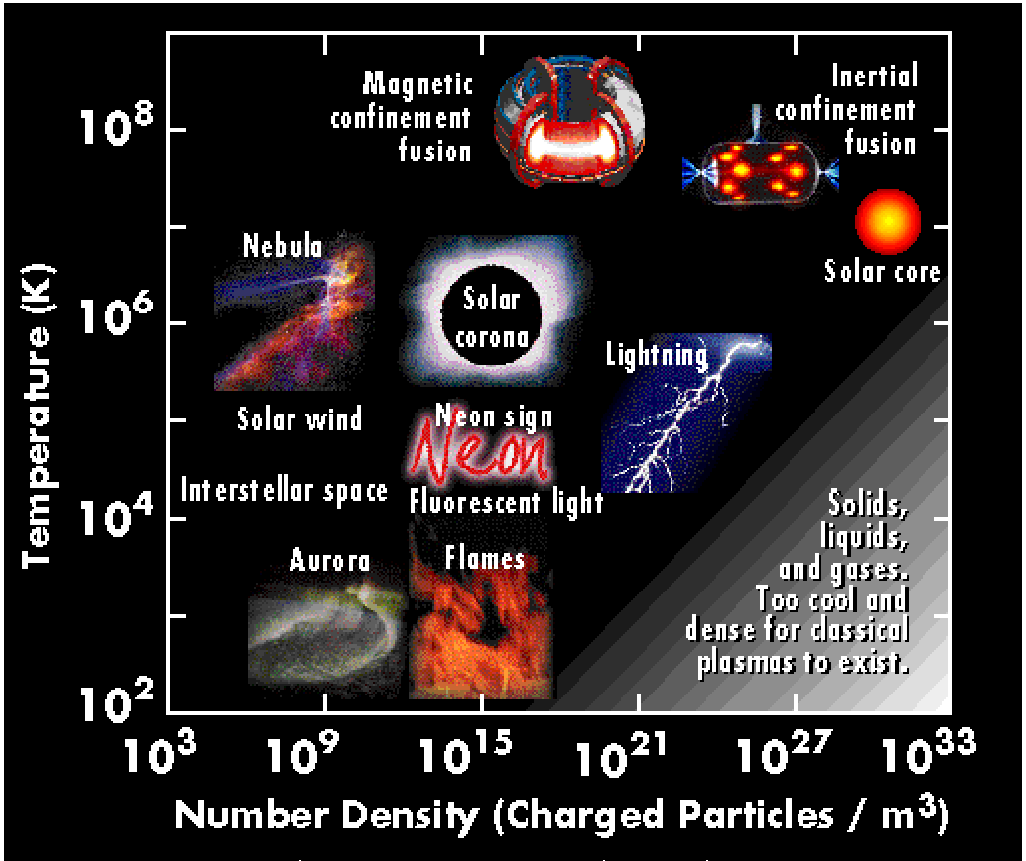
Fundamentals of plasma physics in a nutshell

Plasmas on earth and in the universe



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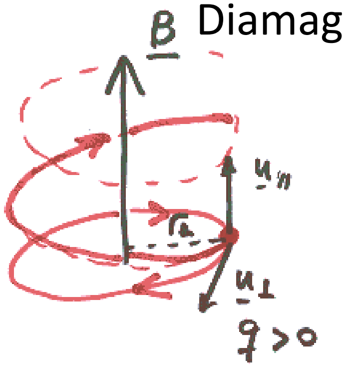
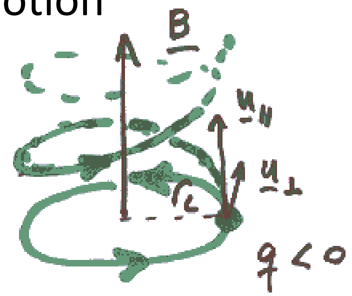
Motion of charges in magnetic fields

