From the Schroedinger Equation to the Dirac Equation

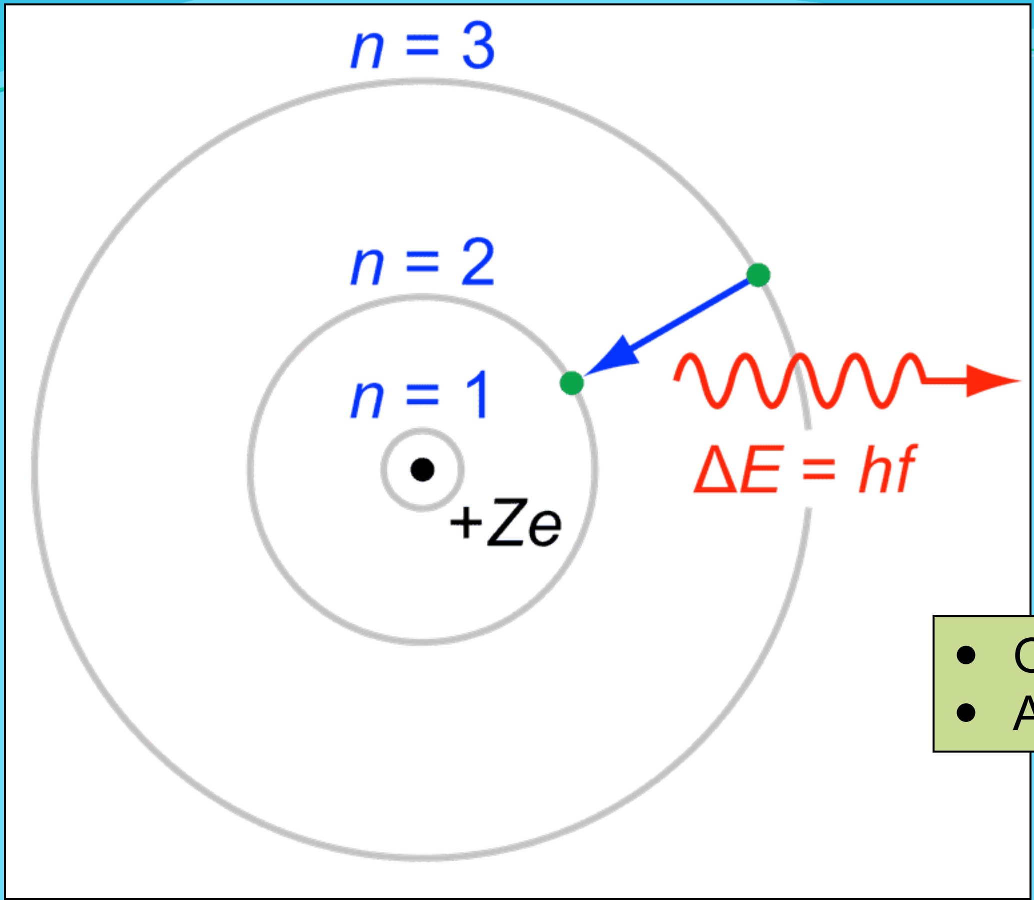
1. From the Schroedinger Equation to the Dirac Equation
2. The discovery of the positron and the antiproton
3. C, P, T symmetries
4. The Standard Model
5. Modern Cosmology
6. Matter-Antimatter asymmetry in the Universe

A crisis in classical physics
Some of the milestones:

- 1900: Planck and the Blackbody Radiation problem
- 1905: Einstein and the Photoelectric Effect
- 1913: Bohr model of the Atomic system
- 1915: Rutherford model of the Atom
- 1915: Moseley experiment

Here is where our story starts. With the emergence of new concepts.

- 1932: Discovery of the Neutron (validation of the modern Atom model)



The Bohr (Rutherford) «planetary» model of the atom

- Only some allowed orbits
- Angular momentum quantization

$$L = r\,p = n\hbar$$

PROs

- It is attractive
- Compatible with Rutherford Atom
- It explains the spectral lines

CONs

- It is against the laws of physics
- Not explaining fine structures and other (relativistic) features

A «deduction» of the Schroedinger Equation
The introduction of wave-particle concepts

$$\psi = Ae^{i(kr-\omega t)}$$

A «wave function». A solution of a wave equation

$$\frac{\partial^2\psi}{\partial x^2} = -k^2\psi = -\frac{4\pi^2}{\lambda^2}\psi$$

$$\frac{\partial\psi}{\partial t} = -i\omega\psi = -i2\pi\nu\psi$$

De Broglie: $\lambda = h/p$

Einstein: $E = h\nu$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{4\pi^2 p^2}{h^2}\psi$$

$$\frac{\partial\psi}{\partial t} = -i2\pi\frac{E}{h}\psi$$

A wave function that incorporates corpuscolar concepts

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi^2 p^2}{h^2}\psi$$

$$\frac{\partial \psi}{\partial t} = -i\,2\pi \frac{E}{h}\psi$$

A wave function that incorporates corpuscolar concepts

$$E = \frac{p^2}{2m} + V \qquad p^2 = 2m(E - V)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{2m(E - V)}{\hbar^2}\psi$$

$$\frac{\partial \psi}{\partial t} = -i\frac{E}{\hbar}\psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi$$

$$i\hbar\frac{\partial \psi}{\partial t} = E\psi$$

$$i\hbar\frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi$$

Schroedinger (1926) Equation

Particle-like quantities and Wave-like quantities. Crossing the boundary

	Particles	Waves
Energy E	$\frac{\vec{p}^2}{2m}, \sqrt{\vec{p}^2+m^2}, \gamma mc^2$	$E = h\nu$
Frequency, Wavelength ν , λ	$\lambda = \frac{h}{\lvert\vec{p}\rvert}$	$\lambda\nu = c$

Einstein (1905) : waves behave like particles with a given energy , (confirmed by the Compton effect)

De Broglie (1924) : particles also have a wavelength like waves , (confirmed by the Davisson & Germer experiment)

	Particles	Waves
Energy E	$\frac{\vec{p}^2}{2m}, \sqrt{\vec{p}^2+m^2}, \gamma mc^2$	$\nu = E/h$
Frequency, Wavelength ν , λ	$\lambda = \frac{h}{\lvert\vec{p}\rvert}$	$\lambda\nu = c$

Is it all consistent?

