Nuevos métodos matemáticos y computacionales para las ciencias y la ingeniería

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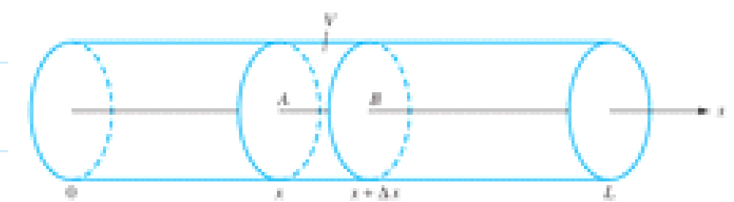
Warm-up and review. Separation of variables.

Simplest example: One-dimensional heat equation



$$\begin{cases} u_t(x,t) = \beta u_{xx}(x,t), & 0<x<L, \quad t>0, \\ u(0,t) = u(L,t) = 0, & t>0, \\ u(x,0) = f(x), & 0<x<L. \end{cases}$$

Separation of variables

$$u(x,t) = X(x)T(t)$$

$$u_t = X(x)T'(t), \qquad u_{xx} = X''(x)T(t)$$

$$\rightsquigarrow X(x)T'(t) = \beta X''(x)T(t), \text{ or}$$

$$\frac{T'(t)}{\beta T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\rightsquigarrow \quad X'' = -\lambda X \quad \text{and} \quad T' = -\lambda \beta T$$

From the PDE boundary conditions we have

$$X(0) = X(L) = 0$$

<u>Case $\lambda < 0$:</u> General solution X is

$$X(x) = C_1 e^{\sqrt{-\lambda}\cdot x} + C_2 e^{-\sqrt{-\lambda}\cdot x}$$

Boundary conditions $\Rightarrow C_1 = C_2 = 0 \Rightarrow X = 0$

<u>Case $\lambda = 0$:</u> General solution X is

$$X(x) = C_1 + C_2 x$$

Boundary conditions $\Rightarrow C_1 = C_2 = 0 \Rightarrow X = 0$

<u>Case $\lambda > 0$:</u> General solution X is

$$X(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

The boundary conditions now tell us that

$$C_1 = 0 \quad \text{and} \quad C_2 \sin(\sqrt{\lambda}\, L) = 0$$

$$\rightsquigarrow \sqrt{\lambda}\, L = n\pi, \quad \text{or} \quad \lambda = \left(\frac{n\pi}{L}\right)^2 \qquad n \in \mathbb{N}$$

$$X = X_n(x) = a_n \sin\left(\frac{n\pi x}{L}\right)$$

and

$$T = T_n(t) = b_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 \cdot t}$$

$\rightsquigarrow$ we have a PDE solution

$$u_n(x,t) = X_n(x)\, T_n(t) = c_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 \cdot t} \sin\left(\frac{n\pi x}{L}\right)$$

This is a solution with initial condition function

$$f(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad (n = 1, 2, \ldots)$$

How about other functions f?

We can easily obtain solutions for linear combinations such as

$$f(x) = 3\sin\left(\frac{\pi x}{L}\right) + 2\sin\left(\frac{4\pi x}{L}\right)$$

$\rightsquigarrow$

$$u(x,t) = 3e^{-\beta\left(\frac{\pi}{L}\right)^2 \cdot t}\sin\left(\frac{\pi x}{L}\right) + 2e^{-\beta\left(\frac{4\pi}{L}\right)^2 \cdot t}\sin\left(\frac{4\pi x}{L}\right)$$

What is the most general function f we can obtain in this manner?

<u>ANS</u>: "All" functions! (using, in general, infinite sums):

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

- Does this mean that, necessarily $f(0) = f(L) = 0$?
- How about cosines? How does the series converge?
- Can we differentiate the series termwise?
- How come we can do this for all f?
- Can we do this for other equations? (Simplest example $\beta = \beta(x)$.) Higher dimensions?
- General domains?

Many important questions!

We will address these questions, and we will exploit the resulting answers to great effect — both theoretically and as a basis of powerful numerical methods.

Convergence of trigonometric Fourier Series

Guiding fact concerning the trigonometric functions in their periodicity interval: orthogonality

For example, assuming a periodicity interval of length 2π, the relevant trigonometric functions are

$$1,\ \cos(x),\ \sin(x),\ \cos(2x),\ \sin(2x),\ \ldots$$

They are pairwise orthogonal in any interval of length 2π (we use $[-\pi,\pi]$); e.g.

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0,$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0 \qquad (m \neq n)$$

etc.

Also
$$\int_{-\pi}^{\pi} 1^2\,dx = 2\pi, \quad \int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi$$

$n = 1, 2, \ldots$

Orthogonality allows us to evaluate the necessary coefficients <u>easily</u> (at least formally):

$$f(x) \sim \frac{1}{2}a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots =$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

$\rightsquigarrow$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$n = 0, 1, 2, \ldots$

Convergence? To study this we proceed as follows.

The partial sum S_N is given by

$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}\left(a_n\cos(nx) + b_n\sin(nx)\right) =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left\{\frac{1}{2} + \sum_{n=1}^{N}\left(\cos(nx)\cos(nt) + \sin(nx)\sin(nt)\right)\right\}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left\{\frac{1}{2} + \sum_{n=1}^{N}\cos[n(x-t)]\right\}dt.$$

The sum can be evaluated explicitly:

$$\left\{\frac{1}{2}+\sum_{n=1}^{N}\cos[ny]\right\}\cdot\sin\left(\frac{1}{2}y\right)=$$

$$=\frac{1}{2}\left\{\sin\left(\frac{1}{2}y\right)+\sum_{n=1}^{N}\left[\sin\left(\left(n+\frac{1}{2}\right)y\right)-\sin\left(\left(n-\frac{1}{2}\right)y\right)\right]\right\}$$

$$=\frac{1}{2}\sin\left(\left(N+\frac{1}{2}\right)y\right)\longrightarrow$$

$$\left\{\frac{1}{2}+\sum_{n=1}^{N}\cos[ny]\right\}=\frac{\sin\left(\left(N+\frac{1}{2}\right)y\right)}{2\cdot\sin\left(\frac{1}{2}y\right)}$$

$$\longrightarrow\quad S_N(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\,\frac{\sin\left(N+\frac{1}{2}\right)(x-t)}{\sin\frac{1}{2}(x-t)}\,dt$$

$$z=t-x\longrightarrow$$

$$S_N(x)=\frac{1}{2\pi}\int_{-\pi-x}^{\pi-x}f(x+z)\,\frac{\sin\left(N+\frac{1}{2}\right)z}{\sin\frac{1}{2}z}\,dz$$

The quotient of sin functions is 2π periodic.

