Práctica I

1.- Consider the partial sums

$$r_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \tag{1}$$

and

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}, \tag{2}$$

with complex numbers a_k, b_k and c_k. Establish that $r_n = s_n$ if and only if

$$a_0 = 2c_0 \quad ; \quad a_k = c_k + c_{-k} \quad \text{and} \quad b_k = i(c_k - c_{-k}) \quad \text{for} \quad k \geq 1.$$

[EASY!]

2.- Fourier series and the Gibbs phenomenon.

(a) Let $f(x)$, $-L \leq x < L$ be a function whose periodic extension of period $2L$, which will also be denoted by f, is piecewise differentiable but discontinuous at a finite number of points in the interval $-L \leq x < L$ (with consequent discontinuities in the periodic extension). Show that, denoting by a_n, b_n and c_n the cosine, sine and exponential Fourier coefficients of f, we have $c_n = \mathcal{O}(1/n)$, $a_n = \mathcal{O}(1/n)$ and $b_n = \mathcal{O}(1/n)$ as $n \to \infty$. Generalize: what can you say if the k-th derivative $f^{(k)}$ is piecewise continuous while all the previous derivatives are continuous?

[Review Re. point (a): A function $f : [a, b] \to \mathbb{R}$ is said to be k times piecewise continuously differentiable in an interval $[a, b]$ if and only if f is $k - 1$ times continuously differentiable in $[a, b]$, and there exists a finite sequence of points $x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b$ such that f is k times continuously differentiable within each closed interval $[x_{i-1}, x_i]$, including the endpoints ($1 \leq i \leq n$).]

(b) What do the results of item (a) imply about the rate of convergence of the corresponding Fourier series at various points x in the interval $[-L, L]$? Obtain a simple estimate by means of a direct application of the triangle inequality to the Fourier-series sum in conjunction with point (a) above.

(c) With reference to point (b), obtain improved estimates of the rate of convergence via integration by parts in the integral expression for $s_N - f$ that was used in class as part of the derivation of Dini's test.

3.- Using the result established in class describing the convergence of the Fourier series of a certain step function, establish the convergence properties of an arbitrary piece-wise continuously differentiable function around a point of discontinuity.

4.- Let $\{s_n(x)\}$ be the sequence of partial sums of the trigonometric Fourier series for $f(x)$, and define σ_n as the mean of the first n partial sums:

$$\sigma_n = \frac{s_0(x) + \cdots + s_{n-1}(x)}{n}.$$

This is called the Cesáro sum of the Fourier series.

(i) Using the Dirichlet kernel

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\pi\sin(x/2)}$$

which, as was shown in class, can be used to produce $s_n(x)$ via a certain integration process, obtain a corresponding expression for $\sigma_n(x)$ in terms of the Féjer kernel

$$\frac{1}{2\pi n}\frac{\sin^2(nx/2)}{\sin^2(x/2)}.$$

(ii) Show for any 2π-periodic *continuous* function f, $\sigma_n(x)$ converges to $f(x)$ for all $x \in \mathbb{R}$. (This is in contrast to the Fourier expansions s_n themselves which do not converge to f for certain continuous functions f.)

(iii) Plot the Fourier expansion and the Cesáro sum for one or more discontinuous functions of your choice. Note that, in particular, the Cesáro sum does not suffer from the Gibbs overshoot phenomenon. (Note: This can be explained by exploiting the expressions for s_n and σ_n in terms of the Dirichlet and Fejér kernels, respectively. Such explanatory analyses are not part of the present assigment.)

5.- Trapezoidal rule; relation to fast convergence of Fourier series of smooth/analytic functions.

(a) Evaluate numerically the integral

$$\frac{1}{2\pi}\int_0^{\pi/4}\exp\left(\frac{1}{\sqrt{2}}\sin(x)\right)dx$$

by means of the composite trapezoidal rule with mesh-sizes $h = \pi/8$, $h = \pi/16$ and a few other uniform discretizations and establish that the integration error is a quantity of order $\mathcal{O}(h^2)$. Give a theoretical rationale explaining this fact.

(b) Repeat the numerical test for the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{1}{\sqrt{2}}\sin(x)\right)dx.$$

In this case the resulting error should be much smaller than the theoretical value $\mathcal{O}(h^2)$.