$$m \frac{d\underline{u}}{dt} = q\, \underline{u} \times \underline{B}$$

Diamagnetic motion





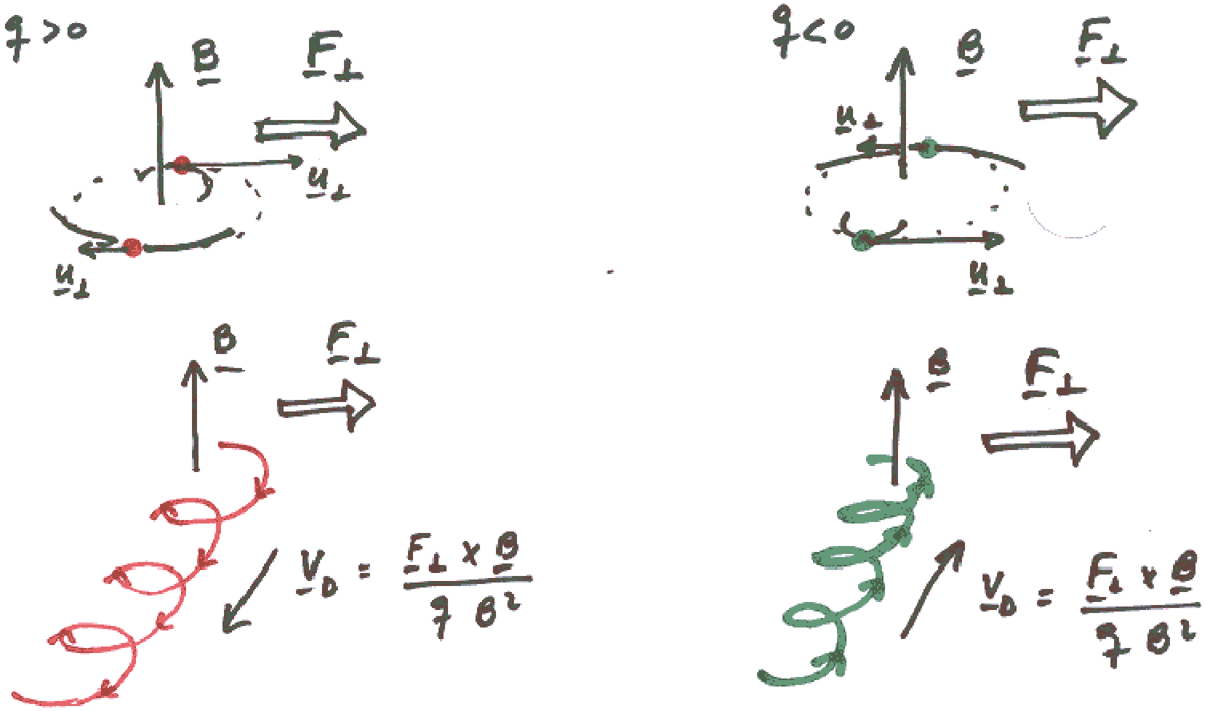
Larmor radius $r_L = \frac{|u_{\perp}|\, m}{|q|\, B}$

$\omega_c = \frac{|q| B}{m}$ Cyclotron frequency

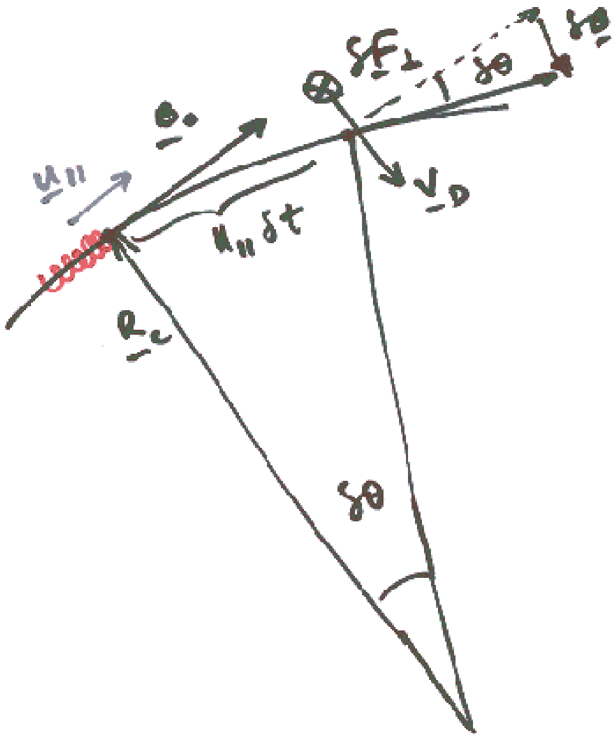
$$|\underline{u}_{\perp}| = \omega_c\, r_L$$

$$\begin{cases} W_{\perp} = \frac{1}{2} m u_{\perp}^2 \\ W_{\parallel} = \frac{1}{2} m u_{\parallel}^2 \end{cases}$$

