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Práctica III

1.- Let f denote a square-integrable 2π periodic function. Determine the values of $\alpha \in \mathbb{R}$ for which the problem

$$\alpha u(x) - u''(x) = f(x), \quad x \in \mathbb{R}$$

with periodic boundary conditions:

$$u(x + 2\pi) = u(x)$$

admits a unique solution.

Let $\alpha = 1$. Using the Fast Fourier Transform algorithm in Matlab to expand the function f in a Fourier series, solve the periodic boundary-value problem described above in the cases $f = f^1$ and $f = f^2$, where f^1 and f^2 are the 2π-periodic functions which coincide with

$$f^1(x) = 10 \sin^3 |x - \pi| - 6 \sin |x - \pi|, \quad \text{and} \tag{1}$$

$$f^2(x) = e^{\sin(x)}(\sin^2(x) + \sin(x)) \tag{2}$$

for $0 \leq x \leq 2\pi$. The solutions are 2π periodic functions given by the closed form expressions

$$u^1(x) = \sin^3 |x - \pi|, \quad \text{and} \tag{3}$$

$$u^2(x) = e^{\sin(x)} \tag{4}$$

for $0 \leq x \leq 2\pi$.

Plot the maximum errors in the numerical solutions as functions of N. Use a log-log plot (resp, a semi-log plot) for the solution corresponding to $f = f^1$ (resp. $f = f^2$). Both error curves should be straight lines. Explain this, and relate the slope of the error curves to the degree of smoothness of the function f (and u) in each case.

How large should N be, in each case, in order to produce the solution with an error of the order of 10^{-4}?

2.- Consider an equation of the form

$$\partial_t u(x, t) - \mathcal{L}[u](x, t) = \mathcal{N}[u](x, t), \quad t > 0, \quad x \in \Omega, \tag{5}$$

where Ω is a d-dimensional "spatial" domain ($d = 1, 2, 3, \dots$), where $\mathcal{L}$ is the "leading" (linear) spatial differentiation operator (that is, a linear operator that contains all derivatives of the highest order in the equation), and where $\mathcal{N}$ is a linear or nonlinear operator containing only spatial derivatives of lower order, and which does not contain any temporal derivatives. Naturally, boundary conditions of some sort, which generally depend in type on the corresponding PDEs themselves, are to be prescribed.

We propose the Crank-Nicolson leap-frog time semi-discretization scheme

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} - \mathcal{L}\left[\frac{u^{n+1} + u^{n-1}}{2}\right] = \mathcal{N}\left[u^n\right], \quad n \geq 1 \tag{6}$$

for equation (5). (A fully discrete numerical scheme for (5) can then be obtained by incorporating an additional *spatial* discretization into (6).)

(a) What is the order of the truncation error for the time discretization (6)?

(b) Note that, in spite of the fact that only an initial condition for u is necessary at time $t = 0$ in (5), the semidiscrete scheme (6) requires initial conditions at times t_0 and t_1. How can this difficulty be addressed?

(c) Note that the scheme (6) is *implicit*—that is, the solution at time t_{n+1} is utilized to approximate the spatial differential operator, and thus, a system of equations needs to be solved at each time-step. As discussed in class, implicit methods tend to yield improved stability, and are sometimes subject to less-strict restrictions on the usable values of the time-step Δt for stability.

[NOTE: The use of an implicit discretization for the linear operator (which is more easily invertible than the nonlinear counterpart) and an explicit discretization for the nonlinear operator (which is assumed to contain lower-order derivatives only) is intended to lessen the time-step restrictions for stability, which often arise from the highest derivatives in the spatial operator. The situation is illustrated in problem 3 below.]

3.- Differentiation in physical and frequency space.

(a) Following the discussion in Section 2.1.3 of STW construct the Fourier-spectral differentiation matrices for differentiation of first and second order. Demonstrate the accuracy of these differentiation methods by applying them to the (2π-periodic) function $f(x) = e^{\cos(x)}$.

(b) Repeat the exercise but applying the FFT "frequency-space" procedure described in Section 2.1.4 of STW.

(c) Compare the accuracy and speed provided by the approaches in points (a) and (b) by applying them to the function $f(x) = e^{\cos(kx)}$ for selected values of k. In order to notice significant speed differences you may need to use sufficiently large discretizations, which may only be meaningful (in terms of the accuracy they provide) for sufficiently large values of k.

4.- Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}.$$

is the natural one-dimensional-one-unknown Navier-Stokes analog.

(a) Use the Crank-Nicolson leap-frog scheme described in pbm. 1 above and the Fourier-spectral method in space solve the Burgers equation up to time $t = 1$. Assume $\varepsilon = 0.03$ as well as the 2π-periodic initial condition

$$u(x, 0) = e^{-10\sin^2(x/2)}$$

and 2π-periodic boundary conditions. Plot the solution at several points of time.

[Hint: to debug your code you can use a "manufactured solution" U. In the present case a MS is a function $U = U(x, t)$ that is arbitrarily prescribed by you, and for which you add the necessary right-hand term as well as well as the correct initial and boundary conditions, as applicable, so that U is a solution of the equation. You can use this exact solution to check for correctness and thus help you debug your code, if necessary.]

(b) Demonstrate how, on the basis of convergence studies, the accuracy of the solution produced in any given run can be estimated. Obtain a sufficiently fine spatio-temporal dicretization to guarantee an accuracy of 10^{-2} for all $t \leq 1$ and all points in space. How expensive is it to obtain an additional one or two correct digits?