$$\hat{a} = (2m\hbar\omega)^{-\frac{1}{2}} (m\omega\hat{q} + i\hat{p}) \tag{2.13}$$

$$\hat{a}^+ = (2m\hbar\omega)^{-\frac{1}{2}} (m\omega\hat{q} \quad i\hat{p}) \tag{2.14}$$

so that the quantized field Hamiltonian can be written as

$$\hat{\mathcal{H}} = \hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2}) \tag{2.15}$$

and the canonical commutation relation as

$$[\hat{a},\hat{a}^+] = 1 \tag{2.16}$$

The operator $\hat{a}$ is called the annihilation, or destruction operator. The operator $\hat{a}^+$ is called the creation operator. Using these operators it follows that we can write the electric field operator for the mode of frequency ω as

$$\hat{E}_x(z,t) = \mathcal{E}(\hat{a} + \hat{a}^+) \sin kz \tag{2.17}$$

where $\mathcal{E}$ given by

$$\mathcal{E} = (\hbar\omega/V\varepsilon_o)^{\frac{1}{2}} , \tag{2.18}$$

is an amplitude which will turn out to be the "electric field per photon" (Sargent, Scully and Lamb 1974). The time-dependence of the annihilation operator $\hat{a}$ and the creation operator $\hat{a}^+$ can be determined using Heisenberg's equation of motion

$$\frac{d}{dt}\hat{a} = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{a}] \tag{2.19}$$

The commutator can be evaluated using eqs.(2.15) and (2.16)

$$[\hat{\mathcal{H}} \quad \hat{a}] = [\hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2}), \hat{a}] = \hbar\omega(\hat{a}^+\hat{a}\hat{a} \quad \hat{a}\hat{a}^+\hat{a})$$

$$= \hbar\omega(\hat{a}^+\hat{a} - \hat{a}\hat{a}^+)\hat{a} = \hbar\omega\hat{a} \tag{2.20}$$

So

$$\frac{d}{dt}\hat{a} = \quad i\omega\hat{a} \tag{2.21}$$

which has the solution

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t}. \tag{2.22}$$

By the same method we find

$$\hat{a}^+(t) = \hat{a}^+(0)e^{i\omega t} \tag{2.23}$$

The product $\hat{a}^+\hat{a}$ has a special significance and is called the number operator $\hat{n}$. Let $|n\rangle$ be an energy eigenstate of the single-mode field with

energy eigenvalue E_n, such that

$$\hat{\mathcal{H}}|n\rangle = \hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2})|n\rangle = E_n|n\rangle \tag{2.24}$$

We can then generate a new eigenvalue equation by multiplying eq.(2.24) from the left by $\hat{a}^+$ to give

$$\hbar\omega(\hat{a}^+\hat{a}^+\hat{a} + \tfrac{1}{2}\hat{a}^+)|n\rangle = E_n\hat{a}^+|n\rangle \tag{2.25}$$

Using the commutator (eq.(2.16)) this can be re-written as

$$\hbar\omega[(\hat{a}^+\hat{a} - 1)\hat{a}^+ + \tfrac{1}{2}\hat{a}^+)|n\rangle = E_n\hat{a}^+|n\rangle$$

or

$$\hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2})\ (\hat{a}^+|n\rangle) = (E_n + \hbar\omega)\ (\hat{a}^+|n\rangle) , \tag{2.26}$$

which is an eigenvalue equation for the eigenstate $(\hat{a}^+|n\rangle)$ with eigenenergy $(E_n + \hbar\omega)$. The application of the creation operator $\hat{a}^+$ has resulted in a new state containing one more quantum of excitation. We may denote this new state as $|n+1\rangle$, but it needs to be normalised. In a similar fashion we find that the operator $\hat{a}$ destroys one unit of excitation, generating a new eigenstate $|n-1\rangle$ with energy $(E_n - \hbar\omega)$. This cannot be repeated indefinitely since the system has a ground state $|0\rangle$ of energy E_o which must be positive. So we demand

$$\hat{\mathcal{H}}(\hat{a}|n\rangle) = (E_n - \hbar\omega)(\hat{a}|n\rangle)$$

$$\hat{\mathcal{H}}(\hat{a}|0\rangle) = (E_o - \hbar\omega)(\hat{a}|0\rangle) , \tag{2.27}$$