Extending f by 2π-periodicity,

$$f(x+2\pi n) = f(x)$$

the complete integrand becomes 2π-periodic

and we obtain

Dirichlet kernel

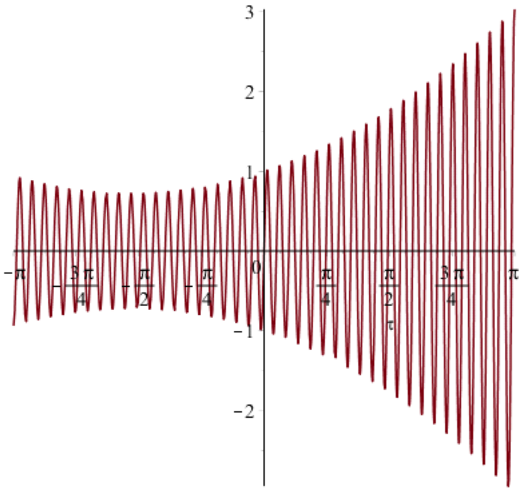
$$s_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+z)\,\frac{\sin\left(N+\frac{1}{2}\right)z}{\sin\frac{1}{2}z}\,dz\,. \quad (1)$$

We wish to obtain the limit of s_N as $N\to\infty$.

The integrand becomes more and more oscillatory

as N grows.

Let us look at a couple of related examples.



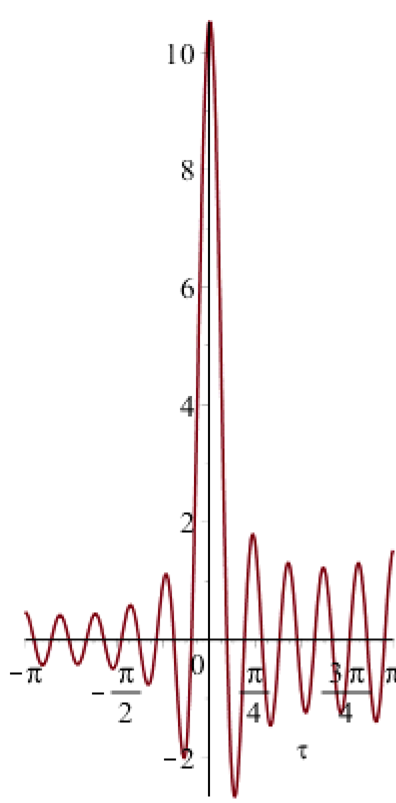
$$\underbrace{\left(\frac{\tau^2}{10}+\frac{\tau}{3}+1\right)}_{g(\tau)}\cdot\sin\left(\left(N+\frac{1}{2}\right)\tau\right)$$

(No vanishing denominator.)

$(N=40)$

$$\left(\frac{\tau^2}{10}+\frac{\tau}{3}+1\right)\cdot\frac{\sin\left(\left(N+\frac{1}{2}\right)\tau\right)}{2\cdot\sin\left(\frac{1}{2}\tau\right)}=$$

$$=\frac{\frac{\tau^2}{10}+\frac{\tau}{3}+1}{2\cdot\sin\left(\frac{1}{2}\tau\right)}\sin\left(\left(N+\frac{1}{2}\right)\tau\right)$$



(Including a vanishing denominator.)

$(N=10)$

In the first case the integral tends to zero, in view of cancellations. To see this we integrate by parts:

$$\int_{-\pi}^{\pi} \underbrace{g(z)}_{v} \cdot \overbrace{\sin\left(\left(N+\tfrac{1}{2}\right)z\right)}^{u'} dz = -\frac{\cos\left(\left(N+\frac{1}{2}\right)z\right)}{\left(N+\frac{1}{2}\right)} g(z)\Bigg|_{z=-\pi}^{z=\pi}$$

$$+ \frac{1}{\left(N+\frac{1}{2}\right)} \int_{-\pi}^{\pi} \underbrace{g'(z)}_{v} \cdot \cos\left(\left(N+\tfrac{1}{2}\right)z\right) dz \xrightarrow[N\to\infty]{} 0$$

(In fact, the Riemann Lebesgue's lemma, which we will establish later, tells us that the integral tends to zero for general functions g satisfying

$$\int_{-\pi}^{\pi} |g(z)|\, dz < \infty .)$$

We also see oscillations in the second function, but because of the vanishing denominator there is a peak at the origin. Note that the integrand is smooth! (The numerator vanishes too.)

But integration by parts as above $\longrightarrow$ infinite integrals.

While most oscillations may cancel each other out, the peak at the origin does not get cancelled! (Note: the width and height of the peak are $\sim \frac{1}{N}$ and N, respectively!)

We can slightly manipulate our integral (1) for $s_N(x)$ and reduce the problem to an integral determined by cancellations.

To do this we note that the representation (1) for the truncated sum

$$s_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

is valid, in particular, for the function

$$f(x) = 1$$

(for which $s_N(x) = 1$ for all N). We thus have

$$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left(N+\frac{1}{2}\right)z}{\sin\frac{1}{2}z}\,dz, \qquad (3)$$

and, therefore

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\frac{\sin\left(N+\frac{1}{2}\right)z}{\sin\frac{1}{2}z}\,dz \qquad (2)$$

Subtracting (2) from (1) we obtain

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{f(x+z)-f(x)}{\sin\frac{1}{2}z}\right)\sin\left(N+\tfrac{1}{2}\right)z\,dz$$

We now note that, provided f is smooth, the factor $\left(\frac{f(x+z)-f(x)}{\sin\frac{1}{2}z}\right)$ is also smooth!