For waves : $\nu = E/h$, $\lambda\nu = c$ → $c/\lambda = E/h$.

Using the relativistic dispersion law: $E = \lvert\vec{p}\rvert c$ ⇒ $\lvert\vec{p}\rvert = h/\lambda$

We can consistently attribute Energy and Momentum to the photon.

Now, what about massive particles ?

	Particles	**Waves**
Energy E	$\frac{\vec{p}^{\,2}}{2m}, \sqrt{\vec{p}^{\,2}+m^2}, \gamma mc^2$	$\nu = E/h$
Frequency, Wavelength ν , λ	$\lambda = \frac{h}{p}$	$\lambda\nu = c$

For a massive particle : $\lambda = h/p$ ⟹ $\lambda = h/(mv\gamma)$

Using the wavelike equation $\nu = \frac{E}{h} = \frac{\gamma mc^2}{h}$

Note: this attributes a frequency also to a particle at rest !

⟹ $$\lambda\nu = \frac{h}{mv\gamma}\frac{\gamma mc^2}{h} = c\frac{c}{v} = \frac{c}{\beta} = v_P > c$$

Phase velocity. A massive particle is not just a De Broglie wave: it is a wavepacket.

in this slide $p = |\vec{p}|$

The particle velocity is the GROUP velocity

$$\nu = E/h \rightarrow E = \hbar\omega$$
$$\lambda = h/p \rightarrow p = \hbar k$$

Given a dispersion relation:

$$E = E(p)$$
$$\omega = \omega(k)$$

$$\Rightarrow \quad v_G = \frac{\partial \omega}{\partial k} = \frac{\partial E}{\partial p}$$

For a non-relativistic particle :

$$v_G = \frac{\partial E}{\partial p} = \frac{\partial}{\partial p}\left(\frac{p^2}{2m}\right) = \frac{p}{m}$$

For a relativistic particle :

$$v_G = \frac{\partial E}{\partial p} = \frac{\partial}{\partial p}\left(\sqrt{p^2c^2 + m^2c^4}\right) =$$
$$\frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} = \frac{pc^2}{E} = \frac{pc^2}{m\gamma c^2} = \frac{p}{m\gamma}$$

in this slide $p = |\vec{p}|$

Reconsidering angular momentum (Bohr) quantization in the light of de Broglie hypotesis

Bohr's Quantization Condition / standing waves

- Bohr's crucial assumptions concerning his hydrogen atom model was that the angular momentum of the electron-nucleus system in a stationary state is an integral multiple of $h/2\pi$.
- One can justify this by saying that the electron is a standing wave (in an circular orbit) around the proton. This standing wave will have nodes and be an integral number of wavelengths.

$$2\pi r = n\lambda = n\frac{h}{p}$$

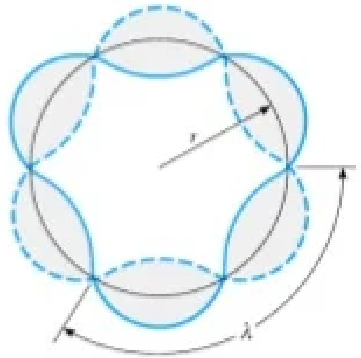


Figure 5.2 Standing waves fit to a circular Bohr orbit. In this particular diagram, three wavelengths are fit to the orbit, corresponding to the $n = 3$ energy state of the Bohr theory.

- The angular momentum becomes:

$$L = rp = \frac{nh}{2\pi} = n\hbar$$

Which is identical to Bohr's crucial assumption

Linear momentum is quantized as well, how come ? because total energy is quantized in bound systems !!

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Direct tests of wave-like nature of particles :

- Electrons) C.J. Davisson, L.H. Germer, *Proc. Natl. Acad. Sci. 14 (1928) 317*.

- Electrons) G.P. Thomson, A. Reid, *Nature 119 (1927) 890*.

- Neutrons) A.V. Overhauser, R. Colella, *Phys. Rev. Lett. 33 (1974) 1237*.

- Single electrons) P.G. Merli, G.G. Missiroli, G. Pozzi, *Am. J. Phys. 44 (1976) 306.*

- Positrons) I.J. Rosberg, A.H. Weiss, K.F. Canter, *Phys. Rev. Lett. 44 (1980) 1139*.

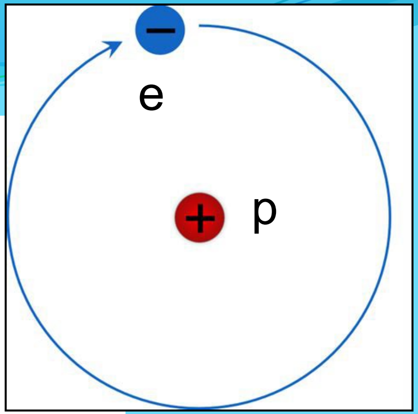
- Single Neutrons) A. Zeilinger, R. Gaehler, C.G. Shull, W. Treimer, W. Mampe, *Rev. Mod. Phys. 60 (1988) 106*.