which defines the ground state through

$$\hat{a}|0\rangle = 0 \tag{2.28}$$

Therefore the eigenvalue equation for the ground state is

$$\hat{\mathcal{H}}|0\rangle = \hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2})|0\rangle = \tfrac{1}{2}\hbar\omega|0\rangle \tag{2.29}$$

so that the ground state of the single mode electromagnetic field has a non-zero energy $\hbar\omega/2$. This zero-point energy reflects our inability to confine a "particle", which here is the mode "oscillator", at rest and still satisfy the Heisenberg uncertainty relation. Since $E_{n+1} = E_n + \hbar\omega$,

$$E_n = \hbar\omega(n + \tfrac{1}{2}),\ n = 0,1,2... \tag{2.30}$$

and

$$\hat{n}|n\rangle = \hat{a}^+\hat{a}|n\rangle = n|n\rangle \tag{2.31}$$

The "number states" $|n\rangle$ are normalised, with

$$\langle n \quad 1|n-1\rangle = 1 = \langle n|n\rangle = \langle n+1|n+1\rangle \tag{2.32}$$

If we write

$$\hat{a}|n\rangle = C_n|n-1\rangle \tag{2.33}$$

then

$$(\langle n|\hat{a}^+)(\hat{a}|n\rangle) = \langle n-1|C_n^* C_n|n\ \ 1\rangle = n$$

so that

$$|C_n|^2 = n$$

The constant C_n can be taken as real without loss of generality, so that

$$C_n = n^{\frac{1}{2}} \tag{2.34}$$

and

$$\hat{a}|n\rangle = \sqrt{n}\ |n-1\rangle \tag{2.35}$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\ |n+1\rangle \tag{2.36}$$

The spectrum of the radiation field is thus of a ladder of equally spaced levels separated by $\hbar\omega$, which one ascends by the action of $\hat{a}^+$ and descends by the action of $\hat{a}$. The number states are generated by the repeated action of the creation operator $\hat{a}^+$:

$$|n\rangle = (\sqrt{n!})^{-1}\ (\hat{a}^+)^n|0\rangle \tag{2.37}$$

The only non-vanishing matrix elements of the annihilation and creation operators are

$$\langle n-1|\hat{a}|n\rangle = \langle n-1|\sqrt{n}\ |n-1\rangle = \sqrt{n} \tag{2.38}$$

$$\langle n+1|\hat{a}^+|n\rangle = \langle n+1|\sqrt{n+1}\ |n+1\rangle = \sqrt{n+1}\ . \tag{2.39}$$

2.3 QUANTUM FLUCTUATIONS OF A SINGLE-MODE FIELD

The number states $|n\rangle$ are energy eigenstates of the single mode field. But they do not describe a field with a well-defined electromagnetic field, as

$$\langle n|\hat{E}_x(z,t)|n\rangle = \mathcal{E}\,\sin kz(\langle n|\hat{a}(t)|n\rangle + \langle n|\hat{a}^+(t)|n\rangle)$$

$$= 0 \tag{2.40}$$

using eqs.(2.38) and (2.39). This does not, of course, mean that the field is zero as,

$$\langle n|\hat{E}_x^2(z,t)|n\rangle = \mathcal{E}^2\sin^2 kz\ \langle n|(\hat{a}^+\hat{a}^+ + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}\hat{a})|n\rangle$$

$$= 2\mathcal{E}^2\sin^2 kz\ (n+\tfrac{1}{2}) \tag{2.41}$$

The r.m.s. electric field in the cavity when only one photon is present ignoring the ever present zero-point-energy is $\sqrt{2}\mathcal{E}\sin kz$. When this is averaged

over spatial position z we see that the "electric field per photon" is indeed $\mathcal{E}$. The number states field can be visualized as having a definite amplitude with a phase randomly distributed over 2π (Loudon 1973). The field energy, or number of quanta n, and the field strength are complementary concepts (Power 1964), because the number operator $\hat{n}$ does not commute with the electric field operator $\hat{E}_x(z,t)$. We see that

$$[\hat{n},\hat{E}_x] = \mathcal{E}\,\sin kz(\hat{a}^+\ \ \hat{a}) = \ \ 2i\hat{p}(2m\mathcal{V}\varepsilon_o)^{\frac{1}{2}}\sin kz \tag{2.42}$$

If the excitation number n is known exactly the fluctuations in $E_x(z,t)$ are of the order of $E_x(z,t)$. Conversely if $E_x(z,t)$ is known accurately the excitation number n is correspondingly uncertain.