(c) Show that the trapezoidal rule with $h = 2\pi/(n+1)$ in the interval $[0, 2\pi]$ is exact for all trigonometric polynomials of period 2π of the form

$$\sum_{k=-n}^{n} c_k e^{ikt}.$$

(d) Show that if $f(t)$ can be approximated by a trigonometric polynomial of degree n so that the magnitude of the error is less than $\varepsilon > 0$ for all $t \in [0, 2\pi]$, then the error resulting from use of the trapezoidal rule with $h = 2\pi/(n+1)$ to evaluate

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)dt$$

is less than 2ε.

(e) Let f be a real-analytic periodic function of period 2π. Show that the Fourier coefficients of f converge to zero exponentially fast. [Hint: Consider the integral expression of the Fourier coefficients of f, and deform the integration path appropriately in the complex plane, taking advantage of the fact that the integral of f on certain horizontal rectangles must vanish (as it follows from Cauchy's theorem). The integrals on vertical segments cancel each other out by periodicity.] What can you conclude about the convergence of Fourier series of analytic functions? Demonstrate your result by comparing plots of the function $\exp\left(\frac{1}{\sqrt{2}}\sin(x)\right)$ and its trigonometric Fourier expansions of various orders over the period $0 \le x \le 2\pi$.

(f) Use points (c) and (d) above to explain the sensationally good performance of the trapezoidal rule that you (should have) observed in point (b) above.

6.-

(a) Let $f(x) = x$ for $x \in [0, \pi]$. By extending f to the interval $[-\pi, \pi]$ as an odd function, obtain series expansions of f in terms of sine functions only. This is called the "sine expansion" of f in the interval $[0, \pi]$. Similarly, extending f to the interval $[-\pi, \pi]$ as an even function, obtain a "cosine expansion" f in the interval $[0, \pi]$. Which of these two expansions provides a better approximation of the function f in the interval $[0, \pi]$? To answer this question, estimate the maximum error in the interval $[0, \pi]$ as truncated versions of these expansions are used as approximations to the function f. Provide both a theoretical analysis and numerical demonstrations supporting your conclusions.

(b) Show that the manipulations described in point (a) concerning extensions to a double-length interval together with our previous knowledge that the trigonometric system $T = \{\sin(nx) : n = 1, 2, \dots\} \cup \{\cos(nx) : n = 1, 2, \dots\}$ is complete in $L^2[-\pi, \pi]$, can be used to establish that the set $S = \{\sin(nx) : n = 1, 2, \dots\}$ is a complete set of orthogonal functions in $L^2[0, \pi]$. Similarly, show that the set $C = \{\cos(nx) : n = 1, 2, \dots\})$ is a complete set of orthogonal functions in $L^2[0, \pi]$.

7.- For given $h > 0$ consider the finite-difference differentiation formulae

$$D_+u(\overline{x}) = \frac{u(\overline{x}+h) - u(\overline{x})}{h}, \quad D_-u(\overline{x}) = \frac{u(\overline{x}) - u(\overline{x}-h)}{h}, \quad D_0u(\overline{x}) = \frac{u(\overline{x}+h) - u(\overline{x}-h)}{2h} \quad \text{and}$$

$$D_3u(\overline{x}) = \frac{1}{6h}\left[2u(\overline{x}+h) + 3u(\overline{x}) - 6u(\overline{x}-h) + u(\overline{x}-2h)\right].$$

Using relevant Taylor expansions, show that these expressions provide approximations to the first derivative $u'(\overline{x})$ with errors less than or equal to orders $O(h^p)$ with $p = 1$, 1, 2 and 3, respectively. Using a log-log plot demonstrate in graphical form the convergence rate of the various approximations for e.g. the function $u(x) = \sin(x)$ at $x = 1$, and for $h = 10^{-1}$, $h = 5 \cdot 10^{-2}$, $h = 10^{-2}, \dots$, $h = 10^{-3}$. (Remark: At the point $x = 0$, for the same example $u(x) = \sin(x)$, some of the approximations mentioned are better than suggested by the estimates indicated above. Verify this, and explain the observed behavior.)

Additionally, test numerically the approximations that result from the various finite-difference formulae for much smaller values of h, say $h = 10^{-6}$, $h = 10^{-8}$, $h = 10^{-10}$, $h = 10^{-12}$, etc. Significant deterioration should result. How con we account for the decreased approximation quality?

