Práctica II

1.- Consider the following vector norms defined on $\mathbb{C}^n$:

- $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$
- $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$
- $\|x\|_1 = \sum_{i=1}^n |x_i|$

Show that

(a) $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$; and

(b) $\|x\|_2 \le \sqrt{n}\,\|x\|_\infty$, and $\|x\|_1 \le \sqrt{n}\,\|x\|_2$.

Further,

(c) For $A \in \mathbb{C}^{n \times n}$ derive simple expressions for the induced norms $\|A\|_1$, $\|A\|_\infty$ and $\|A\|_2$.

Finally, for a given matrix $A \in \mathbb{C}^{n \times n}$, establish the following relations:

$$\frac{1}{n}K_2(A) \le K_1(A) \le nK_2(A) \quad , \quad \frac{1}{n}K_\infty(A) \le K_2(A) \le nK_\infty(A)$$

$$\frac{1}{n^2}K_1(A) \le K_\infty(A) \le n^2 K_1(A).$$

(Here $K_p(A)$ denotes the condition number of A in the p-norm $\|\cdot\|_p$, $1 \le p \le \infty$.)

2.- Let

$$P = \begin{bmatrix} .3 & .9 & .5 & .3 & .6 \\ .1 & .2 & .8 & .2 & .2 \\ .4 & .6 & .9 & .3 & .6 \\ .1 & .5 & .7 & .8 & .4 \\ .5 & .3 & .9 & .4 & .3 \end{bmatrix}$$

and

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix}.$$

The elements of P are numbers from the decimal expansion of π. H is the 5×5 Hilbert matrix with $H_{ij} = 1/(i+j-1)$. Also, define

$$b = (.2, .7, .1, .8, .3)^T,$$
$$\delta b = 0.001 \cdot (.1, .7, .3, .2, .1)^T \quad \text{and}$$
$$(\delta A)_{ij} = 10^{-5} \cdot (-1)^{i+j}.$$

Use the Matlab command "$A\backslash b$" to solve the following systems of equations:

$$Px = b \qquad\qquad Hx = b$$
$$Px_1 = b + \delta b \qquad\qquad Hx_1 = b + \delta b$$
$$(P + \delta A)x_2 = b \qquad\qquad (H + \delta A)x_2 = b.$$

Print the computed solution vectors to machine precision in scientific notation (i.e., use Matlab's "format long e" command). Then, using the matlab command "norm", fill the following table:

	Cond	$\frac{\|\delta x_1\|_\infty}{\|x\|_\infty} / \frac{\|\delta b\|_\infty}{\|b\|_\infty}$	$\frac{\|\delta x_2\|_\infty}{\|x\|_\infty} / \frac{\|\delta A\|_\infty}{\|A\|_\infty}$
P	$\kappa_\infty(P)$	p_1	p_2
H	$\kappa_\infty(H)$	h_1	h_2

How well do the numbers in the table conform the estimates presented in class on the effects of perturbations? Note that the Matlab solutions "$A\backslash b$" are computed by means of Gaussian elimination with partial pivoting.

Experiment along the lines described above with $n \times n$ Hilbert matrices (hilb(n) in Matlab) for larger values of n.

3.-

(a) Implement two methods to enact polynomial interpolation in an interval $[a, b]$ (either in Matlab or in your favorite programming language), namely (i) A method based on inversion of the Vandermonde matrix; and (ii) A method based on use of the Lagrange form of the interpolating polynomial. (To obtain the Lagrange polynomial either use the efficient method described in the paper Berrut-Trefethen 2004, or, for simplicity, use a direct implementation of the Lagrange formula.)

(b) Using the Lagrange form of the interpolating polynomial reproduce the results displayed on page 176 of the class notes for the function

$$f(x) = \frac{1}{1 + 16x^2}. \tag{1}$$

What happens if the Vandermonde method is used instead, as the number of points grow?

(c) Compare the performance of the Lagrange and Vandermonde interpolation methods by testing them on the function (1), and any other function of your choice, but, in contrast to point (b), using the Chebyshev points $-\cos(j\pi/n)$, $0 \le j \le n$ in the interval $[-1, 1]$. This test should make it clear which method of interpolation is preferable in a context in which the interpolation problem itself is theoretically convergent (and, indeed, rapidly so!).