S$ that are close to S. The procedure we used for the SLP fails in the present case, and, in fact, the limit of the DLP is not equal to the DLP of the limit.

In order to evaluate the limit we first perform an explicit calculation for the density $\phi = 1$.

Lemma: (Gauss' law)

$$\int_S \frac{\partial N}{\partial \nu_y}(x,y)\, d\sigma(y) = \begin{cases} 1 & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \{\Omega \cup S\} \\ 1/2 & \text{for } x \in S \end{cases}$$

Proof: For $x \in \mathbb{R}^n \setminus \{\Omega \cup S\}$ we have

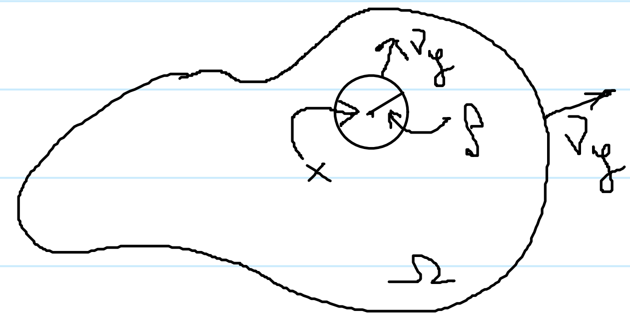
$$\Delta_y N(x,y) = 0$$

for all $y \in \Omega$ and N and $\partial_{\nu_y} N$ are continuous in y up to and including S.

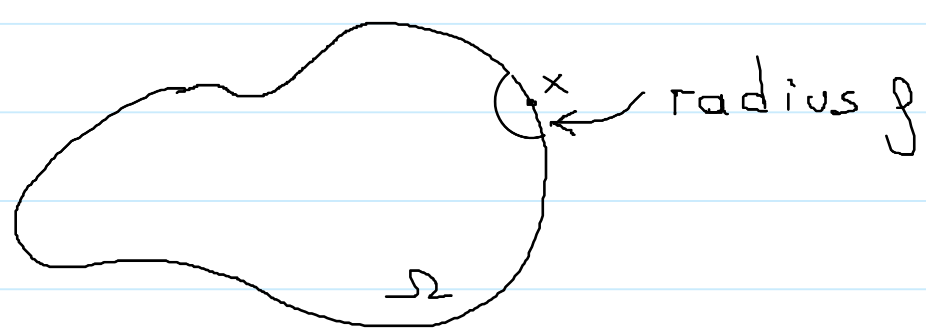
We can thus use the divergence theorem for $\Delta_y N(x,y) = \nabla_y \cdot \nabla_y N(x,y)$ and we obtain

$$0 = \int_\Omega \Delta_y N(x,y)\, dy = \int_S \overbrace{\nu_y \cdot \nabla N(x,y)}^{= \frac{\partial N(x,y)}{\partial \nu_y}}\, d\sigma(y)$$

For $x \in \Omega$ we use the method utilized to obtain Green's formula:



As in Green's formula we obtain a contribution equal to one from the small circle, in the limit as $\rho \to 0$, and the desired result for this case $(DLP = 1)$ follows. For the case $x \in S$ we consider the construction



And the calculation follows similarly. Since only half the circle remains in the limit as $\rho \to 0$ the result is $DLP = 1/2$ in this case, as claimed.

Using Gauss' lemma as a basis, the following more general result can be established

Theorem: Let $\phi \in C(S)$ and define

$$u(x) = \int_S \frac{\partial N}{\partial \nu_y}(x,y)\, \phi(y)\, d\sigma(y)$$

Then, the restriction of u to Ω has a continuous extension to $(\Omega \cup S)$, and the restriction to $\mathbb{R}^n \setminus \{\Omega \cup S\}$ has a continuous extension to $\mathbb{R}^n \setminus \Omega$. Further, defining

$$u_t(x) = u(x + t\nu_x) \quad (x \in S)$$

and letting

$$u_+(x) = \lim_{t \to 0^+} u_t(x) \; ; \quad u_-(x) = \lim_{t \to 0^-} u_t(x)$$

we have

$$u_-(x) = \frac{1}{2}\phi(x) + \int_S K(x,y)\,\phi(y)\,d\sigma(y)$$

$$u_+(x) = -\frac{1}{2}\phi(x) + \int_S K(x,y)\,\phi(y)\,d\sigma(y),$$

or, more compactly,

$$u_- = \frac{1}{2}\phi + T_K\phi$$

$$u_+ = -\frac{1}{2}\phi + T_K\phi$$

We can use this result to solve the Dirichlet problems inside and outside Ω.

