9- The total Hamiltonian

The total Hamiltonian for a single mode field interacting with a two-level atom.

i) the field Hamiltonian $\mathcal{H}_f$

$$\mathcal{H}_f = \hbar\omega_\lambda\left(\hat{a}_\lambda^+\hat{a}_\lambda + 1/2\right)$$

ii) the unperterbed atomic Hamiltonian $\mathcal{H}_a$

$$\mathcal{H}_a = E_1\,(b_1^+b_1) + E_2(b_2^+b_2)$$

If we take the system with energy reference $E_0 = 0$ between the two levels, we have:

$|2\rangle\, E_2$ ——— $\qquad E_1 = -\dfrac{\hbar\,\omega_{21}}{2} \qquad E_2 = \dfrac{\hbar\,\omega_{21}}{2}$

- - - - - - $E=0$

$|1\rangle\, E_1$ ——— $\qquad \mathcal{H}_a = \dfrac{\hbar\,\omega_{21}}{2}\,(b_2^+b_2) - \dfrac{\hbar\,\omega_{21}}{2}\,(b_1^+b_1)$

$$\Rightarrow \quad \mathcal{H}_a = \frac{\hbar\,\omega_{21}}{2}\left[(b_2^+b_2) - (b_1^+b_1)\right] = \hbar\,\omega_{21}\,\hat{\sigma}_3$$

iii) the interaction Hamiltonian ($\mathcal{H}_{int}$) in the dipole approximation is:

$$\mathcal{H}_{int} = -i\sqrt{\frac{\hbar\,\omega_{21}^2}{2\varepsilon V\omega_\lambda}}\Big[\langle\mu\rangle_{21}\,(\hat{a}_\lambda^+ + \hat{a}_\lambda)(b_2^+b_1) + {}$$
$$+ \langle\mu\rangle_{12}\,(\hat{a}_\lambda^+ + \hat{a}_\lambda)(b_1^+b_2)\Big]$$

where we assumed already a linear polarization for the electromagnetic field.

We also assume $\langle\mu\rangle_{21} = \langle\mu_{12}\rangle$ which is true for a "real" two level system, then: $\langle\mu\rangle_{21} = \langle\mu\rangle_{12} = \langle\mu\rangle$

$$\mathcal{H}_{int} = -i\sqrt{\frac{\hbar\,\omega_{21}}{2\varepsilon V}}\,\langle\mu\rangle\left\{(\hat{a}_\lambda^+ + \hat{a}_\lambda)\left[(b_2^+b_1) + (b_1^+b_2)\right]\right\}$$

We have also set $\langle\mu\rangle_{11} = \langle\mu\rangle_{22} = 0$. This choice is appropriate to transitions between states of definite parity.

The commutation rules:

We know already; for the creation and anihilation operators:

$$[\hat{a}, \hat{a}^+] = 1 \Rightarrow [\hat{a}^+, \hat{a}] = -1$$

The principle of exclusion can give us the commutation rules for the operators:

$$\hat{\sigma}_3 = \frac{1}{2}(\hat{b}_2^+\hat{b}_2 - \hat{b}_1^+\hat{b}_1) \quad \text{population diference.}$$

$$\hat{\sigma}^+ = (b_2^+ b_1) \quad \text{and} \quad \hat{\sigma}^- = (b_1^+ b_2) \quad \text{polarization.}$$

We know that:

$$b_1 b_2^+ + b_2^+ b_1 = 0 \qquad b_1 b_1^+ + b_1^+ b_1 = 1 \qquad b_2 b_2^+ + b_2^+ b_2 = 1.$$

$$b_1 b_1 + b_1 b_1 = 0 \qquad b_1 b_2 + b_2 b_1 = 0 \qquad b_2 b_2 + b_2 b_2 = 0$$

$$b_1^+ b_1^+ + b_1^+ b_1^+ = 0 \qquad b_1^+ b_2^+ + b_2^+ b_1^+ = 0 \qquad b_2^+ b_2^+ + b_2^+ b_2^+ = 0.$$