Drift due to perpendicular force



Gyration center follows curved B lines

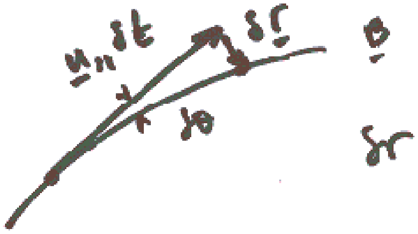


$$\delta\theta = \frac{u_{\parallel} \delta t}{R_c} \quad ; \quad \delta B = \delta\theta \, B_0$$

$$\delta \underline{F}_{\perp} = q \, \underline{u}_{\parallel} \times \delta \underline{B}$$

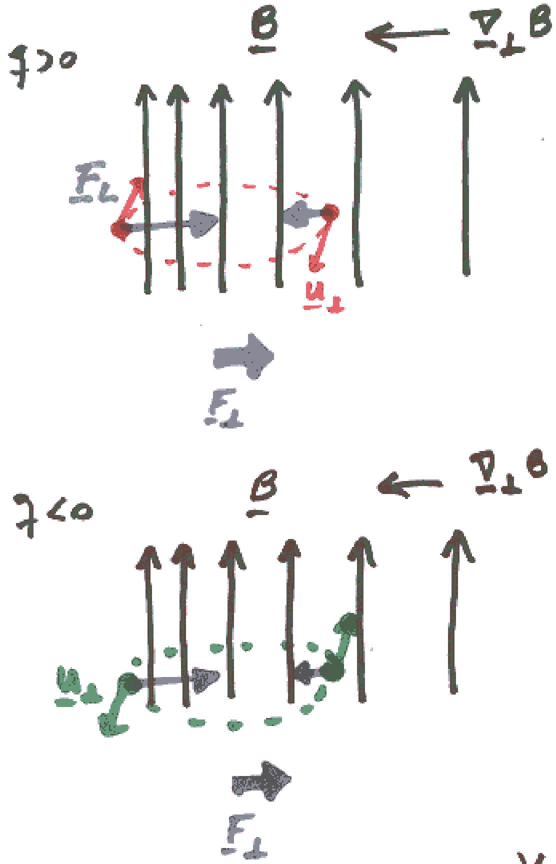
$$\underline{V}_D = \frac{\delta \underline{F}_{\perp} \times \underline{B}_0}{q \, B_0^2}$$

$$\delta \underline{r} = \underline{V}_D \, \delta t \quad ; \quad \delta r = \frac{u_{\parallel}^2 \, \delta t^2}{R_c}$$



$$\delta r = u_{\parallel} \delta t \, \delta\theta$$

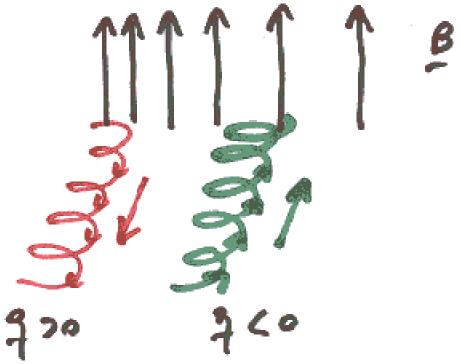
Drift due to varying *B* intensity



$$\underline{F}_L = q\, \underline{u}_\perp \times \underline{B}$$

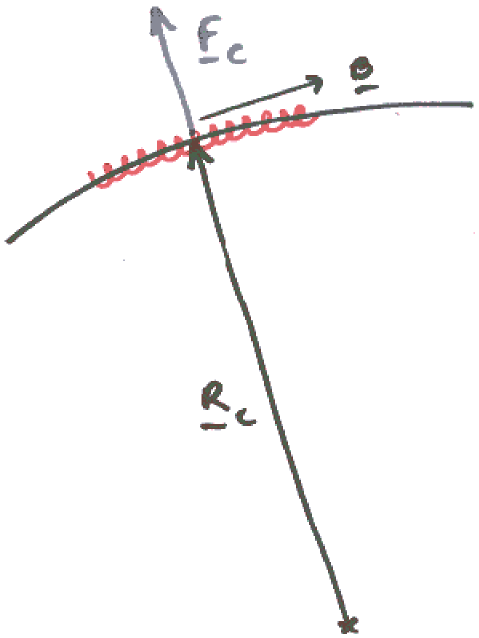
$$\underline{F}_\perp = \langle q\, \underline{u}_\perp \times \underline{B} \rangle =$$