2.4 MULTIMODE FIELDS

So far we have studied only a single mode of a field confined to a cavity. These results can be generalized to describe a multi-mode radiation field. Before we do this, we stress that the quanta in the cavity of energy $\hbar\omega$ are the photons of the system, and as such are not localized particles but are characteristic excitations of the cavity mode and spread over the mode volume. Quantum mechanics introduces not localized corpuscular photons as in the old quantum theory, but rather mode-excitations of the system.

In free space, the electric and magnetic radiation fields can be described in terms of the vector potential $\underline{A}$ which obeys the wave equation

$$\nabla^2\underline{A}\ \ \frac{1}{c^2}\frac{\partial^2\underline{A}}{\partial t^2} = 0 \tag{2.43}$$

where

$$\underline{B} = \underline{\nabla}\ \mathrm{x}\ \underline{A}$$

and

$$\underline{E} = \ \ \frac{\partial}{\partial t}\underline{A}$$

We take running wave solutions to the wave equation of the plane-wave form

$$\underline{A} = \sum_{\underline{k}}\left\{\underline{A}_{\underline{k}}(t)e^{i\underline{k}.\underline{r}} + \underline{A}^*_{\underline{k}}(t)e^{-i\underline{k}.\underline{r}}\right\} \tag{2.44}$$

subject to the periodic boundary conditions (Loudon 1973)

$$k_x = \frac{2\pi}{L}m_x\ ,\quad k_y = \frac{2\pi}{L}m_y\ ,\quad k_z = \frac{2\pi}{L}m_z \tag{2.45}$$

where the "box" volume $V = L^3$ and m_x, m_y, $m_z = 0, \pm1, \pm2$---. The mode density, the number of modes of wavenumber between k and k + dk, is

$$\rho(k)dk = k^2dk/\pi^2 \tag{2.46}$$

Alternatively in frequency space the number of modes of frequency between ω and $\omega + d\omega$ is

$$\rho(\omega)d\omega = \omega^2 d\omega/\pi^2 c^3 \tag{2.47}$$

In free space, the discrete sum in eq.(2.44) is replaced by a continuum integral

$$\sum_{\underline{k}} \rightarrow \frac{V}{\pi^2}\int k^2 dk \tag{2.48}$$

In quantum optics the Coulomb gauge, in which $\underline{\nabla}.\underline{A} = 0$, is mainly used. In this gauge, $\underline{k}.\underline{A}_{\underline{k}}(t) = 0 = \underline{k}.\underline{A}^*_{\underline{k}}(t)$ and each Fourier component satisfies, from eq.(2.43), the harmonic oscillator equation

$$\frac{\partial^2 \underline{A}_{\underline{k}}}{\partial t^2} + \omega_k^2 \underline{A}_{\underline{k}} = 0 \tag{2.49}$$

where $\omega_k = ck$. From eq.(2.49) we find

$$\underline{A}_{\underline{k}}(t) = \underline{A}_{\underline{k}}\, e^{-i\omega_k t}. \tag{2.50}$$

The cycle-averaged energy of mode $\underline{k}$ is

$$\mathcal{E}_k = \frac{1}{2}\int (\varepsilon_o \overline{E}^2_{\underline{k}} + \mu_o \overline{H}^2_{\underline{k}})dV \tag{2.51}$$

where the bar denotes the cycle average. The electric field from eq.(2.43) is,

$$\underline{E}_{\underline{k}} = i\omega_k \left\{ \underline{A}_{\underline{k}}\, e^{-i\omega_k t + i\underline{k}.\underline{r}} \quad \underline{A}^*_{\underline{k}}\, e^{i\omega_k t - i\underline{k}.\underline{r}} \right\} \tag{2.52}$$

and the cavity magnetic field is

$$\underline{H}_{\underline{k}} = (i/\mu_o)\underline{k}\,x \left\{ \underline{A}_{\underline{k}}\, e^{-i\omega_k t + i\underline{k}.\underline{r}} \quad \underline{A}^*_{\underline{k}}\, e^{i\omega_k t - i\underline{k}.\underline{r}} \right\} \tag{2.53}$$

so that

$$\overline{\mathcal{E}}_k = 2\varepsilon_o V\omega_k^2\, \underline{A}_{\underline{k}}.\underline{A}^*_{\underline{k}} \tag{2.54}$$