We can thus apply the integration by parts argument, and we obtain $S_N(x) - f(x) \to 0$, or

$$S_N(x) \to f(x) \qquad \text{as desired!}$$

We can also ensure convergence under milder assumptions.

Indeed, using the Riemann-Lebesgue lemma we see that convergence is guaranteed at the point x provided

$$\int_{-\pi}^{\pi} \left| \frac{\left(f(x+z)-f(x)\right)}{\sin \frac{1}{2} z} \right| dz < \infty .$$

Since $\dfrac{z}{\sin\left(\frac{z}{2}\right)}$ is bounded we may replace this condition by

$$\int_{-\pi}^{\pi} \left| \frac{\left(f(x+z)-f(x)\right)}{z} \right| dz < \infty$$

This is <u>Dini's test</u>. If it holds for a point x then the trigonometric expansion converges at x.

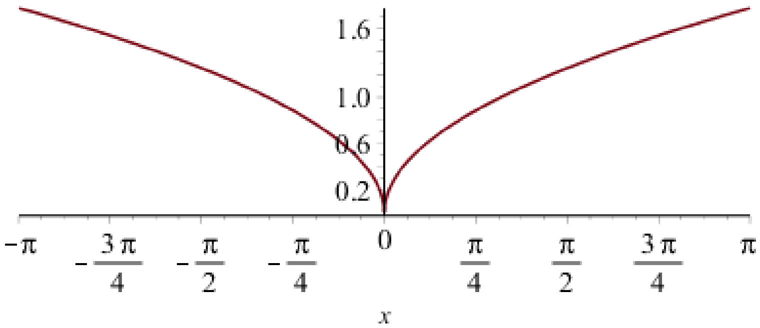
For example, if f is differentiable at x and

and
$$\int_{-\pi}^{\pi} |f(t)|\,dt < \infty$$

then we have convergence at the point x.

More generally, if f is Hölder α at x,
$$|f(x) - f(y)| < M\,|x-y|^{\alpha}$$

for some α $(0 < \alpha \leq 1)$ then there is convergence at x. (Example: $f(x) = \sqrt{|x|}$ even at $x = 0$!)



How about discontinuous f? Say that the limits $f(x+0)$ and $f(x-0)$ exist.

Instead of (3) (p. 14) we may also write

$$\frac{1}{2} = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin\left(N+\frac{1}{2}\right)z}{\sin\frac{1}{2}z}\,dz$$

and

$$\frac{1}{2} = \frac{1}{2\pi}\int_{-\pi}^{0} \frac{\sin\left(N+\frac{1}{2}\right)z}{\sin\frac{1}{2}z}\,dz.$$

Multiplying by $f(x+0)$ and $f(x-0)$ we obtain

$$s_N(x) - \frac{1}{2}\left[f(x+0)+f(x-0)\right] =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} \frac{\left(f(x+z) - f(x-0)\right)}{\sin\frac{1}{2}z} \sin\left(N+\tfrac{1}{2}\right)z \, dz$$

$$+ \frac{1}{2\pi}\int_{0}^{\pi} \frac{\left(f(x+z) - f(x+0)\right)}{\sin\frac{1}{2}z} \sin\left(N+\tfrac{1}{2}\right)z \, dz =$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} \frac{\left(f(x+z) - f(x+0) + f(x-z) - f(x-0)\right)}{\sin\frac{1}{2}z} \sin\left(N+\tfrac{1}{2}\right)z \, dz$$

Improved version of Dini's test: if

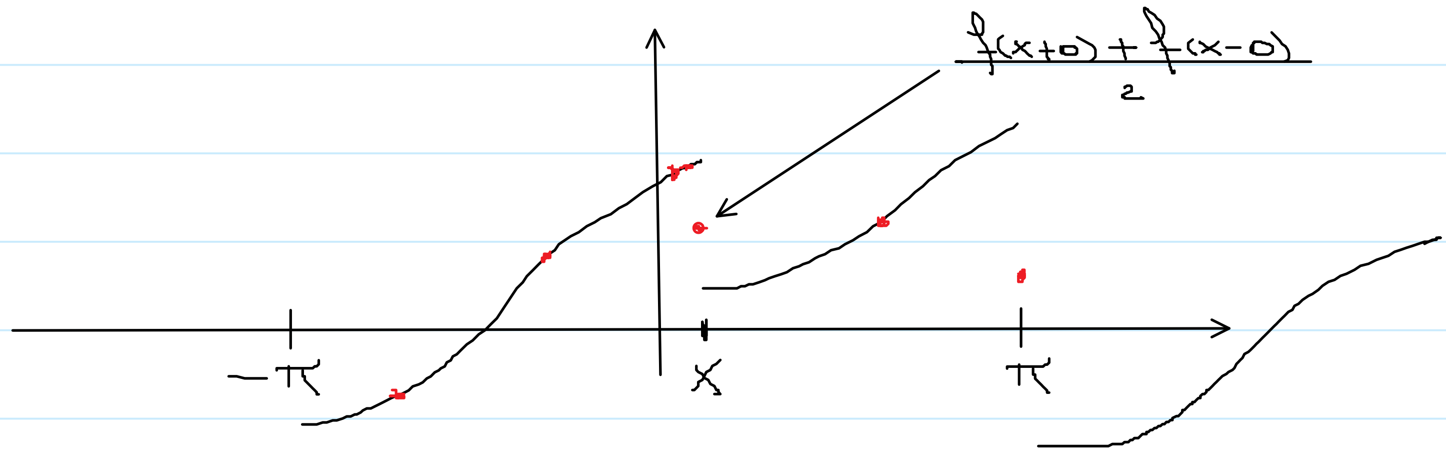
$$\int_0^{\pi} \left| \frac{f(x+z) - f(x+0) + f(x-z) - f(x-0)}{z} \right| dz < \infty$$