8.- The solution of the one-dimensional advection equation

$$u_t(x,t) - au_x(x,t) = 0$$

with initial conditions $u(x,0) = f(x)$ is given by

$$u(x,t) = f(x+at).$$

(a) What is the domain of dependence of this PDE? More precisely, given a point (x_0, t_0) in space time with $t_0 > 0$, what are the values of x for which the given initial condition at $(x,0)$ may influence the values of the exact solution at (x_0, t_0)?

(b) Assuming $a > 0$, construct a lowest-order explicit upwind (resp. downwind) finite-difference approximation for this PDE. In detail, with reference to problem 1, an upwind (resp. downwind) method could be designed by approximating the derivatives at a point (t_n, x_n) by using D_+ in time together with D_+ (resp. D_-) in space. Explain the upwind/downwind terminology, and note that the usage of the terms upwind and downwing should be reversed for negative values of a.

(c) Find the numerical domain of dependence of the upwind and downwind methods, and explain why, in view of the Lax-Richtmyer stability and convergence theory mentioned in class, the downwind approach cannot be stable.

(d) For what values of $\Delta t/\Delta x$ can we be certain, on the sole basis of consideration of exact and numerical domains of dependence, that even the upwind method is unstable?

9.- Notation: let $\nu = \frac{ak}{h}$ (where $h = \Delta x = \frac{1}{m+1}$ and $k = \Delta t$). Also, consider grid functions (column vectors) of the form $V = (V_0, V_1, \ldots, V_m)^T \in \mathbb{C}^{m+1}$ endowed with the regular Euclidean scalar product $(V,W)_2 = \sum_{j=0}^{m} V_j \overline{W_j}$ and the associated 2-norm $\|V\|_2 = \left(\sum_{j=0}^{m} |V_j|^2\right)^{1/2}$.

Using these notations, study the stability of the upwwind scheme for the one-dimensional advection equation (Problem 2) in the interval $[0,1]$ with periodic boundary conditions—by utilizing the following procedure (known as "von Neumann stability analysis").

(a) Let $V = V(\xi) = (V_j(\xi))_j$ denote the grid function given by $V_j = e^{ijh\xi}$, $j = 0, \ldots, m$. Show that the grid functions $V(\xi)$ with $\xi = 2\pi\ell$, $\ell = 0, \ldots, m$, are mutually orthogonal with respect to the scalar product $(\cdot,\cdot)_2$.

(b) Show that any grid function $U = (U_0, U_1, U_2, \ldots, U_m)^T$ can be expressed as a linear combination of the finitely many grid functions $V(\xi)$ with $\xi = 2\pi\ell$, $\ell = 0, \ldots, m$. [Hint: use point (a).]

(c) Letting $U_j^n = V_j(\xi)$ evaluate the grid function U^{n+1} at time t_{n+1} that results from time stepping according to the upwind scheme considered in the previous problem. Show that $U^{n+1} = g(\xi)U^n = g(\xi)V_j(\xi)$ for a certain function $g(\xi)$, and explicitly determine the function g.

(d) Show that $|g(\xi)| \leq 1$ for all relevant values of ξ if and only if the condition $0 \leq \nu \leq 1$ is satisfied. (Note that, per Problem 2 above, $0 \leq \nu \leq 1$ if and only if the exact domain of dependence is contained in the numerical domain of dependence.)

(e) Show that the condition $|g(\xi)| \leq 1$ ensures that the scheme is stable. [Hint: use points (a) and (b) to show that the discrete 2-norm $\|U^{n+1}\|_2$ of the grid function U^{n+1} is less than or equal to the 2-norm of U^n—and, in particular, stability in the 2-norm holds (the norm of the numerical solution is uniformly bounded for all n).]

(f) Explain why Point (e) only establishes the stability of the upwind scheme under *periodic boundary conditions.*

(g) Is the downwind scheme similarly stable in the 2-norm?

[Note: The von Neumann stability analysis can only establish stablity under periodic boundary conditions. Study of stability for more general boundary conditions require use of other methods, including matrix eigenvalue analysis (as in the example considered in class); discrete energy conservation methods; and method-of-lines analyis combined with ODE stability theory, among others. Even though, strictly speaking, it only applies to the periodic case, the von Neumann analysis is often a very important base-line indicator of the stability of a scheme. In particular, a failure to satisfy von Neumann stablity is a clear indication that a scheme is not stable.]