The vector potential can be written in terms of generalized canonical position q_k and momentum p_k variables

$$\underline{A}_{\underline{k}} = (4\varepsilon_o V\omega_k^2)^{-\frac{1}{2}}(\omega_k q_k + ip_k)\underline{\varepsilon}_{\underline{k}} \tag{2.55}$$

$$\underline{A}^*_{\underline{k}} = (4\varepsilon_o V\omega_k^2)^{-\frac{1}{2}}(\omega_k q_k - ip_k)\underline{\varepsilon}_{\underline{k}} \tag{2.56}$$

where $\underline{\varepsilon}_{\underline{k}}$ is the polarization unit vector. In terms of these variables the

cycle-averaged energy is written as

$$\overline{\mathcal{E}}_k = \frac{1}{2}(p_k^2 + \omega_k^2 q_k^2), \tag{2.57}$$

the energy of a unit mass simple harmonic oscillator. The complete free field Hamiltonian is given by the sum of all single-mode terms given by eq. (2.57). The summation is over the wavevectors $\underline{k}$ and polarization directions $\underline{\varepsilon}_{\underline{k}}$.

Each mode $\underline{k}$, represented as a unit mass harmonic oscillator, can be quantized in terms of number states as before. The state of the multimode field can be written as a product of the individual mode states,

$$|\{n_k\}> = |n_{k1}>|n_{k2}>---|n_{kj}>---$$

$$= |n_{k1}, n_{k2},--- n_{kj},--->. \tag{2.58}$$

where n_{kj} represents the number of quanta in mode with wavevector $\underline{k}_j$. The creation operator $\hat{a}_{k1}$ acts on the multimode state

$$\hat{a}^+_{k1}\,|n_{k1}, n_{k2}, ---n_{kj}, ---> = (n_{k1}+1)^{\frac{1}{2}}|n_{k1}+1, n_{k2},-- n_{kj}, --->. \tag{2.59}$$

The field amplitudes $\underline{A}_{\underline{k}}$ of classical theory become

$$\hat{\underline{A}}_{\underline{k}} = (\hbar/2\varepsilon_o V\omega_k)^{\frac{1}{2}}\, \hat{a}_{\underline{k}}\, \hat{\underline{\varepsilon}}_{\underline{k}} \tag{2.60}$$

and

$$\hat{\underline{A}}^*_{\underline{k}} = (\hbar/2\varepsilon_o V\omega_k)^{\frac{1}{2}}\, \hat{a}^+_{\underline{k}}\, \hat{\underline{\varepsilon}}_{\underline{k}} \tag{2.61}$$

such that the vector field operator can be written as

$$\hat{\underline{A}} = \sum_{\underline{k}} (\hbar/2\varepsilon_o V\omega_k)^{\frac{1}{2}}\, \underline{\varepsilon}_{\underline{k}} \left\{ \hat{a}_{\underline{k}}\, e^{-i\omega_k t + i\underline{k}.\underline{r}} + \hat{a}^+_{\underline{k}}\, e^{i\omega t \quad i\underline{k}.\underline{r}} \right\} \tag{2.62}$$

(2.62)

The electric field operator is

$$\hat{\underline{E}}(\underline{r},t) = \sum_{\underline{k}} i(\hbar\omega_k/2\varepsilon_o V)^{\frac{1}{2}}\, \underline{\varepsilon}_{\underline{k}} \left\{ \hat{a}_{\underline{k}}(t) e^{-i\omega_k t + i\underline{k}.\underline{r}} - \hat{a}^+_{\underline{k}}(t) e^{i\omega_k t - i\underline{k}.\underline{r}} \right\} \tag{2.63}$$

and the magnetic field operator

$$\hat{\underline{H}}(\underline{r},t) = \sum_{\underline{k}} i(\hbar c^2/2\mu_o V\omega_k)^{\frac{1}{2}}\, \underline{k}\,x\,\underline{\varepsilon}_{\underline{k}} \left\{ \hat{a}_{\underline{k}}(t) e^{-i\omega_k t + i\underline{k}.\underline{r}} \quad \hat{a}^+_{\underline{k}}(t) e^{i\omega_k t - i\underline{k}.\underline{r}} \right\}. \tag{2.64}$$