- Potassium) J.F. Clauser, S. Li, *Phys. Rev. A 49 (1994) R2213*.

- Single C60) M. Arndt, O. Nairz, J. Vos-Andreae, C. Keller, G. van der Zouw, A. Zeilinger *Nature 401 (1999) 680*.

- Single Positrons) S. Sala, A. Ariga, A. Ereditato, R. Ferragut, M. Giammarchi, M. Leone, C. Pistillo, P. Scampoli, *Science Adv. 5 (2019) eaav7610*.

Schroedinger Equation for the Hydrogen Atom



Coulomb energy

total energy

$$\left(\frac{mv^2}{2} - \frac{e^2}{4\pi\varepsilon_0\, r}\right)\psi_H = \mathbf{E}\,\psi_H$$

kinetic energy

constant !

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi_H + \left(-\frac{e^2}{4\pi\varepsilon_0\, r}\right)\psi_H = \mathbf{E}\,\psi_H$$

de Broglie

Hydrogen wavefunction

Another way to deduce the Schroedinger Equation: the operatorial formalism

Deriving the Schrodinger Equation using operators

The energy is: $E = K + V = \frac{p^2}{2m} + V \quad \Rightarrow E\Psi = \frac{p^2}{2m}\Psi + V\Psi$

Non-relativistic

Substituting operators:

E: $E\Psi = i\hbar \frac{\partial \Psi}{\partial t}$

$E \rightarrow i\hbar \frac{\partial}{\partial t}$

$K+V$: $\frac{p^2}{2m}\Psi + V\Psi = \frac{1}{2m}\left(-i\hbar \frac{\partial}{\partial x}\right)^2 \Psi + V\Psi$

$p_i \rightarrow -i\hbar \frac{\partial}{\partial x^i}$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

Substituting: $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$

Schroedinger non-relativistic

Description of a state, evolution, measurement

Postulates of Quantum Mechanics

1. Normalized **ket vector** $|\Psi\rangle$ contains all the information about the state of a quantum mechanical system.

2. **Operator** ***A*** describes a physical observable and acts on kets.

3. One of the **eigenvalues** $\boldsymbol{a_n}$ of A is the only possible result of a measurement.

4. The **probability** of obtaining the eigenvalue a_n : $P = |\langle a_n|\psi\rangle|^2$

5. **State vector collapse** : $|\psi'\rangle = \dfrac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$

6. **Schrödinger Equation** : $i\hbar\dfrac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$

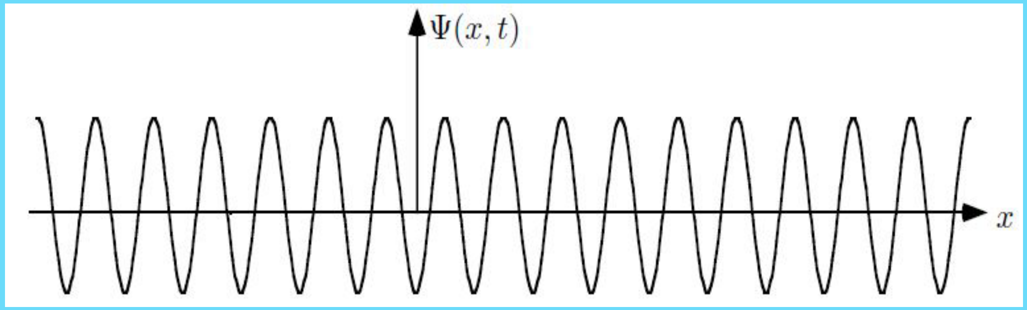
Time evolution of a quantum system

Uncertainty Principle(s)

It all starts by the idea of representing particles with waves!
This makes an Uncertainty Principle unavoidable

Fourier Theorem

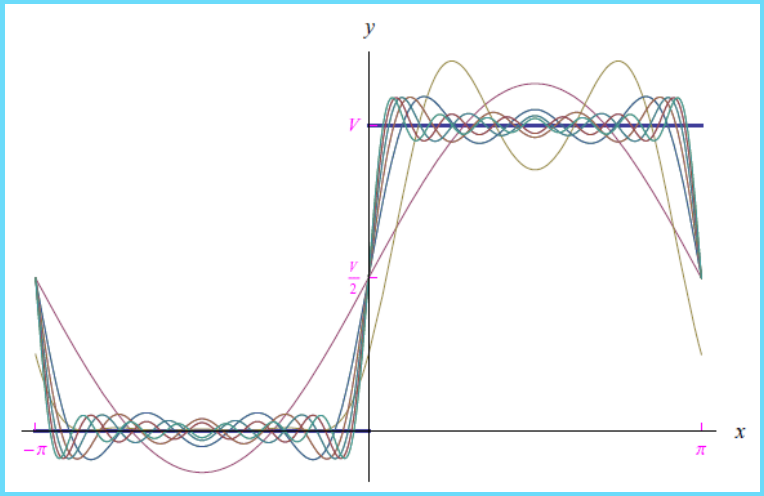
An ideal monochromatic wave to represent a particle?