Indeed, letting

$$u(x) = \int_S \frac{\partial N}{\partial \nu_y}(x,y)\, \phi(y)\, d\sigma(y),$$

we know that $\Delta u = 0$ both within and outside S.

Thus, in view of the previous theorem, u solves the problem

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u(x) = f(x) & x \in S \end{cases}$$

iff and only iff ϕ satisfies the integral equation

$$\frac{1}{2}\phi(x) + \int_S K(x,y)\, \phi(y)\, d\sigma(y) = f(x) \qquad x \in S$$

The exterior problem can be solved similarly.

We could instead have proposed

$$u(x) = \int_S N(x,y)\,\phi(y)\,d\sigma(y)$$

and we would have been lead to solve

$$\int_S N(x,y)\,\phi(y)\,d\sigma(y) = f(x) \qquad x \in S$$

In view of the Fredholm theory (next class) the preferred approach for the Dirichlet problem is not this one, but the previous one based on the DLP.

The SLP provides the preferred approach for the Neumann problem. Indeed, to solve the Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_x} = g & \text{on } S \end{cases} \qquad (2)$$

we may propose

$$u(x) = \int_S N(x,y)\, \phi(y)\, d\sigma(y) \qquad (3)$$

From a calculation related to the one on p. 280

we obtain

$$\frac{\partial u}{\partial \nu_x}(x) = -\frac{1}{2}\phi(x) + \int_S \frac{\partial N}{\partial \nu_x}(x,y)\, \phi(y)\, d\sigma(y)$$

which leads to the equation

$$-\frac{1}{2}\phi(x) + \int_S \frac{\partial N}{\partial \nu_x}(x,y)\, \phi(y)\, d\sigma(y) = g(x) \qquad (4)$$

Solving this equation and substituting in (3) yields the desired solution u.

But we know that the condition

$$\int_S g(x)\, dS_x = 0$$

is necessary for the problem (2) to admit a solution (see p. 239). As we will discuss next class, this is precisely the condition necessary for the integral equation (4) to admit a solution.

Conclusions from last class

We may seek to solve both the Dirichlet and Neumann problems, either inside or outside Ω, by means of integral equations.

We consider the Dirichlet problem first.

Letting

$$u(x) = \int_S \frac{\partial N(x,y)}{\partial \nu_y} \phi(y)\, d\sigma(y), \qquad (0)$$

u solves the $\frac{\text{exterior}}{\text{interior}}$ Dirichlet problem for the Laplace equation with boundary values f iff ϕ satisfies the integral equation

$$\mp \frac{1}{2}\phi(x) + \int_S K(x,y)\,\phi(y)\,d\sigma(y) = f(x), \quad x \in S$$

where

$$K(x,y) = -\frac{(x-y)\cdot \nu_y}{\omega_n |x-y|^n}.$$

The Neumann problem can be treated similarly:

Letting

$$u(x) = \int_S N(x,y)\,\phi(y)\,d\sigma(y)$$

u solves the $\frac{\text{exterior}}{\text{interior}}$ Neumann problem

for the Laplace equation with Neumann

boundary values g iff ϕ satisfies the integral equation

$$\pm\frac{1}{2}\,\phi(x)+\int_S K^*(x,y)\,\phi(y)\,d\sigma(y)=g(x),\quad x\in S,$$

where

$$K^*(x,y)=\frac{(x-y)\cdot\nu_x}{\omega_n|x-y|^n}.$$

(Note that $K^*(x,y)=K(y,x)$. This is a fact of importance, which implies that the operators T_K and T_{K^*} are adjoints of each other.)

But do these equations have solution? If so, are the solutions unique?

How can such solutions be obtained in practice?

Review of related linear algebra

(finite - dimensional)

Let us recall the following notions, for a given matrix $A \in \mathbb{C}^{m \times m}$ (and similarly for $A \in \mathbb{R}^{m \times m}$):

$$\mathrm{Range}(A) = \{ Ax \; : \; x \in \mathbb{C}^m \}$$

$$\mathrm{Null}(A) = \{ x \in \mathbb{C}^m \; : \; Ax = 0 \}.$$

- The matrix operator $x \longrightarrow Ax$ is injective iff $\mathrm{Null}(A) = \{0\}$.
- The matrix operator $x \longrightarrow Ax$ is surjective iff $\mathrm{Range}(A) = \mathbb{C}^m$

- $\dim \mathrm{Null}(A) + \dim \mathrm{Range}(A) = m$

In particular the problem $Ax = y$ can be solved for all $y \in \mathbb{C}^m$ iff $\mathrm{Null}(A) = \{0\}$.