$$= -\frac{1}{2} |q|\, r_L^2\, \omega_L\, \underline{\nabla}_\perp B$$



$$\underline{V}_0 = \frac{\underline{F}_\perp \times \underline{B}}{q B^2} = -W_\perp \frac{\underline{\nabla}_\perp B \times \underline{B}}{q B^3} \equiv \underline{V}_{GRAD}$$

Drift due to curved B lines

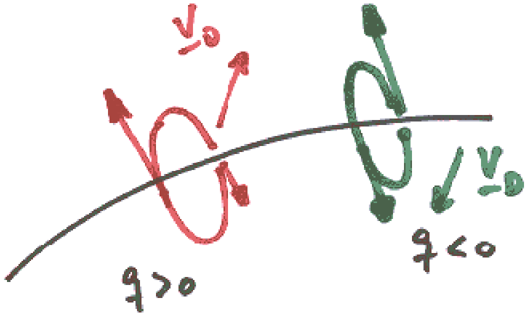


$\underline{F}_c$ in system of moving center

$$F_c = m\frac{u_{\parallel}^2}{R_c} = \frac{2W_{\parallel}}{R_c}\ ;\ \underline{F}_c = F_c\frac{\underline{R}_c}{R_c}$$

$$\underline{V}_D = \frac{\underline{F}_c\times\underline{B}}{q\,B^2} = \frac{2W_{\parallel}}{q}\,\frac{\underline{R}_c\times\underline{B}}{R_c^2\,B^2} \equiv$$

$$\equiv \underline{V}_{\text{curv.}}$$

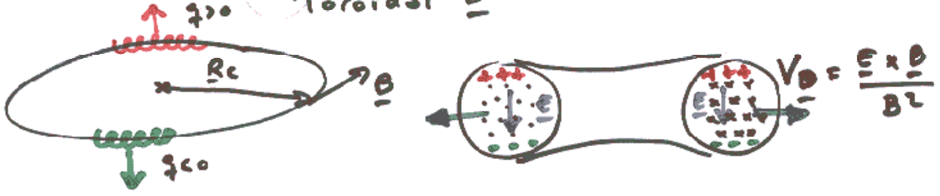


Combined drifts

If $\underline{\nabla} \times \underline{B} = 0 \Rightarrow \frac{\underline{\nabla}_\perp B \times \underline{B}}{B} = -\frac{\underline{R}_c \times \underline{B}}{R_c^2}$

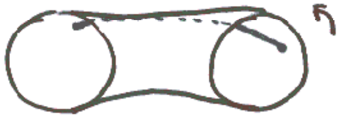
$$\Rightarrow \underline{V}_{GRAD} + \underline{V}_{CURV.} = (W_\perp + 2W_\parallel)\frac{\underline{R}_c \times \underline{B}}{q R_c^2 B^2}$$