$$\Delta(h/\lambda)\Delta x = (0)\,(\infty)$$

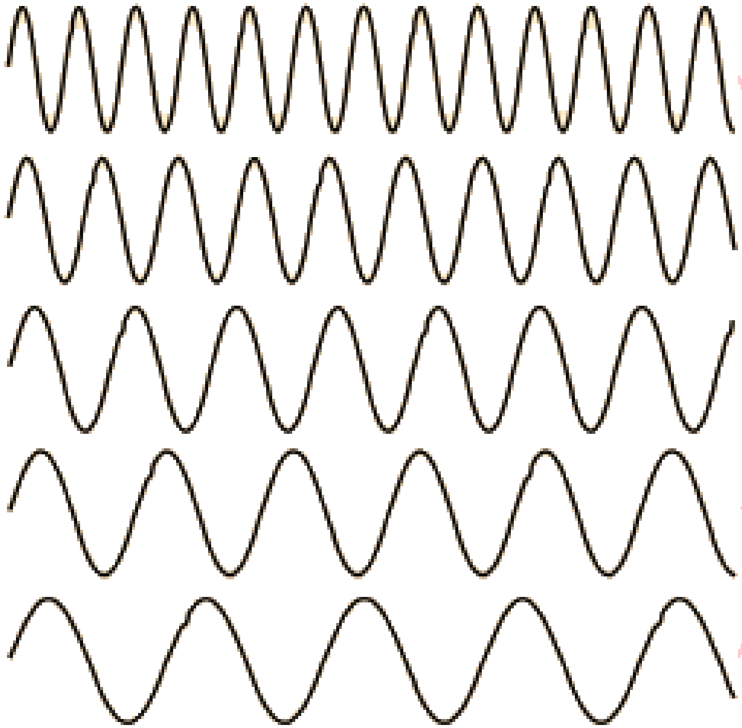
$$\Delta p\,\Delta x = (0)\,(\infty)$$

To «localize» the particle, one has to use many plane waves

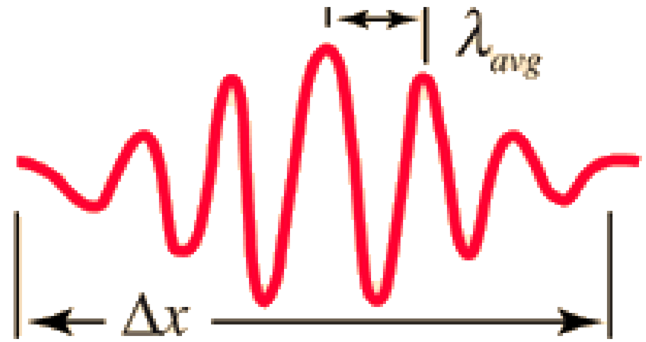


$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Fourier Transform



Adding several waves of different wavelength together will produce an interference pattern which begins to localize the wave.



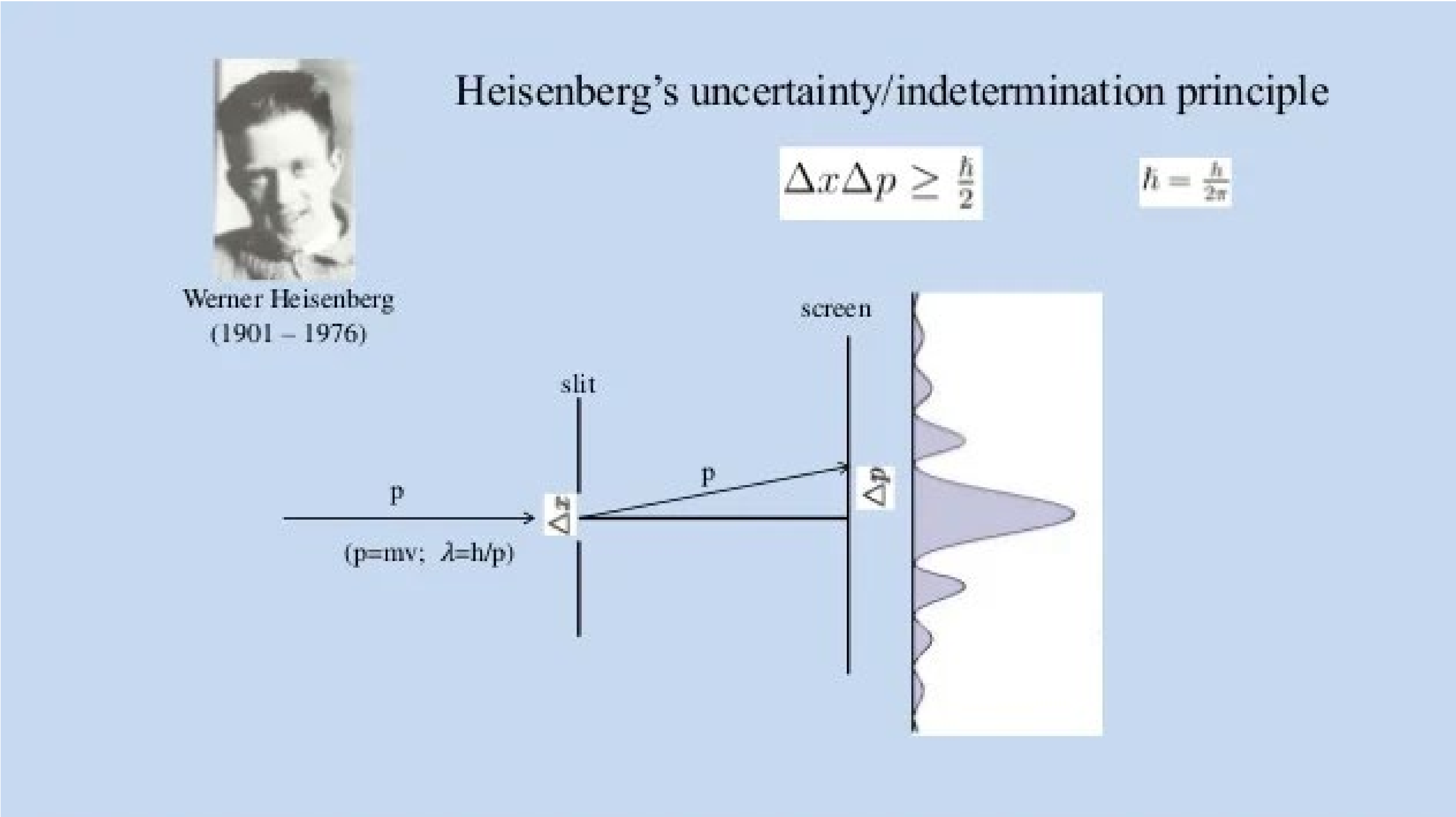
But that process spreads the wave number k values and makes it more uncertain. This is an inherent and inescapable increase in the uncertainty Δk when Δx is decreased.

$$\Delta k \Delta x \approx 1$$

If this is to represent a particle, the constant should include the quantum scale

$$\Delta p \, \Delta x \geq \frac{1}{2} \hbar$$

Interference / diffraction experiments ?