The quantum expression for a single-mode plane wave field from eq.(2.63) is

$$\hat{\underline{E}}(\underline{r},t) = i(\hbar\omega/2\varepsilon_o V)^{\frac{1}{2}}\,\underline{\varepsilon}_k \left\{ \hat{a}_k e^{-i\omega_k t + i\underline{k}.\underline{r}} \quad \hat{a}_k^+ e^{i\omega_k t - i\underline{k}.\underline{r}} \right\} \qquad (2.65)$$

In much of quantum optics the spatial variation of the electromagnetic field mode over the dimensions of the atomic system may be neglected. In this case the exponential factors $e^{\pm i\underline{k}.\underline{r}}$ may be set equal to unity. This single-mode expression can be used either in the interaction representation in which the operator $\hat{a}(t)$ and its conjugate $\hat{a}^+(t)$ have the time development shown in eq. (2.65) or in the Schrödinger representation in which $\hat{a}$ and $\hat{a}^+$ have no time development. The expressions for the electric field operator in these two pictures coincide at $t = 0$.

2.5 ZERO-POINT ENERGY AND VACUUM FLUCTUATIONS

In Section (2.3) we saw that the quantized radiation field fluctuates. For a single-mode field described by a number state $|n\rangle$, the r.m.s. deviation in electric field strength $\Delta E = (\langle E^2\rangle - \langle E\rangle^2)^{\frac{1}{2}}$, using eq.(2.40) and (2.41) is

$$\Delta E = \sqrt{2}\mathcal{E}\sqrt{n+\tfrac{1}{2}}\,\sin kz \ . \qquad (2.66)$$

If the single-mode field is unoccupied, $n = 0$, then the r.m.s. fluctuation in the vacuum field strength is $\Delta E(\text{vac}) = \mathcal{E}\sin kz$. Vacuum fluctuations and zero-point energy have a common origin in the non-commutability of the field annihilation and creation operators $\hat{a}$ and $\hat{a}^+$. The occurrence of the zero-point energy term and its associated vacuum fluctuation presents severe problems in quantum field theory. In practice there are an infinite number of radiation field modes, each with a finite zero-point energy. The total zero-point energy therefore diverges unless the high frequency modes are excluded from the sum. Yet the zero-point energy seems to lead to observable consequences (Casimir 1948, Power 1964) and cannot be ignored. A stimulating analysis of the role of vacuum fluctuations may be found in the article by Jaynes (1978) in the proceedings of the 1977 Rochester Coherence and Quantum Optics conference.

The major observable effect attributed to the existence of vacuum fluctuations is the Lamb shift. In 1947, Lamb and Retherford (Reprinted Paper 4) used a microwave frequency method to examine the fine structure of the $n = 2$ level of atomic hydrogen. Earlier high resolution optical studies of the H_α line seemed to indicate a discrepancy between experiment and the Dirac relativistic theory of the hydrogen atom. The Dirac theory predicts that the $2^2S_{1/2}$ and $2^2P_{1/2}$ levels should be degenerate. The early optical work suggested that these states were not in fact degenerate but separated by about $0.033\ \text{cm}^{-1}$. Lamb and Retherford used an elegant combination of atomic beam and microwave techniques and showed that the $2^2S_{1/2}$ state is higher in energy than the $2^2P_{1/2}$ state by about 1000 MHz. The lifting of the $2^2P_{1/2}$ and $2^2S_{1/2}$ degeneracies is explained by the interaction of the bound electron with the vacuum fluctuations (Bethe 1947). In the paper by Power (Reprinted Paper 5), the Lamb shift, or radiative level shift of the $2^2S_{1/2}$ level, is calculated directly from the change in zero-point energy due to the interaction of the vacuum field with the hydrogen atom.