then $\displaystyle \lim_{N\to\infty} S_N(x) = \frac{f(x+0) + f(x-0)}{2}$

The criterion is verified if

$$|f(x\pm z) - f(x\pm 0)| < M z^{\alpha} \qquad z \to 0^{\pm}$$

$$(0 < \alpha \le 1)$$



Convergence to the mid point of a discontinuity

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Gibbs Phenomenon

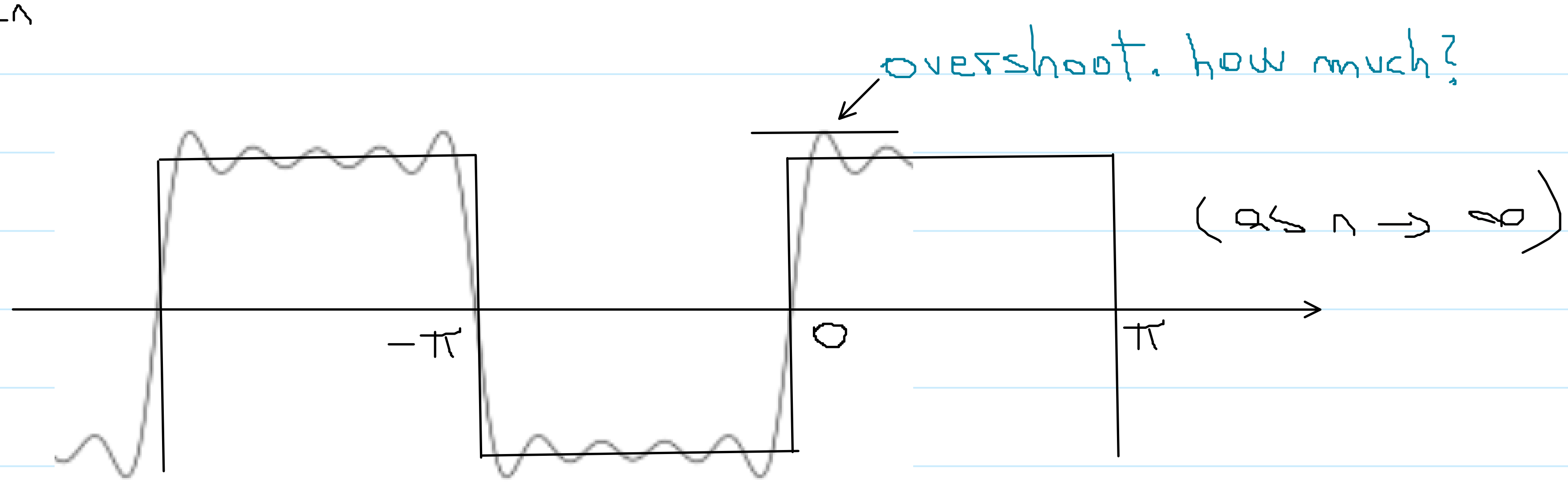
What sort of convergence results around a point of discontinuity?

To understand this we consider a particular case (from which the general case then follows, as shown in a homework problem).

Let $f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 \leq x \leq \pi \end{cases}$

f is odd, the cosine is even $\rightsquigarrow$ only sin coefficients arise (and, then, only the odd ones):

$$\underset{\substack{\parallel \\ S_{2n}}}{S_{2n-1}(x)} = \frac{4}{\pi}\left[\sin(x) + \frac{\sin(3x)}{3} + \cdots + \frac{\sin(2n-1)x}{2n-1}\right]$$



Let us compute the local maximum closest to the origin:

$$0 = S'_{2n-1} =$$

$$= \frac{4}{\pi}\left[\cos(x) + \cos(3x) + \cdots + \cos(2n-1)x\right]$$

Mult. by $\sin(x)$ and using

$$\sin(x)\cos(kx) = \frac{1}{2}\left[\sin(k+1)x - \sin(k-1)x\right]$$

$\rightsquigarrow$

$$\pi \sin(x)\, S'_{2n-1}(x) = 2\sin(2nx) = 0$$

$$\Longrightarrow 2nx = \pm\pi, \pm 2\pi, \ldots, \pm 2n\pi .$$

$\Longrightarrow$ $x = \dfrac{\pi}{2n}$ is the x-coordinate of the maximum that is closest to the origin (for $x > 0$).

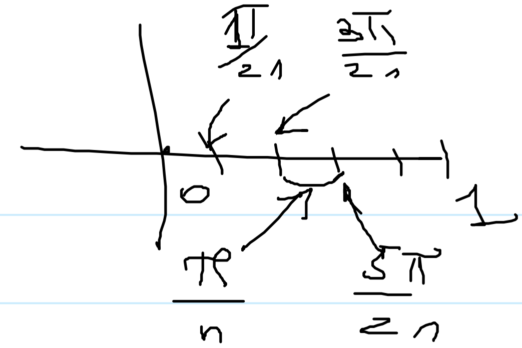
$$S_{2n-1}\left(\frac{\pi}{2n}\right) =$$

$$\frac{4}{\pi}\left[\sin\left(\frac{\pi}{2n}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2n}\right) + \dots + \frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi}{2n}\right)\right]$$

$$\xrightarrow[n\to\infty]{(a)} \frac{2}{\pi}\int_0^{\pi} \frac{\sin(x)}{x}\,dx \overset{(b)}{\cong} 1.18\dots$$

[(b) E.g., expanding $\sin(x)$ in Taylor around $x = 0$ up to x^6, the error in the integral is smaller than 10^{-2} (first two digits right).]

This means that, as $n \to \infty$, the overshoot approaches $0.18 =$ 9% of the jump.