We can now calculate:

$$[\sigma^+, \sigma^-] = [(b_2^+ b_1), (b_1^+ b_2)] = b_2^+ b_1 b_1^+ b_2 - b_1^+ b_2 b_2^+ b_1 =$$

$$= b_2^+ b_2 - b_2^+ b_1^+ b_1 b_2 - b_1^+ b_1 + b_1^+ b_2^+ b_2 b_1 =$$

$$= (b_2^+ b_2) - (b_1^+ b_1) - b_2^+ b_1^+ b_1 b_2 - b_2^+ b_1^+ b_2 b_1 =$$

$$= 2\hat{\sigma}_3 - b_2^+ b_1^+ b_1 b_2 + b_2^+ b_1^+ b_1 b_2$$

$$\Rightarrow [\sigma^+, \sigma^-] = 2\hat{\sigma}_3 \quad \text{x} \quad [\sigma^-, \sigma^+] = -2\hat{\sigma}_3$$

In a similar way we can calculate

$$[\hat{\sigma}^+, \hat{\sigma}_3] \quad \text{and} \quad [\hat{\sigma}^-, \hat{\sigma}_3]$$

$$[\hat{\sigma}^+, \hat{\sigma}_3] = \left\{[b_2^+ b_1, b_2^+ b_2] - [b_2^+ b_1, b_1^+ b_1]\right\} \cdot 1/2 =$$

$$= 1/2 \cdot \left\{ b_2^+ b_1 b_2^+ b_2 - b_2^+ b_2 b_2^+ b_1 - b_2^+ b_1 b_1^+ b_1 + b_1^+ b_1 b_2^+ b_1 \right\} =$$

$$= 1/2 \cdot \left\{ b_2^+ b_1 b_2^+ b_2 - b_2^+ b_1 + b_2^+ b_2^+ b_2 b_1 - b_2^+ b_1 + b_2^+ b_1^+ b_1 b_1 + b_1^+ b_1 b_2^+ b_1 \right.$$

$$= 1/2 \cdot \left\{ - b_2^+ b_2^+ b_1 b_2 - b_2^+ b_1 + b_2^+ b_2^+ b_2 b_1 - b_2^+ b_1 + b_2^+ b_1^+ b_1 b_1 + b_1^+ b_2^+ b_1 b_1 \right.$$

$$= 1/2 \left\{ b_2^+ b_2^+ b_2 b_1 - 2 b_2^+ b_1 + b_2^+ b_2^+ b_2 b_1 + b_2^+ b_1^+ b_1 b_1 + b_2^+ b_1^+ b_1 b_1 \right\}$$

$$= 1/2 \cdot \left\{ \cancel{(b_2^+ b_2^+ + b_2^+ b_2^+) b_2 b_1} - 2 b_2^+ b_1 + \cancel{b_2^+ b_1^+ (b_1 b_1 + b_1 b_1)} \right\}$$

$$\Rightarrow [\hat{\sigma}^+, \hat{\sigma}_3] = - \hat{\sigma}^+ \quad \Rightarrow [\hat{\sigma}_3, \hat{\sigma}^+] = \hat{\sigma}^+$$

$$[\hat{\sigma}^-, \hat{\sigma}_3] = 1/2 \left\{ [b_1^+ b_2, b_2^+ b_2] - [b_1^+ b_2, b_1^+ b_1] \right\} =$$

$$= 1/2 \left\{ b_1^+ b_2 b_2^+ b_2 - b_2^+ b_2 b_1^+ b_2 - b_1^+ b_2 b_1^+ b_1 + b_1^+ b_1 b_1^+ b_2 \right\} =$$

$$= 1/2 \cdot \left\{ b_1^+ b_2 - b_1^+ b_2^+ b_2 b_2 + b_2^+ b_1^+ b_2 b_2 - b_1^+ b_2 b_1^+ b_1 + b_1^+ b_2 - b_1^+ b_1^+ b_1 b_2 \right\}$$

$$= 1/2 \left\{ 2 b_1^+ b_2 - b_1^+ b_2^+ b_2 b_2 - b_1^+ b_2^+ b_2 b_2 + b_1^+ b_1^+ b_2 b_1 + b_1^+ b_1^+ b_2 b_1 \right\} =$$

$$= 1/2 \left\{ 2 b_1^+ b_2 - \cancel{b_1^+ b_2^+ (b_2 b_2 + b_2 b_2)} + \cancel{(b_1^+ b_1^+ + b_1^+ b_1^+) b_2 b_1} \right\} -$$

$$\Rightarrow [\hat{\sigma}^-, \hat{\sigma}_3] = \hat{\sigma}^- \quad \Rightarrow [\hat{\sigma}_3, \hat{\sigma}^-] = - \hat{\sigma}^-$$

Physical interpretation: $\hat{\sigma}^+$ represents the transition between state $|1\rangle$ and state $|2\rangle$, while $\hat{\sigma}^-$ the opposite one knowing also that

$$[\sigma_3, a] = 0 = [\hat{\sigma}_3, \hat{a}^+] = [\sigma^\pm, a^+] = [\sigma^\pm, a]$$

we can obtain the dynamical evolution of the operators.