Explains the formation of diffraction pattern of particles (even single particles) !

Uncertainty Principle(s)

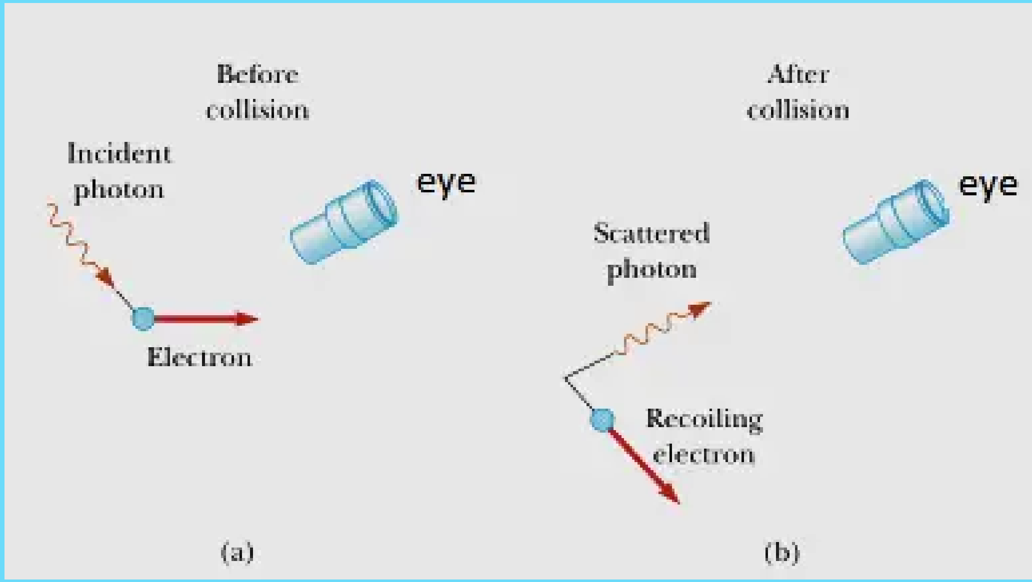
Conjugate variables | **Operatorial structure**

$$\Delta x\,\Delta p_x \geq \frac{\hbar}{2} \qquad \left[x, p_x\right] = i\hbar$$

$$\Delta E\,\Delta t \geq \hbar$$

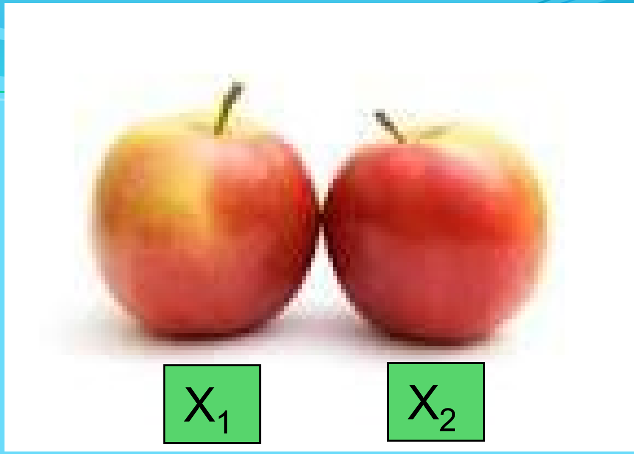
t not a dynamical variable represented by an hermitian operator

$$\Delta\Phi\,\Delta n \geq \hbar$$



Uncertainty relations are often depicted as (unavoidable) perturbations upon measuring a quantum system

Quantum Statistics: Particles are not Apples

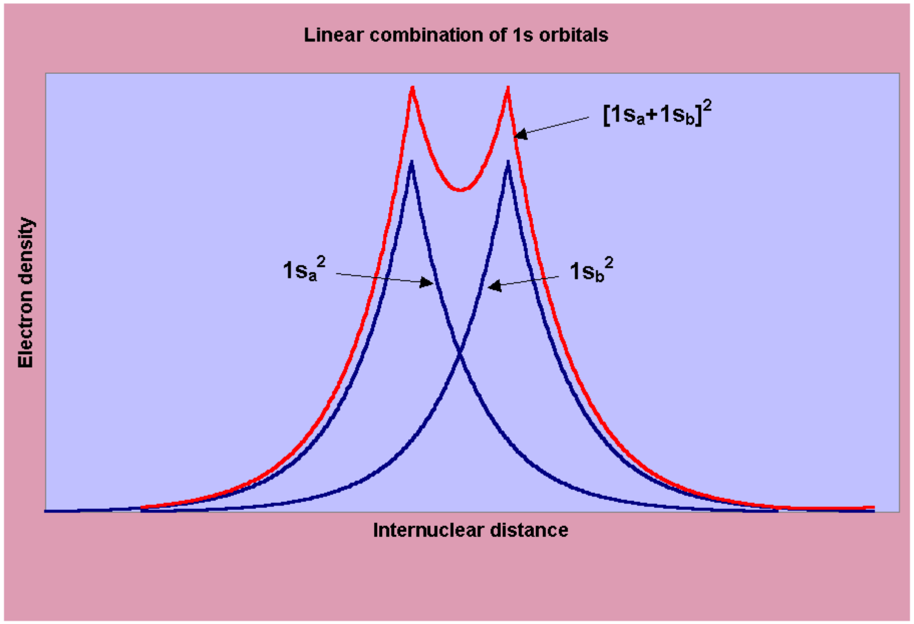


Why are these two apples distinguishables ?