(a) Evaluation of sum

$$\frac{4}{\pi}\left[\sin\left(\frac{\pi}{2n}\right)+\frac{1}{3}\sin\left(\frac{3\pi}{2n}\right)+\dots+\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi}{2n}\right)\right]$$

$$=\frac{4}{\pi}\underbrace{\sum_{k=1}^{n}\sin\left(\pi\frac{2k-1}{2n}\right)}_{S}\cdot\frac{1}{2k-1}$$

But the sum S itself equals

$$S=\frac{1}{2}\sum_{k=1}^{n}\underbrace{\sin\left(\overbrace{\pi\frac{2k-1}{2n}}^{x_k}\right)\cdot\frac{1}{\pi\frac{2k-1}{2n}}}_{\frac{\sin(x)}{x}\text{ at } x_k=\pi\frac{2k-1}{2n}}\cdot\frac{\pi}{n}$$

$$\xrightarrow[n\to\infty]{}\frac{1}{2}\int_0^{\pi}\frac{\sin(x)}{x}\,dx.$$

PDE Discretization

Simplest approach to PDE discretization: finite differences

Example: let us consider the advection PDE

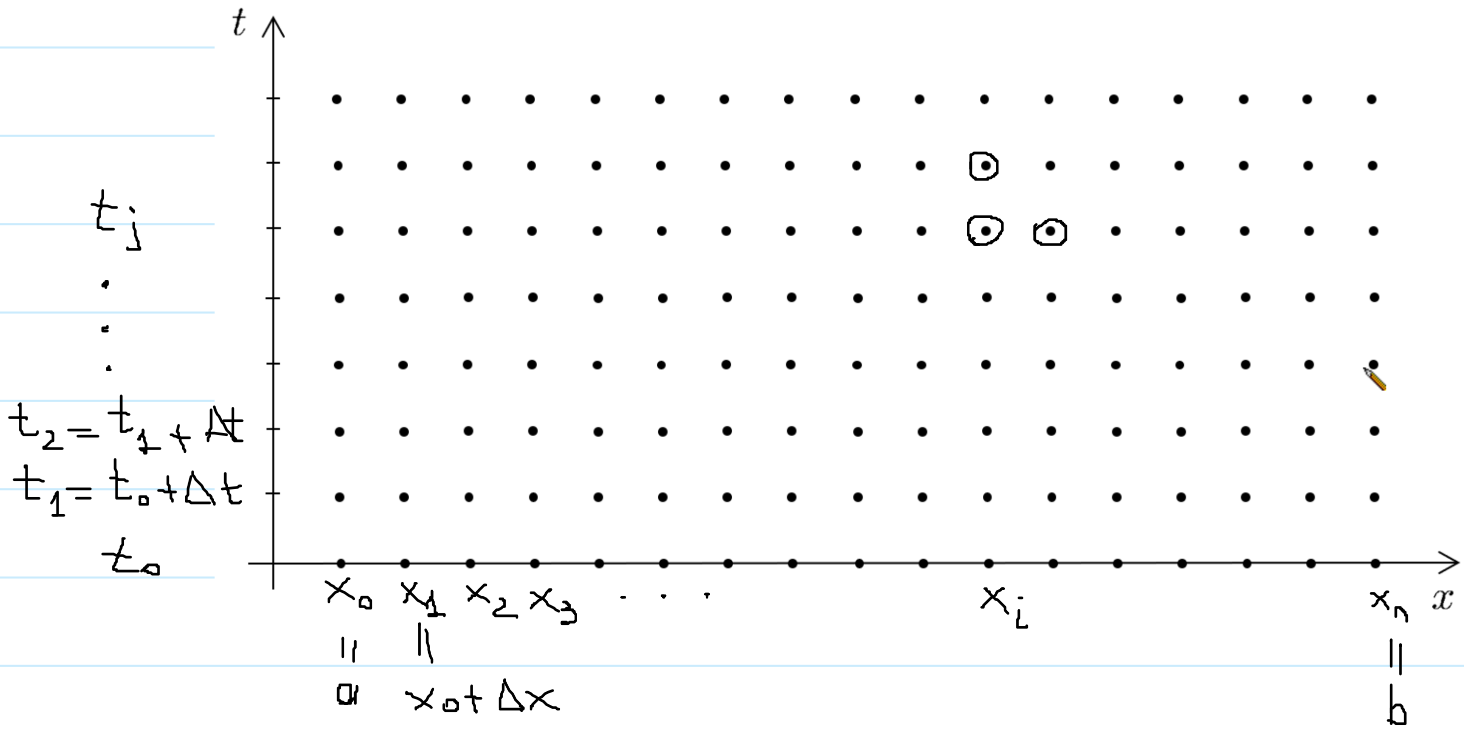
$$u_t(x,t) - u_x(x,t) = 0$$

for $a \le x \le b$; $t \geqslant t_0$

Finite difference paradigm: substitute

$$u_t(x,t) \sim \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

$$u_x(x,t) \sim \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}$$



Letting

$$u_{i,j} \cong u(x_i, t_j)$$

use of finite differences leads to the finite-difference approximation

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x}$$

If we know

$$u_{i,0} = u(x_i, t_0) \quad (1 \le i \le n) \quad \text{[initial condition]}$$

and

$$u_{0,j} = u(x_0, t_j) \quad (j = 1, 2, \cdots) \quad \text{[boundary cond.]}$$

we can obtain $u_{i,j}$ for all i, j:

$$u_{i,j+1} = u_{i,j} + \frac{\Delta t}{\Delta x}\left(u_{i+1,j} - u_{i,j}\right)$$

Error? Convergence?