The Dynamical equations for polarization and population difference.

Using Heisenberg's equation:

$$\dot{\sigma}_3 = -\frac{i}{\hbar}\left[\hat{\sigma}_3, H\right]$$

$$\left[\hat{\sigma}_3, \hat{a}^+\hat{a}\right] = \left[\hat{\sigma}_3, \hat{a}^+\right]\hat{a} + \hat{a}^+\left[\hat{\sigma}_3, \hat{a}\right] = 0.$$

$$\left[\hat{\sigma}_3, \hat{\sigma}_3\right] = 0.$$

$$\left[\hat{\sigma}_3, -i\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\langle\mu\rangle(\hat{a}_\lambda^+ + \hat{a}_\lambda).(\sigma^+ + \sigma^-)\right] =$$

$$= -i\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\langle\mu\rangle\left\{\cancel{\left[\hat{\sigma}_3, (a_\lambda^+ + a_\lambda)\right]}(\sigma^+ + \sigma^-) + (a_\lambda^+ + a_\lambda)\left[\hat{\sigma}_3, (\hat{\sigma}^+ + \hat{\sigma}^-)\right]\right.$$

$$= -i\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\langle\mu\rangle\left\{(a_\lambda^+ + a_\lambda)\left\{[\sigma_3, \sigma^+] + [\sigma_3, \sigma^-]\right\}\right.$$

$$= -i\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\langle\mu\rangle\left\{(\hat{a}_\lambda^+ + \hat{a}_\lambda)(\hat{\sigma}^+ - \hat{\sigma}^-)\right\}$$

$$= -i\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\langle\mu\rangle\left\{\hat{a}^+\hat{\sigma}^+ - \hat{a}^+\hat{\sigma}^- + \hat{a}\,\hat{\sigma}^+ - \hat{a}\,\hat{\sigma}^-\right\}$$

Then:

$$\frac{d\sigma_3}{dt} = -\frac{1}{\hbar}\langle\mu\rangle\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\left\{(\hat{a}^+\hat{\sigma}^+ - \hat{a}^-\hat{\sigma}^-) + \hat{a}\,\hat{\sigma}^+ - \hat{a}^+\hat{\sigma}^-\right\}$$

$$\dot{\hat{\sigma}}^+ = -i/\hbar\left[\hat{\sigma}^+, \hat{H}\right]$$

$$\left[\hat{\sigma}^+, \hat{a}^+\hat{a}\right] = 0.$$

$$\left[\hat{\sigma}^+, \hbar\omega_{21}\hat{\sigma}_3\right] = \hbar\omega_{21}\left[\hat{\sigma}^+, \hat{\sigma}_3\right] = -\hbar\omega_{21}\hat{\sigma}^+$$

$$\left[\hat{\sigma}^+, -i\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\langle\mu\rangle(\hat{a}^+ + \hat{a})(\hat{\sigma}^+ + \sigma^-)\right] = -i\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\langle\mu\rangle\left[\hat{\sigma}^+, (\hat{a}^+ + \hat{a})(\hat{\sigma}^+ + \hat{\sigma}^-)\right]$$

$$= -i\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\langle\mu\rangle\left\{\underbrace{\cancel{\left[\hat{\sigma}^+, (\hat{a}^+ + \hat{a})\right]}}_{=0}(\hat{\sigma}^+ + \hat{\sigma}) + (a^+ + a)\left[\hat{\sigma}^+, (\hat{\sigma}^+ + \hat{\sigma}^-)\right]\right\}$$

Before dealing with such equations, we have to look closer to the interaction Hamiltonian:

The electric field can be defined as:

$$\vec{E} = i\sqrt{\frac{\hbar\omega_\lambda}{2\epsilon V}}\left(\hat{a}_\lambda e^{i(k_\lambda \cdot r - \omega t)} - \hat{a}_\lambda^+ e^{-i(k_\lambda \cdot r - \omega t)}\right)$$

and the potential vector is:

$$\vec{A} = \sqrt{\frac{\hbar}{2\epsilon\omega_\lambda V}}\left(\hat{a}_\lambda e^{i(k_\lambda r - \omega t)} + \hat{a}_\lambda^+ e^{-i(k_\lambda r - \omega t)}\right)$$