Because we can assign coordinates to them !

Classical Particles (like apples) can be distinguished → Boltzmann Statistics

But we cannot assign coordinates to Quantum Particles ! They cannot be distinguished



Quantum Particles cannot be distinguished

They obey a quantum statistics

For a single-valued many-particle wave function

The wave function must have the correct symmetry under interchange of identical particles. If 1, 2 are identical particles :

$$\left|\psi(x_1, x_2)\right|^2 = \left|\psi(x_2, x_1)\right|^2$$

(probability must be conserved upon exchange of identical particles)

$$\psi \rightarrow +\psi \qquad (1 \leftrightarrow 2)$$

Identical Bosons (symmetric)

$$\psi \rightarrow -\psi \qquad (1 \leftrightarrow 2)$$

Identical Fermions (antisymm.)

} Spin-Statistics Theorem

A consequence of the Spin/Statistics Theorem: for two identical Fermions 1,2 in the same quantum state x:

$$\psi(x_1,x_2) = \psi(x_2,x_1) = -\psi(x_1,x_2) \Rightarrow \psi(x_1,x_2) = 0$$

Because identical

Spin/Statistics Theorem

Pauli Exclusion Principle!

Towards the Dirac Equation

Back to the «demonstration» of the Schroedinger Equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V\psi=E\psi$$

$$i\hbar\frac{\partial\psi}{\partial t}=E\psi$$

$$E=\frac{p^2}{2m}+V$$

A classical potential law

A relativistic dispersion relation

$$E^2=p^2c^2+m^2c^4$$

$$i\hbar\frac{\partial\psi}{\partial t}=\sqrt{p^2c^2+m^2c^4}\;\psi$$

$$\sqrt{\hat{p}^2c^2+m^2c^4}=\alpha_1\hat{p}_x+\alpha_2\hat{p}_y+\alpha_3\hat{p}_z+\beta mc$$

A linearization that can work only if α_i β are 4x4

Towards the Dirac Equation: the Gamma Matrices notation

Gamma Matrices

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{\hbar c}{i}\left(\alpha_1\frac{\partial\psi}{\partial x_1}+\alpha_2\frac{\partial\psi}{\partial x_2}+\alpha_3\frac{\partial\psi}{\partial x_3}\right)+\beta mc^2\psi$$

Rewrite:
$$i\hbar\left(\gamma^0\frac{\partial}{c\partial t}+\gamma^1\frac{\partial}{\partial x_1}+\gamma^2\frac{\partial}{\partial x_2}+\gamma^3\frac{\partial}{\partial x_3}\right)\psi-mc\psi=0$$

$$\gamma^0=\beta=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\qquad \gamma^i=\beta\alpha_i=\begin{bmatrix}0&\sigma_i\\-\sigma_i&0\end{bmatrix}$$ (Not Hermitian)

$$\left(\gamma^0\right)^2=1\qquad \left(\gamma^i\right)^2=-1$$

$$\left\{\gamma^\mu,\gamma^\nu\right\}=2g^{\mu\nu}I$$

Feymann slash notation: $\not{A}=\gamma^\mu A_\mu$ $\quad\not{p}=\gamma^\mu p_\mu$ $\quad\not{\partial}=\gamma^\mu\partial_\mu\equiv\not{\nabla}$

$$\left(i\hbar\not{\partial}-mc\right)\psi=0$$

Lorentz-invariant

The Dirac Equation

Schroedinger Non Rel equation: first order in time, second order in space
Klein-Gordon equation: second order in space/time. It describes relativistic scalar particles (in the modern interpretation of negative-energy solutions).

Dirac Equation : Relativistic equation first order in time and first order in space

$$i\frac{\partial}{\partial t}\psi = \left[-i\sum_k \alpha_k \frac{\partial}{\partial x_k} + \beta m\right]\psi$$

Requiring consistency with the relativistic dispersion relation (iterate the Dirac equation and require the Klein-Gordon condition) implies that α's and β are 4x4 matrices.

Setting :

$$\gamma^0 = \beta \qquad \gamma^k = \beta\alpha^k$$

One has the covariant form of the Dirac equation :

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

A 4-component spinor :

$$\psi = \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}$$

The discovery of the positron and the antiproton

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Schroedinger non relativistic

$$i\hbar\frac{\partial\psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+V\right)\psi$$

Dirac (relativistic)

$$i\frac{\partial}{\partial t}\psi=\left[-i\sum_k\alpha_k\frac{\partial}{\partial x_k}+\beta m\right]\psi$$

Let us know show that the Dirac equation contains some relevant Physics:

- Continuity equation (fermion propagation)
- Neutrino properties

$$\left(i\gamma^\mu\partial_\mu-m\right)\psi=0$$

Electromagnetism continuity equation

Charge Current Continuity Equation

Current I out of a volume is equal to rate of decrease of charge Q contained in that volume:

$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

$$\oint_S \mathbf{J}\cdot d\mathbf{s} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

$$\oint_S \mathbf{J}\cdot d\mathbf{s} = \int_{\mathcal{V}} \nabla\cdot\mathbf{J}\, d\mathcal{V} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