A simple intuitive interpretation of the Lamb shift was given by Welton (1948). Each field mode contains $h\nu/2$ zero-point energy. The number of

modes in a cavity of volume V with frequency between ν and $\nu + d\nu$ is $(8\pi/c^3)\nu^2 d\nu V$. The zero-point field energy is

$$\left(\frac{8\pi}{c^3}\nu^2 d\nu V\right)\frac{1}{2}h\nu = \frac{1}{8\pi}\int_V (E_\nu^2 + B_\nu^2)\,dV = \frac{1}{8\pi}E_\nu^2 V \qquad (2.67)$$

where E_ν is the amplitude of the electric field component of frequency ν. The square of the vacuum electric field is therefore

$$E_\nu^2 = \frac{32\pi^2}{c^3}h\nu^3 d\nu \ . \qquad (2.68)$$

The electron bound in the hydrogen atom interacts with the fluctuating zero-point electric field and with the Coulomb potential of the proton $-e^2/r$. The perturbation of the electron from its standard "orbit" is described by the fluctuating electron position Δr

$$r \to r + \Delta r \qquad (2.69)$$

The change in potential energy is $\Delta V = V(r + \Delta r) - V(r)$, which by Taylor's theorem is

$$\Delta V = \Delta x\frac{\partial V}{\partial x} + \Delta y\frac{\partial V}{\partial y} + \Delta z\frac{\partial V}{\partial z} + \frac{1}{2}(\Delta x)^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}(\Delta y)^2\frac{\partial^2 V}{\partial y^2} + \frac{1}{2}(\Delta z)^2\frac{\partial^2 V}{\partial z^2} + \cdots$$

The fluctuations are isotropic, so that $\langle\Delta x\rangle = \langle\Delta y\rangle = \langle\Delta z\rangle = 0$ and $\langle(\Delta x)^2\rangle = \langle(\Delta y)^2\rangle = \langle(\Delta z)^2\rangle = \langle(\Delta r)^2\rangle/3$. Then

$$\langle\Delta V\rangle = \frac{1}{6}\langle(\Delta r)^2\rangle\,\nabla^2 V \qquad (2.70)$$

The perturbation $\langle\Delta V\rangle$ changes the energy of an atomic state $|n\ell m_\ell\rangle$ by an amount Δ. To first order the energy shift is

$$\Delta = \langle n\ell m_\ell|(\langle\Delta V\rangle)|n\ell m_\ell\rangle \qquad (2.71)$$

$$= \frac{1}{6}\langle(\Delta r)^2\rangle\,\langle n\ell m_\ell|\nabla^2 V|n\ell m_\ell\rangle \qquad (2.72)$$

Using $V(r) = e^2/r$ and $\nabla^2(1/r) = 4\pi\delta(r)$, we find

$$\langle n\ell m_\ell|\nabla^2 V|n\ell m_\ell\rangle = 4\pi e^2\,|\psi_{n\ell m_\ell}(r=0)|^2 \qquad (2.73)$$

Non-relativistic atomic wavefunctions vanish at the origin except for s-states with $\ell = 0$, where

$$|\psi_{noo}(r=0)|^2 = 1/\pi n^3 a_o^3 \qquad (2.74)$$

and a_o is the Bohr radius. For p-states the wavefunction vanishes and hence so does the energy shift. The mean square displacement $\langle(\Delta r)^2\rangle$ is obtained by assuming that the important field frequencies greatly exceed the atomic resonance frequencies; lower frequencies are shielded by the atomic binding and cannot influence the motion of the electron. The electron-field interaction

leads to an equation of motion for the displacement induced by the fluctuating field. The displacement induced with frequency between ν and $\nu + d\nu$ is determined by

$$\frac{d^2}{dt^2}(\Delta r_\nu) = \frac{eE_\nu}{m} \exp(2\pi i\nu t) \tag{2.75}$$

with the solution

$$\Delta r_\nu = \frac{e}{m}\frac{E_\nu}{4\pi^2\nu^2} \exp(2\pi i\nu t) \tag{2.76}$$

The mean square displacement induced by these modes is

$$\langle(\Delta r_\nu)^2\rangle = \frac{e^2}{m^2}\frac{E_\nu^2}{32\pi^4\nu^4} = \frac{e^2\hbar}{\pi^2 m^2 c^3}\frac{d\nu}{\nu} \tag{2.77}$$

using eq.(2.68). We obtain the energy shift for s-states from eqs.(2.72) - (2.77) summed over all frequencies.