The two-level atom - single mode field interaction Hamiltonian is:

$$\mathcal{H}_{int} = -i\sqrt{\frac{\hbar\omega_\lambda}{2\epsilon V}}\langle\mu\rangle\left[\hat{a}\, e^{i(\bar{k}\bar{r} - \omega t)} + \hat{a}^+ e^{-i(\bar{k}\bar{r} - \omega t)}\right] \times$$

$$\times\left[\hat{\sigma}^+ e^{-i(k_\lambda r - \omega t)} + \hat{\sigma}^- e^{+i(kr - \omega t)}\right]$$

The inclusion of the exponential terms in $\hat{\sigma}^+, \hat{\sigma}^-$ is justified automatically by the fact that the positions are different for each atom.

$$\mathcal{H}_{int} = -i\sqrt{\frac{\hbar\omega_\lambda}{2\epsilon V}}\langle\mu\rangle\left\{\hat{a}\hat{\sigma}^+ + \hat{a}\hat{\sigma}^- e^{2i(\bar{k}\bar{r} - \omega t)} + \hat{a}^+\hat{\sigma}^+ e^{-2i(\bar{k}\bar{r} - \omega t)} + \right.$$

$$\left. + \hat{a}^+\hat{\sigma}^-\right\}.$$

The terms $e^{\pm 2i(\bar{k}\bar{r} - \omega t)}$ in the Hamiltonian implied a non-conservative term because it creates or anihilates a photon at double the frequency of the E field. In fact the process $\hat{a}\hat{\sigma}^-$ implies the simultaneous transition of an electron from upper to lower level and the anihilation of a photon. The term $\hat{a}^+\hat{\sigma}^+$ the opposite, but both ~~are~~ have a low probability. Then the ~~total~~ interaction Hamiltonian $\mathcal{H}_{int} = -i\sqrt{\frac{\hbar\omega}{2\epsilon V}}\langle\mu\rangle\left\{\hat{a}\hat{\sigma}^+ + \hat{a}^+\hat{\sigma}^-\right\}$

Therefore:

$$\frac{d\sigma_3}{dt} = -\frac{\langle\mu\rangle}{\hbar}\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\left[\hat{a}\,\hat{\sigma}^+ - \hat{a}^+\hat{\sigma}^-\right]$$

Then:

$$\dot{\hat{\sigma}}^+ = -\frac{i}{\hbar}\left[\hat{\sigma}^+, \mathcal{H}\right] = i\,\omega_{21}\,\hat{\sigma}^+ - \frac{\langle\mu\rangle}{\hbar}\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\left\{\left[\hat{\sigma}^+, \hat{a}\hat{\sigma}^+\right] + \left[\hat{\sigma}^+, \hat{a}^+\hat{\sigma}\right.\right.$$

$$= i\omega_{21}\hat{\sigma}^+ - \frac{\langle\mu\rangle}{\hbar}\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\left\{\hat{a}\,\cancel{\left[\hat{\sigma}^+,\hat{\sigma}^+\right]} + \cancel{\left[\hat{\sigma}^+,\hat{a}\right]\hat{\sigma}^+} + \cancel{\left[\hat{\sigma}^+,\hat{a}^+\right]\hat{\sigma}} + \hat{a}^+\left[\hat{\sigma}^+,\right.\right.$$

$$\Rightarrow \dot{\hat{\sigma}}^+ = i\,\omega_{21}\,\hat{\sigma}^+ - \frac{\langle\mu\rangle}{\hbar}\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\left\{+\hat{a}^+\,2\,\hat{\sigma}_3\right\}$$

$$\dot{\hat{\sigma}}^+ = i\,\omega_{21}\,\hat{\sigma}^+ - \frac{\langle\mu\rangle}{\hbar}\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\;2\left(\hat{a}^+\hat{\sigma}_3\right)$$

$$\Rightarrow \frac{d\hat{\sigma}^+}{dt} = i\,\omega_{21}\,\hat{\sigma}^+ - 2\frac{\langle\mu\rangle}{\hbar}\sqrt{\frac{\hbar\omega_{21}}{2\varepsilon V}}\;\hat{a}^+\hat{\sigma}_3.$$

Two remarks are important:

1) The field is proportional to $\hat{a}^+$. As $\hat{a}$ is the complex conjugate of $\hat{a}^+$, the electromagnetic field is fully described by $\hat{a}^+$ (in principle).

2) The RWA is valid because we took $\omega_{21} \simeq \omega$.