Used Divergence Theorem

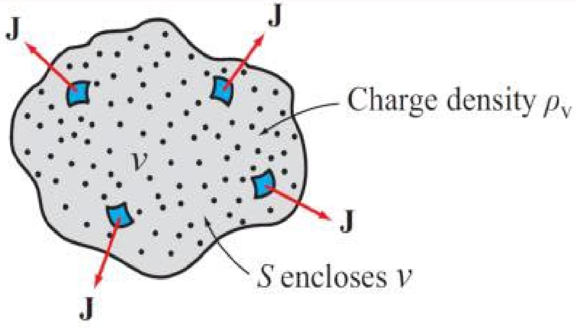

Figure 6-14: The total current flowing out of a volume $\mathcal{V}$ is equal to the flux of the current density **J** through the surface S, which in turn is equal to the rate of decrease of the charge enclosed in $\mathcal{V}$.

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_{\mathrm{v}}}{\partial t}, \qquad (6.54)$$

which is known as the ***charge-current continuity relation***, or simply the ***charge continuity equation***.

Recall the Schroedinger continuity equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t} \quad\Rightarrow\quad -\frac{\hbar^2}{2m}\psi^*\nabla^2\psi = i\hbar\psi^*\frac{\partial\psi}{\partial t}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t} \Rightarrow\quad -\frac{\hbar^2}{2m}\psi\nabla^2\psi^* = -i\hbar\psi\frac{\partial\psi^*}{\partial t}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = i\hbar\frac{\partial}{\partial t}|\psi|^2$$

$$|\psi|^2 \equiv \psi^*\psi$$

$$\psi^*\nabla^2\psi - \psi\nabla^2\psi^* \equiv \underline{\nabla}\bullet\left(\psi^*\underline{\nabla}\psi - \psi\underline{\nabla}\psi^*\right)$$

$$\frac{\partial}{\partial t}|\psi|^2 + \underline{\nabla}\cdot\left(\frac{\hbar}{2im}\left(\psi^*\underline{\nabla}\psi - \psi\underline{\nabla}\psi^*\right)\right) = 0$$

ρ $\qquad \underline{J}$

Electron: $\quad \rho = e|\psi|^2 \qquad \underline{j} = \rho\underline{v} = \frac{\hbar}{2im}\left(\psi^*\underline{\nabla}\psi - \psi\underline{\nabla}\psi^*\right)$

$$\|\psi\|^2 = \iiint|\psi|^2\, dV = 1 \qquad e\iiint|\psi|^2\, dV = e$$

Dirac continuity equation

$$\gamma^0\gamma^0 = 1 \qquad \gamma^{k+} = \gamma^0\gamma^k\gamma^0$$

$$\overline{\psi} = \psi^+\gamma^0 \qquad J^\mu = \overline{\psi}\,\gamma^\mu\,\psi$$

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0$$

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0$$

$$\left[i\gamma^\mu\partial_\mu\psi\right]^+ - m\psi^+ = 0$$

$$-i\left[\partial_\mu\psi\right]^+\gamma^{\mu+} - m\psi^+ = 0$$

$$\left(-i\,\partial_\mu\psi^+\gamma^0\gamma^\mu\gamma^0 - m\psi^+\right)\gamma^0 = 0$$

$$-i\,\partial_\mu\overline{\psi}\gamma^\mu - m\overline{\psi} = 0$$

Dirac equation

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0$$

$$i\,\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} = 0$$

Adjoint Dirac equation

$$\overline{\psi}\left[i\gamma^\mu\partial_\mu\psi - m\psi\right] = 0 \quad \rightarrow \quad i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi = 0$$

$$\left[i\,\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi}\right]\psi = 0 \quad \rightarrow \quad i\left(\partial_\mu\overline{\psi}\right)\gamma^\mu\psi + m\overline{\psi}\psi = 0$$

$$\partial_\mu\left[\overline{\psi}\gamma^\mu\psi\right] = 0$$

The two-component theory of the (massless) Neutrino

The spin-1/2 pointlike particle wave function obeys the Dirac Equation :

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0$$

Four components : two spin states of particle
two spin states of antiparticle

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}$$

- Massive particle: both spin states must be described by the same wavefunction because the spin direction is not Lorentz-invariant.
- Massless particle: it always travel at the speed of light, so its spin direction can be defined in a Lorentz-covariant way (parallel or antiparallel to the direction of the momentum, i.e. positive or negative helicity).

In the Weyl representation of the Gamma Matrices:

$$\gamma^0 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}$$

Introducing the bispinors (upper and lower components) :

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0$$

$$-i\frac{\partial u}{\partial t} - i\vec{\sigma}\vec{\nabla}u = mv$$

$$-i\frac{\partial v}{\partial t} + i\vec{\sigma}\vec{\nabla}v = mu$$

Dirac Equation in the Weyl representation

For a massles fermion, the upper and lower components are decoupled :

$$i\frac{\partial u}{\partial t} = -i\vec{\sigma}\vec{\nabla}u$$

$$i\frac{\partial v}{\partial t} = +i\vec{\sigma}\vec{\nabla}v$$

For a massles particle, E= p

$$E = +\vec{\sigma}\vec{p} \quad for \;\; u \quad \Rightarrow \quad \vec{\sigma}\vec{p} \approx +\vec{p}$$

$$E = -\vec{\sigma}\vec{p} \quad for \;\; v \quad \Rightarrow \quad \vec{\sigma}\vec{p} \approx -\vec{p}$$

$$\psi_L = \begin{pmatrix} u \\ 0 \end{pmatrix}$$

Left-handed spinor

$$\psi_R = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

Right-handed spinor

A snapshot on Quantum Relativistic Equations

How to obtain a quantum mechanical wave equation? One simple way recipe:

- Take a dispersion relation. For instance, a classical one: $E = \frac{\vec{p}^2}{2m}$
- Make the transition to operators. Energy and momentum become operators acting on a state living in a suitable Hilbert space : $\hat{E}\,\psi = \frac{\hat{\vec{p}}^2}{2m}\psi$
- Use the appropriate form for the operators

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \qquad \vec{p} \rightarrow -i\hbar\nabla \quad \Longrightarrow \quad i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$$

The non-relativistic Schroedinger equation !