Let us now consider the somewhat more general equation

$$\begin{cases} u_t + a u_x = 0 \\ u(x,0) = f(x) \end{cases}$$

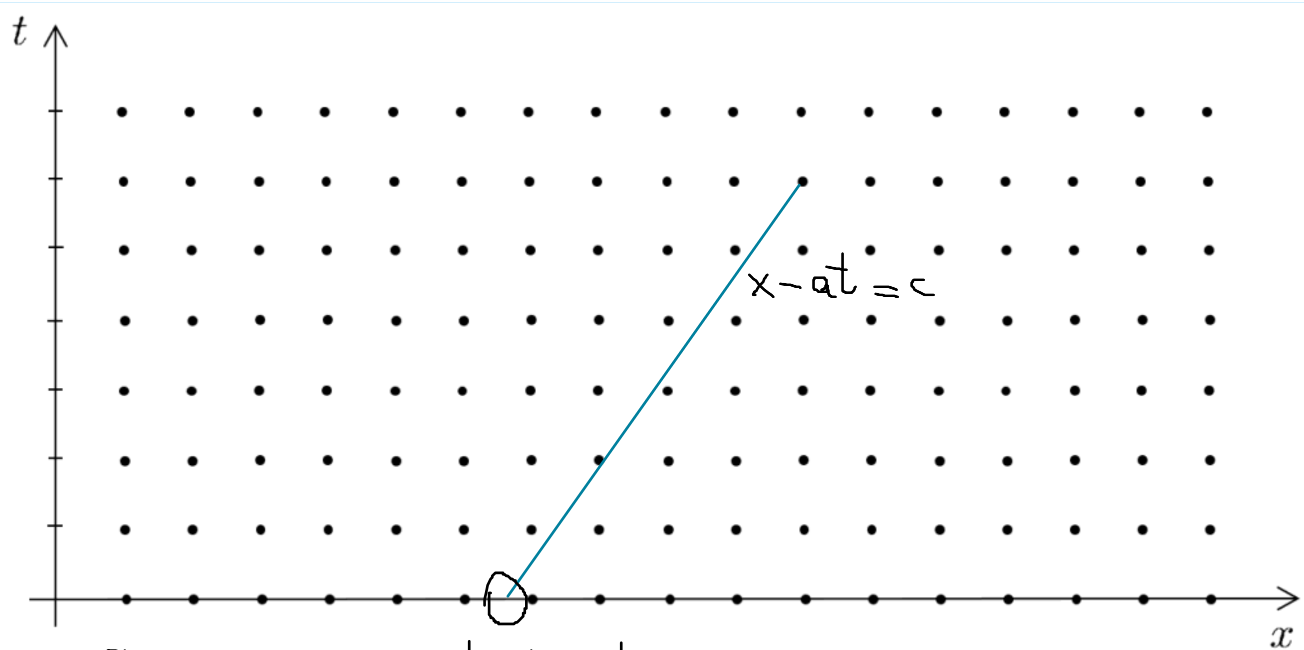
Exact solution

$$u(x,t) = f(x-at).$$

The solution is constant along the lines

$$x - at = c$$

for all c.

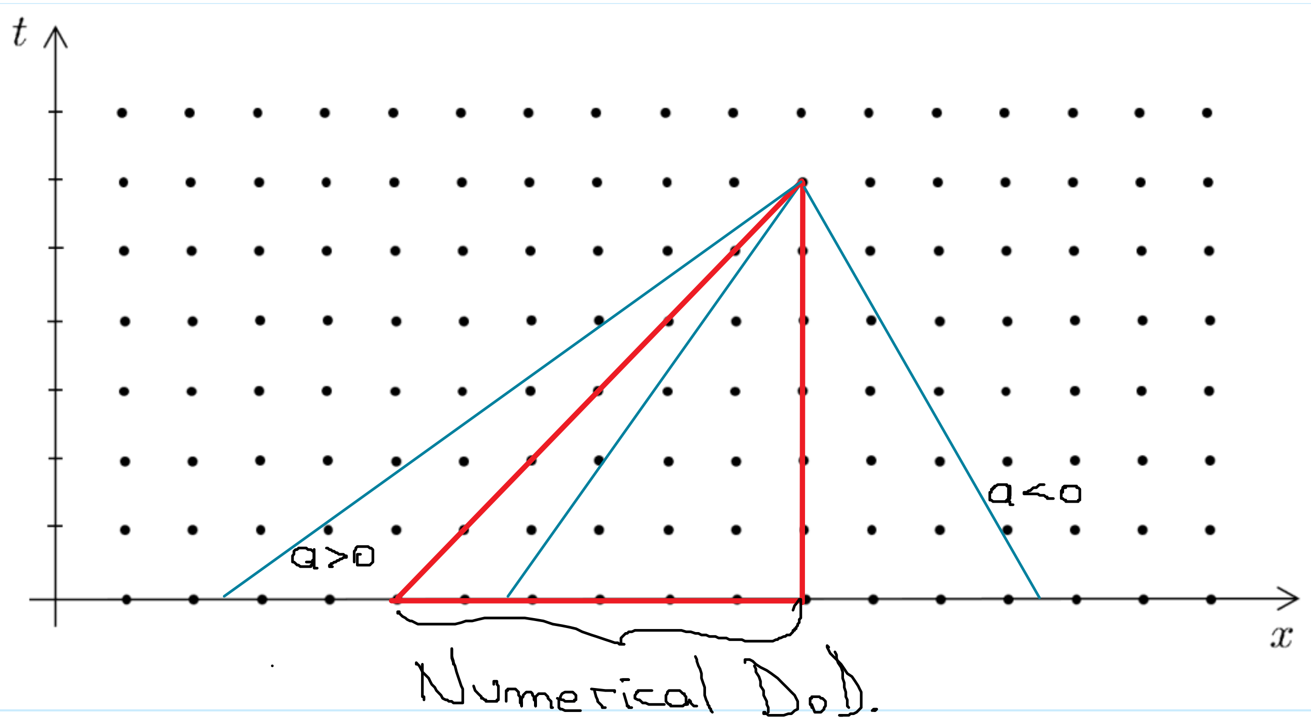


Domain of dependence

(one point, in this case, a region in the domain, for general time dependent PDE).

Numerical domain of dependence (in red).

and true PDE domain of dependence



$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} + a\,\frac{u_{i,j} - u_{i-1,j}}{\Delta x} = 0$$

(upwind, if $a>0$)

For convergence to take place, the numerical DoD must contain the true PDE domain of dependence:

upwind and $|a|\dfrac{\Delta t}{\Delta x} \le 1$.

We cannot possibly expect convergence if this DoD condition is not satisfied: the numerical DoD must contain the exact DoD.