$$\Delta = \frac{2}{3}\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{\hbar}{mc}\right)^2 \frac{\hbar c}{\pi^2 n^3 a_o^3}\int\frac{d\nu}{\nu} \tag{2.78}$$

where $(e^2/\hbar c) = \alpha$ the fine structure constant, and $(\hbar/mc) = \lambda_c$ the Compton wavelength of the electron. The divergent frequency integral is cut off at both high and at low frequencies. At low frequencies the atom does not respond to the fluctuating electric field and a natural cut-off is the frequency of the electron in its orbit, $\nu_o = e^2/\hbar a_o^3 n^3$. The analysis also breaks down at high frequencies where relativistic effects affect the electron's motion. The preceding analysis is limited to velocities $v << c$ (Power 1964),

$$\frac{v}{c} = \frac{(p/m)}{c} = \frac{pc}{mc^2} = \frac{\hbar k}{mc} \lesssim 1$$

which restrict wavenumber k to less than $(mc/\hbar)$ and angular frequencies to less than $(mc^2/\hbar)$ in the integral in eq.(2.78). For the $2^2S_{1/2}$ state in hydrogen using $a_o = \hbar^2/me^2$, the energy shift is

$$\Delta = \frac{1}{6\pi}\alpha^3\frac{me^4}{\hbar^2}\log\left(\frac{mc^2}{h\nu_o}\right), \tag{2.79}$$

giving $\Delta \sim 1000$ MHz. The $2^2P_{1/2}$ state is unaffected to this order by the radiative corrections leaving the Lamb shift $\Delta(2^2S_{1/2}) - \Delta(2^2P_{1/2}) \approx 1000$ MHz. A review of the current status of the Lamb shift theory and experiment is to be found in the paper by Newton et al. (1979). As well as shifting the atomic levels, the fluctuating vacuum field can be thought of an "inducing" spontaneous decay from excited atomic states (Schiff 1955, Milonni 1976).

The fully-quantized approach outlined in this chapter can be used to derive the wave and particle parts of the Einstein fluctuation formula in a very simple way. The mean square photon number fluctuation of a single-mode field is

$$\langle(\Delta n)^2\rangle = \langle\hat{a}^+\hat{a}\hat{a}^+\hat{a}\rangle - \langle\hat{a}^+\hat{a}\rangle^2 \tag{2.80}$$

The first term on the right hand side of eq.(2.80) can be written in normal order, that is with creation operators to the left, annihilation operators to the right, using the commutator $[\hat{a},\hat{a}^+] = 1$. We find

$$\begin{aligned}\langle(\Delta n)^2\rangle &= \langle\hat{a}^+(\hat{a}^+\hat{a} + 1)\hat{a}\rangle - \langle\hat{a}^+\hat{a}\rangle^2 \\ &= \langle\hat{a}^+\hat{a}\rangle + (\langle\hat{a}^+\hat{a}^+\hat{a}\hat{a}\rangle - \langle\hat{a}^+\hat{a}\rangle^2) \\ &= \langle n\rangle + \langle(\Delta n)^2\rangle_{\text{waves}} .\end{aligned} \tag{2.81}$$

The second term has the character of the fluctuations in intensity of a classical wave. This is the only term present in a classical analysis where $\hat{a}$ and $\hat{a}^+$ are taken to commute and the classical wave amplitude is proportional to $\hat{a}$. The first or particle-like term in eq.(2.81) arises from the non-commutability of $\hat{a}$ and $\hat{a}^+$ (or $\hat{p}$ and $\hat{q}$). The single-mode zero-point energy is also attributed to this noncommutability

$$\begin{aligned}\hat{\mathcal{H}} &= \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2) = \frac{1}{2}\hbar\omega\,(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+) \\ &= \hbar\omega\,(\hat{a}^+\hat{a} + \frac{1}{2})\end{aligned} \tag{2.82}$$

And so, as noticed first by Born, Heisenberg and Jordan (1926), the zero-point energy and the particle fluctuation term in $\langle(\Delta n)^2\rangle$ are closely related.