We can take now average values of the variables, defined as:

$$D = \frac{1}{N}\sum_{m=1}^{N}(\sigma_3)_m \quad \text{and} \quad \mathcal{P} = \frac{1}{N}\sum_{m=1}^{N}\hat{\sigma}^+$$

Note that in the definition $\mathcal{P}$ we did not include the spatial part: e^{ikz}. This simplification is not alway valid, mainly if we take into account the factor $\hat{p}^2$ in the atomic hamiltonian. It becomes exact if the term $\hat{p}^2$ is negligible and if we consider only one mode of the e.m. field. or that the length of the medium is smaller than the wavelength of the e.m. field.

At this point we can analyze the ~~evolution~~ response of an atomic medium to an external field (coherent) under the assumption of a scalar classical field.

$$E_{ext} = -i\sqrt{\frac{\hbar\omega}{2\epsilon V}}\,\hat{a}^+ e^{-i(kz-\omega t)} \cancel{+c.c.}$$

Thus:

$$\frac{1}{N}\sum_n \frac{d(\sigma_3)_n}{dt} = -\frac{\langle\mu\rangle}{\hbar}\left\{\sqrt{\frac{\hbar\omega_{21}}{2\epsilon V}}\,\hat{a}\,\frac{1}{N}\sum_n \hat{\sigma}^+_n - \sqrt{\frac{\hbar\omega_{21}}{2\epsilon V}}\,\hat{a}^+\frac{1}{N}\sum_n \hat{\sigma}^-_n\right\}$$

$$E^*_{ext} = i\sqrt{\frac{\hbar\omega}{2\epsilon V}}\,\hat{a}\, e^{i(kz-\omega t)} \cancel{+c.c.}$$

$$\sqrt{\frac{\hbar\omega}{2\epsilon V}}\,\hat{a} = -iE^*_{ext}\, e^{-i(kz-\omega t)} \quad , \quad \sqrt{\frac{\hbar\omega}{2\epsilon V}}\,\hat{a}^+ = iE_{ext}\, e^{i(kz-\omega t)}$$

If we define $P = \wp\, e^{-i(kz-\omega t)} = \frac{1}{N}\sum_n \sigma^+_n\, e^{-i(kz-\omega t)}$

$$\Rightarrow \frac{1}{N}\sum_n \hat{\sigma}^+_n = P e^{i(kz-\omega t)} \quad , \quad \frac{1}{N}\sum_n \hat{\sigma}^-_n = P^* e^{-i(kz-\omega t)}$$

$$\Rightarrow \frac{dD}{dt} = -\frac{\langle\mu\rangle}{\hbar}\left\{-iE^*_{ext}P - iE_{xt}P^*\right\} = i\frac{\langle\mu\rangle}{\hbar}\left\{E^*_{ext}P + E_{ext}P^*\right\}$$

$$\frac{1}{N}\sum_n \frac{d\sigma^+_n}{dt} = i\omega_{21}\frac{1}{N}\sum_n \hat{\sigma}^+ - \frac{\langle\mu\rangle}{\hbar}\sqrt{\frac{\hbar\omega_{21}}{2\epsilon V}}\,\hat{a}^+\sum_n(2\sigma_3)_n\left(\frac{1}{N}\right)$$

$$= i\omega_{21} P e^{i(kz-\omega t)} - \frac{\langle\mu\rangle}{\hbar}\, iE_{ext}\, e^{i(kz-\omega t)}\, 2D.$$

$$\frac{d}{dt}\left(Pe^{i(kz-\omega t)}\right) = i\omega_{21} P e^{i(kz-\omega t)} - 2i\frac{\langle\mu\rangle}{\hbar}E_{ext}\, e^{i(kz-\omega t)} D.$$

$$\left(\frac{dP}{dt} - i\omega P\right)\cancel{e^{i(kz-\omega t)}} = \left(-i\omega_{12}P - 2i\frac{\langle\mu\rangle}{\hbar}E_{ext}D\right)\cancel{e^{i(kz-\omega t)}}$$

$$\begin{cases} \dfrac{dP}{dt} = i(\omega-\omega_{12})P - 2i\dfrac{\langle\mu\rangle}{\hbar}E_{ext}D \\[2ex] \dfrac{dD}{dt} = i\dfrac{\langle\mu\rangle}{\hbar}\left\{E^*_{ext}P + E_{ext}P^*\right\}. \end{cases}$$

This system of ~~the~~ equations is the conservative version of the atomic response to an e.m. field.