Práctica V

1.- Consider the well known expression

$$E_1(t) = (t-a)^2 \int_0^1 (1-r) f''(r(t-a)+a) dr \tag{1}$$

for the remainder E_1 in the first order Taylor expansion

$$f(t) = f(a) + f'(a)(t-a) + E_1(t)$$

of a twice continuously differentiable function f in a neighborhood of the point $t = a$. Using this expansion and the form (1) of the remainder show that, as mentioned in class, for a twice continuously differentiable curve $S \subset \mathbb{R}^2$, the kernel

$$K(x,y) = -\frac{(x-y)\cdot \nu(y)}{2\pi |x-y|^2} \quad (x, y \in S)$$

in the double-layer operator T_K is in fact a *continuous function of* x, y for $x, y \in S$. [HINT: Up to a rotation you may use a parametrizations of the form $x = (s, f(s))$, $y = (t, f(t))$ for points x and y on the curve S in a neighborhood of any given point $x_0 \in S$. Taylor expand $f(s)$ around $s = t$.] Additionally, show that, as indicated in problem 3(a) in HW Set VIII, the kernel K is infinitely differentiable if the curve S is itself infinitely differentiable.

2.- By direct calculation show that the double layer potential

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{\partial}{\partial \nu_y} \log(|x-(s,0)|) ds$$

indeed jumps by an amount consistent with the calculations presented in class as $x = (x_1, x_2)$ crosses the open segment $\{(s,0) : -1 < s < 1\}$.

3.-

a) Let

$$N(x,y) = \frac{1}{2\pi} \log(|x-y|)$$

for $(x, y) \in \mathbb{R}^2$, and let

$$\Gamma = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = R^2 \right\}$$

Find the kernels of the integral equations of the second kind associated with the Dirichlet and Neumann problems for Laplace's equation in the interior of Γ, and note that these kernels are in fact *constant*.

b) Write down integral equations on Γ for the interior Dirichlet and Neumann problems (D_i and N_i).

c) Solve the integral equation for D_i. Write the corresponding formula for the solution u of the problem D_i, and show that it coincides with Poisson's integral formula formula (Theorem 2.48 p. 97 in Folland's text).

d) Solve the integral equation for N_i (up to an element in the nullspace of the left-hand operator which, in the context of the present problem, is the space of constant functions on Γ). Derive Dini's formula for the solution u of the interior Neumann problem:

$$u(x) = \frac{R}{\pi}\int_{-\pi}^{\pi} \psi(w)\log\left(\frac{1}{r}\right) dw + C$$

(C constant), where ψ is related to the boundary Neumann data $f(x)$ via

$$\psi(w) = f(R\cos(w), R\sin(w)).$$

4.- Let K be a 2π periodic function of a real variable z whose restriction to the interval $[0, 2\pi]$ is square integrable (that is $K \in L^2[0, 2\pi]$), and let T_K denote the integral operator

$$T_K[\varphi](x) = \int_0^{2\pi} K(x-y)\varphi(y)dy. \tag{2}$$

Show that T_K defines a continuous linear operator from $L^2[0, 2\pi]$ to $L^2[0, 2\pi]$. [Hint: Use Cauchy-Schwarz.]

In what follows we consider the first-kind integral equation

$$T_K[\varphi] = f \tag{3}$$

as well as the second kind integral equation

$$(I + T_K)[\varphi] = g, \tag{4}$$

where $f, g \in L^2[0, 2\pi]$. The Fourier series of K will be denoted by

$$K(z) = \sum_{j=-\infty}^{\infty} K_j e^{ij\,z}. \tag{5}$$

(a) Show that, for any operator A (examples of which are $A = T_K$ and $A = (I + T_K)$), the injectivity of A is a necessary condition for its invertibility. [Easy!]

(b) Show that the integral operator in equation (3) is injective if and only if the Fourier series of K is "complete" in the sense that none of the Fourier coefficients K_j vanish. What is the corresponding condition for equation (4)?

(c) Assuming that the relevant completeness condition mentioned in point (b) for equation (3) holds, is it true that equation (3) admits a solution $\varphi \in L^2[0, 2\pi]$ for every $f \in L^2[0, 2\pi]$? [Hint: the answer is no!] Are there solutions $\varphi \in L^2[0, 2\pi]$ for some $f \in L^2[0, 2\pi]$? If so, are such solutions unique? Explain.

(d) Same as point (c) but for equation (4): Assuming that the relevant injectivity condition mentioned in point (b) for equation (4) holds, is it true that this equation admits a solution $\varphi \in L^2[0, 2\pi]$ for every $g \in L^2[0, 2\pi]$? Are the solutions unique?

[This problem illustrates, in a simple example, how integral equations of the second kind can be very different in character from integral equations of the first kind. In general, second-kind equations

$(I+T)[\varphi] = g$ with T compact are uniquely solvable provided the operator $(I+T)$ is injective. This is not strictly true of first-kind equations, as indicated by point (c) above, but the situation is somewhat subtle. Confer with an instructor if you would like more details in this regard.]

5.- (a) Design and implement a numerical algorithm along the lines described in class, and based on use of the double-layer operator

$$D[\varphi](x) = \frac{1}{2\pi} \int_\Gamma \frac{\partial}{\partial \nu_y} \log(|x-y|)\varphi(y) d\ell_y$$

together with discretization by means of the trapezoidal rule, for the solution of the interior Dirichlet problem for the Laplace equation in a domain $\Omega \subset \mathbb{R}^2$ bounded by an infinitely differentiable curve Γ. [Hint: recall that, per problem 1 in Problem Set VIII, the kernel $\frac{\partial}{\partial \nu_y} \log(|x-y|)$ is a smooth function of x and y for $x, y \in \Gamma$ provided Γ is itself smooth.]

(b) Using the parameter t, $0 \le t \le 2\pi$, consider the curves Γ_1, Γ_2 and Γ_3, given by parametrizations $C_1(t) = (\cos(t), \sin(t))$ (circle), $C_2(t) = (\cos(t) + 0.65\cos(2t) - 0.65, \sin(t))$ (kite), and $C_3(t) = (1 + 0.3\cos(5t))(\cos(t), \sin(t))$ (star), respectively. Solve the Laplace Dirichlet problem inside each one of these curves for some non-trivial Dirichlet boundary values of your choice. In order to verify your code use at first boundary values given by the real part u of some simple but nontrivial holomorphic function, for which the solution is the function u itself, and compare the exact results to the numerical solutions. Considering the results obtained in HW Set II on the properties of the trapezoidal rule, we may expect very fast convergence in these numerical solutions. Verify that the expected fast convergence is indeed observed.