Additionally, what if $\mathrm{Null}(A) \neq \{0\}$?

- The equation $Ax = y$ admits a solution iff $y \perp \mathrm{Null}(A^*)$.

Proof of this fact: If $Ax = y$, then, for all $z \in \mathbb{C}^m$ we have

$$\begin{array}{ccc} (z, Ax) & = & (z, y) \\ \parallel & & \\ (A^* z, x) & & \end{array}$$

Thus, if $y \in \text{Range}(A) \rightsquigarrow y \perp \text{Null}(A^*)$

$$\rightsquigarrow \underbrace{\text{Range}(A)}_{m-\dim[\text{Null}(A)]} \subseteq \underbrace{[\text{Null}(A^*)]^{\perp}}_{m-\dim[\text{Null}(A^*)]}$$

$$m-\dim[\text{Null}(A^*)] = m-\dim[\text{Null}(A)]$$

Thus $\text{Range}(A) = [\text{Null}(A^*)]^{\perp}$, as claimed

Is the solution unique? No! (Unless $\text{Null}(A) = \{0\}$).

Infinite-dimensional context

Compact operators and Fredholm theory

Let us consider an operator equation of the form

$$(I+T)[\phi] = f \qquad (1)$$

where T is an integral operator of the kind we have been considering, and where I denotes the identity operator

$$I[\phi] = \phi$$

This is a "second-kind" equation. (A first-kind equation is an equation of the form $T\phi = f$.)

Idea concerning second-kind equations (1): we could attempt to use a geometric series.

Formally,

$$\phi = (I+T)^{-1} f =$$

$$= (I - T + T^2 - \dots) f \quad !$$

Does it converge? Possibly not ...

But imagine we can approximate T by an operator T_j of finite rank.

E.g., if
$$T\phi = \int_0^{2\pi} \sum_{k=-\infty}^{\infty} a_k e^{ik(x-y)} \phi(y)\,dy,$$

then this is achieved easily, if the series converges fast enough. For in that case we have

$$T\phi = \sum_{k=-\infty}^{\infty} a_k e^{ikx} \int_0^{2\pi} e^{-iky} \phi(y)\,dy$$

which can be approximated (in norm) by the operator

$$T_j\phi = \sum_{k=-j}^{j} a_k e^{ikx} \int_0^{2\pi} e^{-iky}\phi(y)\,dy.$$

This is a finite-rank operator, with range spanned by $\{e^{-ijx}, e^{-i(j-1)x}, \ldots, e^{i(j-1)x}, e^{ijx}\}$.

Using such a finite rank operator we can proceed as follows:

$$I + T = (I + (T - T_j) + T_j) =$$

$$= \underbrace{\Big[I + \underbrace{T_j \overbrace{(I + (T - T_j))^{-1}}^{\text{power series}}}_{\substack{\text{Finite rank}\\ \text{(regular linear}\\ \text{algebra)}}}\Big]}_{\text{(regular finite-dimensional linear algebra)}} \underbrace{\Big[I + (T - T_j)\Big]}_{\substack{\text{power series}\\ ((T-T_j)\text{ is small})}}$$

On the basis of this structure, the Fredholm theory reduces the problem of inversion of $I + T$ to solution of a regular finite-dimensional linear algebra problem (to which the second result on page 289 applies) and the straightforward sumation of a convergent power series of operators.

But, for all of this to be possible it is necessary to ensure that the operator T may be approximated, in operator norm, and to a prescribed accuracy, by an operator T_j of finite rank. Such an operator T is said to be compact.

Fredholm Theorem: for a compact operator $T: H \to H$ on a Hilbert space H (e.g. $H = L^2(S)$) we have, as in page 289 for finite-dimensional problems, that the equation $(I + T)\,x = y$ admits a solution iff $y \perp \mathrm{Null}(I + T^*)$.

It is not difficult to show that the integral operators we have considered are compact operators from $L^2(S)$ to $L^2(S)$.