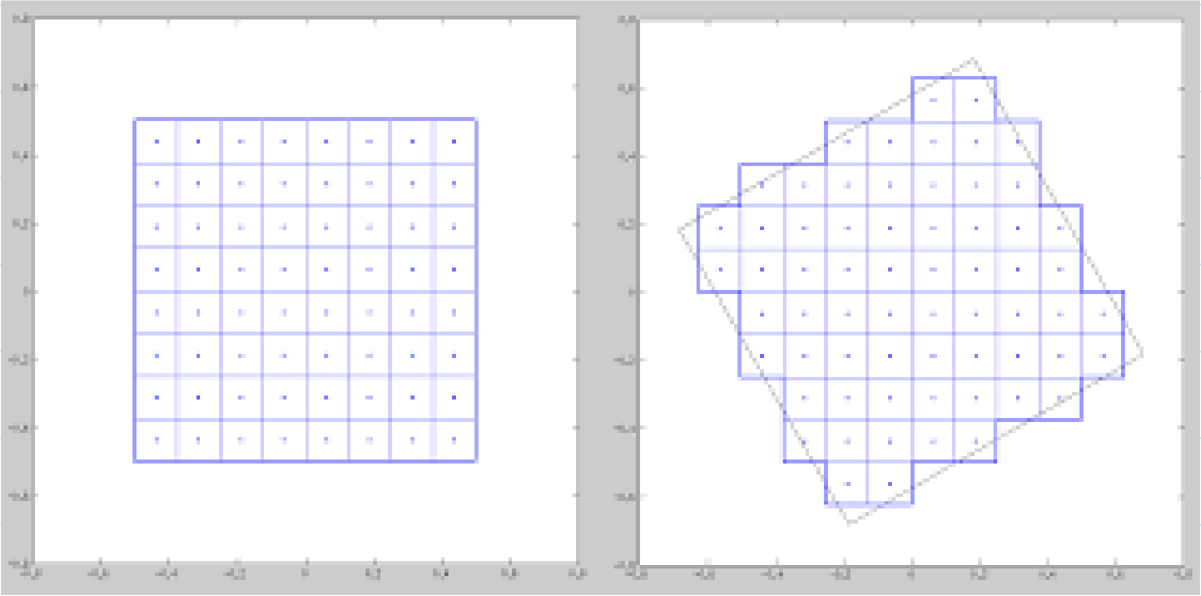
Yet, the error from one time-step to the next is (comparably) small, independently of whether the DoD condition is satisfied!

Rationale: the error factor grows without bound if the DoD condition is not satisfied.

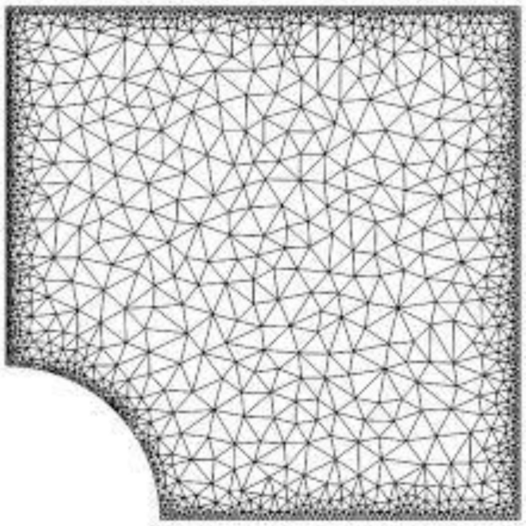
How about higher spatial dimensions?

Eg., for a domain in two dimensions use

a two-dimensional grid



Note the "staircasing" that arises (except for a square domain. Alternative: finite element meshes



Error decay as the mesh-size tends to zero?

Issues: Accuracy, Stability and Speed.

1) Accuracy: Produce accurate derivatives from the available data.

2) Stability: how do errors accumulate? Avoid catastrophic accumulation!

3) Speed/size: Fast numerical implementation. Acceleration (e.g. FFT).

We will consider issues in all three categories. We start by considering the accuracy of various finite difference differentiation strategies.

Differentiation I: finite differences and differentiation of polynomial interpolants.

(See e.g. Leveque's text.)

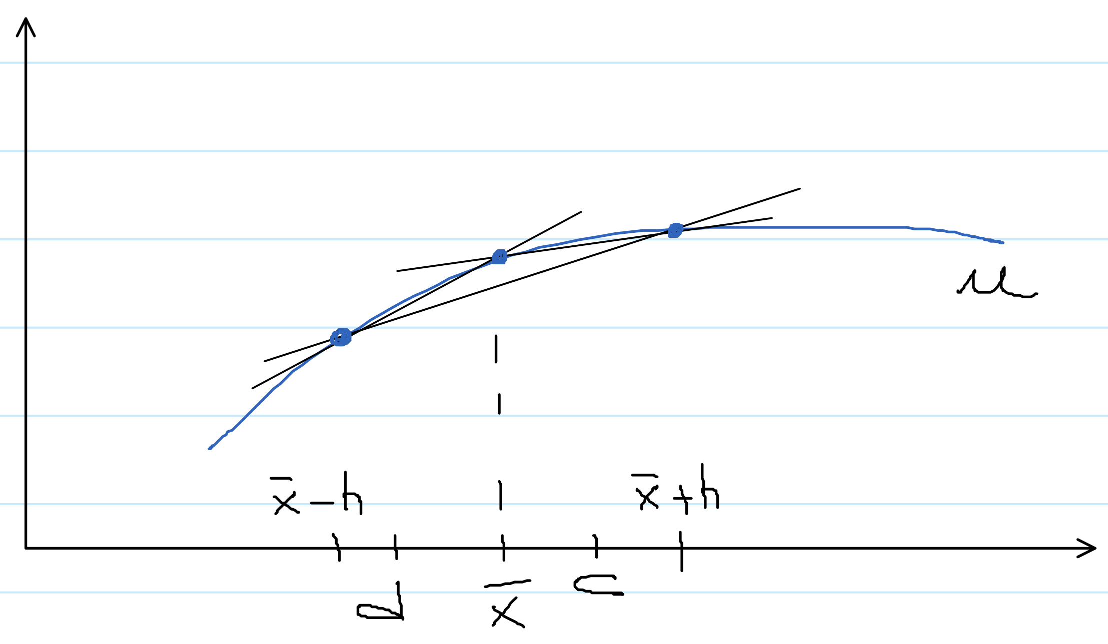
Define

$$D_+ u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x})}{h}$$

$$D_- u(\bar{x}) = \frac{u(\bar{x}) - u(\bar{x}-h)}{h}$$

$$D_0 u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h}$$

Are some of these alternatives more accurate than the others?