2.6 MODE OCCUPATION AND PHOTONS

The electric field operator in an arbitrary enclosure can be written, generalising eq.(2.63) as

$$\hat{\underline{E}}(\underline{r},t) = i\sum_{\underline{k}}(\hbar\omega_k)^{\frac{1}{2}}[\hat{a}_{\underline{k}}(t)\underline{F}_{\underline{k}}(\underline{r}) - \hat{a}^+_{\underline{k}}(t)\underline{F}^*_{\underline{k}}(\underline{r})] \tag{2.83}$$

where the mode function $\underline{F}_{\underline{k}}(\underline{r})$ satisfies the wave-equation subject to the usual boundary conditions used in the classical theory. Interference effects are determined by the spatial mode functions $\underline{F}_{\underline{k}}(\underline{r})$ and are precisely those of classical theory. The size of the interference effects however depends on the expectation value of combinations of field mode annihilation and creation operators. For example, the intensity distribution depends on the probability of detecting a *single* photon at position $\underline{r}$ and time t. This involves terms of the form $\langle\hat{a}^+_{\underline{k}}(t)\hat{a}_{\underline{k}}(t)\rangle|\underline{F}_{\underline{k}}(\underline{r})|^2$ and cross-terms between the different contributing modes. The interference pattern is built up from a superposition of one-photon probability patterns (Taylor (Reprinted Paper 1)). Pfleegor and Mandel (Reprinted Paper 6) and Magyar and Mandel (Reprinted Paper 7) have demonstrated that the fields from the independent lasers produce interference fringes. This may be difficult to understand using a crude billiard ball interpretation of the "photon". Since however the photon is an occupation of one of the normal modes of the whole system including both lasers, the quantum explanation is straightforward. The detector measures the occupation of normal modes. In this experiment there is no way of telling from which laser the energy derived and fringes would be expected.

If it is known that only a single photon is present in the apparatus at any one time, an attempt to divide the photon with a beam splitter can be made. Photon division is precluded by quantum theory but is allowed by semi-classical radiation theory, or by any theory in which the photon is viewed as a packet of electromagnetic energy. This experiment was performe by Clauser in 1974 (Reprinted Paper 8) who found no evidence that photons ca be split, in clear contradiction with semi-classical theory. Such disagree ments between semi-classical and quantum theories of coherence are rare: in Chapter 6 we discuss them further.

CHAPTER 3

Absorption and Emission of Radiation

3.1 INTERACTION OF AN ATOM WITH A RADIATION FIELD

In this Chapter we review the application of quantum mechanical perturbation theory to the absorption and emission of electromagnetic radiation by atoms. For details, the reader should refer to a standard quantum mechanics text (e.g. Merzbacher 1961, Matthews 1974). We are concerned here with drawing out certain important features which play a key role in quantum optics. Suppose at time $t < 0$ that we have an unperturbed atomic system described by the Schrödinger equation

$$H_o|k\rangle = \hbar\omega_k|k\rangle. \tag{3.1}$$

We will assume that the eigenstates $|k\rangle$ and their energies $\hbar\omega_k$ are in some way known, and that the atom is prepared in a particular state at $t \leq 0$. For most situations this will be the atomic ground state which we will label $|i\rangle$. Then at time t_o we imagine the interaction between the atom and a classical electromagnetic field is switched on. For example a light pulse might be injected into a vapour cell containing the atoms, or an atomic beam intersects a beam of optical radiation. In this way the interaction can be switched on and off in a controlled way. We will calculate in this section the probability of the incident radiation exciting a transition to an excited state of the atom.

The coupling of an atom to an electromagnetic field can be described in a number of different ways. The radiation field can be described in terms of the vector potential $\underline{A}$ which couples to the atomic electron through the "minimal substitution" Hamiltonian (Power 1964)

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{\underline{p}} - e\underline{A}(\underline{r},t)\right)^2 + \hat{V}(r) \tag{3.2}$$

where $V(r)$ is the Coulomb interaction which binds the electron to the atomic nucleus. The unperturbed part of the Hamiltonian is

$$\hat{\mathcal{H}}_o = \frac{\hat{p}^2}{2m} + \hat{V}(r) \tag{3.3}$$

and the external perturbation due to the radiative interaction is

$$\hat{\mathcal{H}}' = \hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2 = \frac{e}{m}\hat{\underline{p}}.\underline{A}(\underline{r}) + \frac{e^2}{2m}\underline{A}^2(\underline{r}), \tag{3.4}$$

where the external radiation is described in the Coulomb gauge by the wave-equation