From these results it follows that the numerical solution of our second-kind integral equations can be successful provided the corresponding integral operators can be evaluated accurately, and with reasonable cost.

In particular, the integral equation for the interior Dirichlet problems admits a unique solution for any $f \in L^2(S)$.

For the interior Neumann problem a solution of the integral equation exists provided g is ortogonal to the subspace of constants, i.e., provided $\int_S g(y)\,d\sigma_y = 0$ — precisely the necessary condition for existence of solution of the Neumann pbm.

Similar approaches can be used for other equations. For the Helmholtz equation (acoustics)

$$\Delta u + k^2 u = 0$$

in $n=2$ and $n=3$ dimensions, for example, the analysis and equations remain almost unchanged, once use of the relevant fundamental solution

$$N_k(x,y) = \begin{cases} \dfrac{-1}{4\pi} \dfrac{e^{ik|x-y|}}{|x-y|} & (n=3) \\[2ex] \frac{1}{2\pi} H_0^1 \left(k|x-y| \right) & (n=2) \end{cases}$$

is made. Here $H_0^1 = H_0^1(x)$ denotes the Hankel function, $H_0^1 = J_0 + Y_0$, which is a solution of the Bessel equation.

As an addional example we mention the Maxwell equations

$$\begin{cases} \nabla \times E = i\omega\mu H \\ \nabla \times H = -i\omega\varepsilon E \end{cases}$$

under a given incident field (E^{inc}, H^{inc}) (which could be a plane wave, a point source, or some other solution of Maxwell's equations).

The total field

$$(E, H) = (E^{inc} + E^{scat}, H^{inc} + H^{scat})$$

is thus determined by the scattered field

$$(E^{scat}, H^{scat}).$$

(Note that both the incident and scattered fields are solutions of Maxwell's equations.)

In this context, instead of the single-layer and double layer representation of the solution we used in the Laplace case, we use the representation

tangential current (unknown)

$$H^{scat}(x) = \frac{1}{4\pi} \int_S J_s(y) \times \nabla_y N_k(x-y) \, .$$

The current J_s may be obtained as a solution of the "Magnetic field integral equation"

normal

$$J_s(x) - \frac{1}{2\pi} \nu(x) \times \int_S J_s(y) \times \nabla_y N_k(x-y) \, d\sigma_y =$$

$$= 2\, \nu(x) \times H^{inc}(x) \, .$$

(Certain caveats require use of a modified equation in some cases, but we do not discuss such details here.)

Numerical solution of integral equations

We demonstrate a possible approach for the integral equation

$$\frac{1}{2}\Phi(x) + \int_S K(x,y)\,\Phi(y)\,d\sigma(y) = f(x), \quad x \in S \qquad (1)$$

for the interior Laplace Dirichlet problem (pp. 285-286).

Assuming, for simplicity, the two-dimensional case (for which $S \subseteq \mathbb{R}^2$ is a curve), to solve this problem we utilize a discretization

$$y_1, y_2, \dots, y_m$$

of the surface (curve) S, together with (unknown) approximate values

$$(\phi_1, \phi_2, \dots, \phi_m)$$

of the surface density ϕ where ϕ_j is an approximation of $\phi(y_j)$.

Using the discrete values ϕ_j we evaluate an approximation of the integral in (1), of the form

$$\int_S K(y_\ell, y)\, \phi(y)\, d\sigma(y) \sim \sum_{j=1}^{m} K(y_\ell, y_j)\, \phi_j\, w_j.$$

(In principle, any quadrature rule could be used. But the more accurate the quadrature rule, the more accurate will be the solution.)

In the present context, the simple Trapezoidal rule may be used, since the kernel is smooth. (See problem sets VIII and IX.)

Substituting in (1) we obtain the system of linear equations

integration weights

$$\frac{1}{2}\phi_\ell + \sum_{j=1}^{m} K(y_\ell, y_j)\,\phi_j\, w_j = f(y_\ell).$$

Solving this system we obtain ϕ_j $(j = 1, \ldots, m)$.

We may then utilize these values together with equation (o) (p. 285) and a quadrature rule (e.g. the trapezoidal rule) to obtain the solution $u(x)$ (approximately) at any point $x \in \Omega$:

$$u(x) \sim \sum_{j=1}^{m} \frac{\partial N}{\partial \nu_y}(x, y_j)\, \phi_j\, w_j .$$