How would we know?

Guiding principle:

"more accurate" $\longleftrightarrow$ "converges faster to $u'(\bar{x})$ as $h \to 0$."

How fast do these approximations converge?

We can study this by considering Taylor expansions around $x = \bar{x}$.

We have

$$u(\bar{x}+h) = u(\bar{x}) + u'(\bar{x})h + u''(c)\frac{h^2}{2}$$

$$u(\bar{x}-h) = u(\bar{x}) - u'(\bar{x})h + u''(d)\frac{h^2}{2}$$

$\Longrightarrow$

$$\frac{u(\bar{x}+h)-u(\bar{x})}{h} = u'(\bar{x}) + \mathcal{O}(h)$$

$$\frac{u(\bar{x})-u(\bar{x}-h)}{h} = u'(\bar{x}) + \mathcal{O}(h)$$

($\mathcal{O}(h)$ proportionality constant estimated by $\max\{u''\}$ in relevant regions.)

$$\frac{u(\bar{x}+h)-u(\bar{x}-h)}{2h} = ?$$

Need to use one more term in the expansion.

$$u(\bar{x}+h) = u(\bar{x}) + u'(\bar{x})h + u''(\bar{x})\frac{h^2}{2} + u'''(c_1)\frac{h^3}{3!}$$

$$u(\bar{x}-h) = u(\bar{x}) - u'(\bar{x})h + u''(\bar{x})\frac{h^2}{2} - u'''(d_1)\frac{h^3}{3!}$$

$$\Longrightarrow \quad \frac{u(\bar{x}+h)-u(\bar{x}-h)}{2h} = u'(\bar{x}) + \mathcal{O}(h^2)$$

Better! (Although constant $\sim u'''$ could be worse)

Example: approximation of the derivatives of

$u(x) = \sin(x)$ at $\bar{x} = 1$. $u'(1) = \cos(1) = 0.5403023\ldots$

h	$D_+u(\bar{x})$	$D_-u(\bar{x})$	$D_0u(\bar{x})$	$D_3u(\bar{x})$
1.0e−01	−4.2939e−02	4.1138e−02	−9.0005e−04	6.8207e−05
5.0e−02	−2.1257e−02	2.0807e−02	−2.2510e−04	8.6491e−06
1.0e−02	−4.2163e−03	4.1983e−03	−9.0050e−06	6.9941e−08
5.0e−03	−2.1059e−03	2.1014e−03	−2.2513e−06	8.7540e−09
1.0e−03	−4.2083e−04	4.2065e−04	−9.0050e−08	6.9979e−11

$$D_3u(\bar{x}) = \frac{1}{6h}\left[2u(\bar{x}+h) + 3u(\bar{x}) - 6u(\bar{x}-h) + u(\bar{x}-2h)\right]$$

Easy to check

$$D_+u(\bar{x}) - u'(\bar{x}) \approx -0.42h,$$
$$D_0u(\bar{x}) - u'(\bar{x}) \approx -0.09h^2,$$
$$D_3u(\bar{x}) - u'(\bar{x}) \approx 0.007h^3,$$

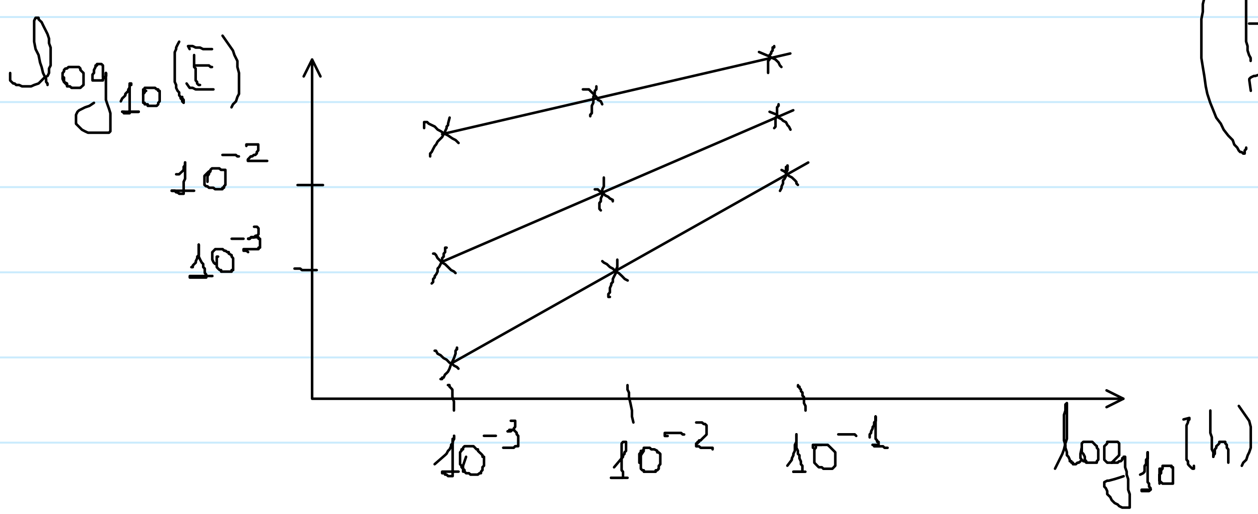
Errors of order

1	D_+, D_-
2	D_0
3	D_3

In a log-log plot the errors are linear functions of $\log(h)$, of slopes 1, 2 and 3, resp.:

If $E \simeq Ch^p$, then

$$\log(E) \simeq p\log h + \log(C)$$



(HW Pbm.)