The Klein-Gordon Equation

By analogy, taking a relativistic dispersion relation for a free particle $E^2 = \vec{p}^2 + m^2$

and using the same recipe

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0$$

The Spin-Statistics Theorem in Quantum Field Theory

The requirement of MICROCAUSALITY : the requirement that two field operators A(x), B(y) be compatible if x-y is a spacelike interval.

A,B solutions of a relativistic equation (Klein-Gordon, Dirac)

$$\left[A(x),B(y)\right]_{\pm}=0 \qquad for \ \ (x-y)^2<0$$

(anti)commutator

This prescription generates the right statistics for Bosons and Fermions

The Pauli Exclusion Principle is an ansatz (ad-hoc assumption) in Non-Relativistic Quantum Mechanics.

It can be demonstrated based on Microcausality in Quantum Field Theory.

A snapshot about our description of the world of “simple” systems (material points in classical physics, particles in quantum physics)

	Observables	Operators	States
Classical Mechanics	$\vec{x}, \vec{v}, t$		$\vec{x}(t)$ t *parameter*
Non Rel Quantum Mechanics	$\langle \psi \vert \hat{A} \psi \rangle$	$\hat{x}, \hat{p}$	$\psi(t)$ *in Hilbert space* t *parameter*
Quantum Field Theory	$\vert \langle A \vert \phi \vert B \rangle \vert^2$	$\psi(x),\ \phi(x),\ A(x)$ *spin* $1/2, 0, 1$ *operators* $x = (\vec{x}, t)$ *parameters*	$\vert n\rangle$ *Fock states* $\vert \alpha\rangle$ *Coherent states* …….

For a scalar (Klein-Gordon) field

$$\left[\phi(x),\phi(y)\right]=0 \qquad for\ \ (x-y)^2<0$$

commutator

satisfying the Klein-Gordon equation

$$\left(\partial_t^2-\nabla^2+m^2\right)\phi=0$$

$$\phi(x)\cong\int d\vec{k}\left\{a(\vec{k})e^{-ikx}+a^+(\vec{k})e^{ikx}\right\}$$

Annihilation/Creation operators

$$\left[a(\vec{k}),a(\vec{k}')\right]=0 \qquad \left[a^+(\vec{k}),a^+(\vec{k}')\right]=0$$

$$\left[a(\vec{k}),a^+(\vec{k}')\right]=\delta_{\vec{k}\vec{k}'}\,i\hbar$$

The number operator $N(\vec{k})=a^+(\vec{k})a(\vec{k})$ has eigenvalues $n(\vec{k})=0,1,2...$

and satisfies $a^+(\vec{k}')a^+(\vec{k})|0\rangle=a^+(\vec{k})a^+(\vec{k}')|0\rangle$

Symmetric under the interchange of particle labels

For a spin 1/2 (Dirac) field

$$\left[\psi(x),\overline{\psi}(y)\right]_{+}=0 \qquad for \quad (x-y)^2<0$$

anticommutator

satisfying the (4x1) Dirac equation

$$(i\gamma^{\mu}\partial_{\mu}-m)\,\psi\ =\ 0$$

$$\psi\ =\begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}$$

$$\psi(x)\cong\sum_{r}\int d\vec{p}\left\{c_r(\vec{p})\,u_r(\vec{p})\,e^{-ipx}+d_r^{+}(\vec{p})\,v_r(\vec{p})\,e^{ipx}\right\}$$

Sum over spins

Annihilation/Creation operators

$$\left[c_r(\vec{p}),c_s^{+}(\vec{p}\,')\right]_{+}=\left[d_r(\vec{p}),d_s^{+}(\vec{p}\,')\right]_{+}=\delta_{rs}\,\delta_{\vec{p}\vec{p}'}\,i\hbar$$

all other anticommutators 0

The number operators

$$N_r(\vec{p})\ =\ c_r^{+}(\vec{p})\,c_r(\vec{p})$$

$$\overline{N}_r(\vec{p})=d_r^{+}(\vec{p})\,d_r(\vec{p})$$

A number operator of the Dirac field has eigenvalues $n_r(\vec{p})\ =\ 0,1$

and $c_s^{+}(\vec{k}\,')\,c_s^{+}(\vec{k})\left|0\right\rangle=-c_s^{+}(\vec{k})\,c_s^{+}(\vec{k}\,')\left|0\right\rangle$

Antiymmetric under the interchange of particle labels

Particles and Antiparticles: the “birth” of Particle Physics

1928: Dirac Equation, merging Special Relativity and Quantum Mechanics.
A relativistic invariant Equation for spin ½ particles. E.g. the electron

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi = 0$$

Rest frame solutions: 4 independent states:

- E>0, s=+1/2
- E>0, s=-1/2
- E<0, s= +1/2
- E<0, s= -1/2

Upon reinterpretation of negative-energy states as antiparticles of the electron:

Electron, s=+1/2
Electron, s=-1/2
Positron, s=1/2
Positron, s=-1/2

The positron, a particle identical to the electron e^- but with a positive charge: e^+. The first prediction of the relativistic quantum theory.

This interpretation holds for every spin-1/2 fundamental particle

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