Toroidal $\underline{B}$

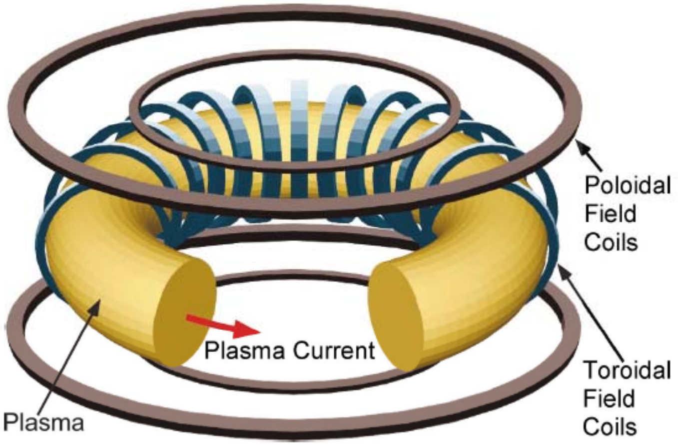


$\underline{V}_D = \frac{\underline{E} \times \underline{B}}{B^2}$

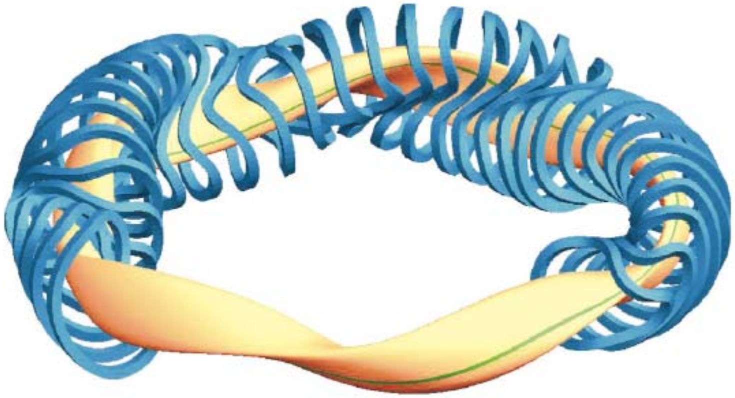
Solution :



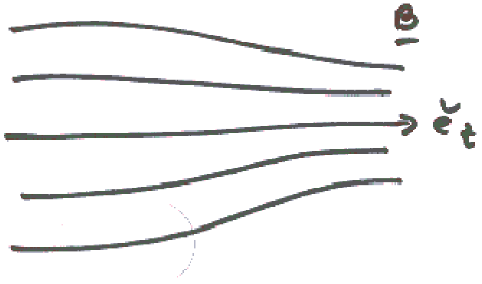
tokamak



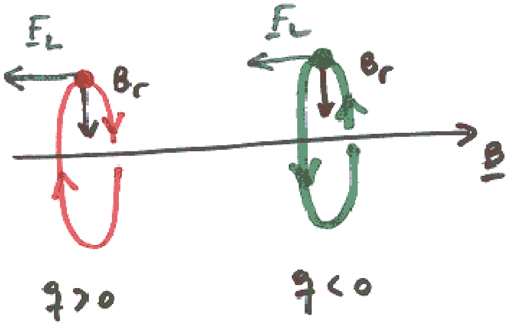
stellarator



Magnetic mirror



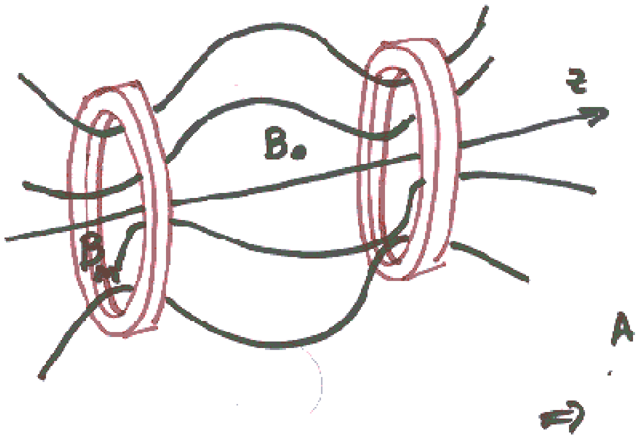
$$\underline{\nabla} \cdot \underline{B} = 0 \Rightarrow B_r \simeq -\frac{r}{2} \frac{dB_z}{dz}$$



$$\underline{F}_L = q\, \underline{u}_\perp \times B_r\, \check{e}_r$$

$$F_L = |q| \frac{\omega_c r_L^2}{2} \left| \frac{dB_z}{dz} \right|$$

$$= \frac{W_\perp}{B} \left| \frac{dB}{dz} \right|$$



$$m\frac{du_{\parallel}}{dt} = \underbrace{m\,u_{\parallel}\frac{du_{\parallel}}{dz}}_{\frac{dW_{\parallel}}{dz}} = -\frac{W_{\perp}}{B}\frac{dB}{dz}$$

Also $W_{\parallel} + W_{\perp} = \text{constant} \Rightarrow$

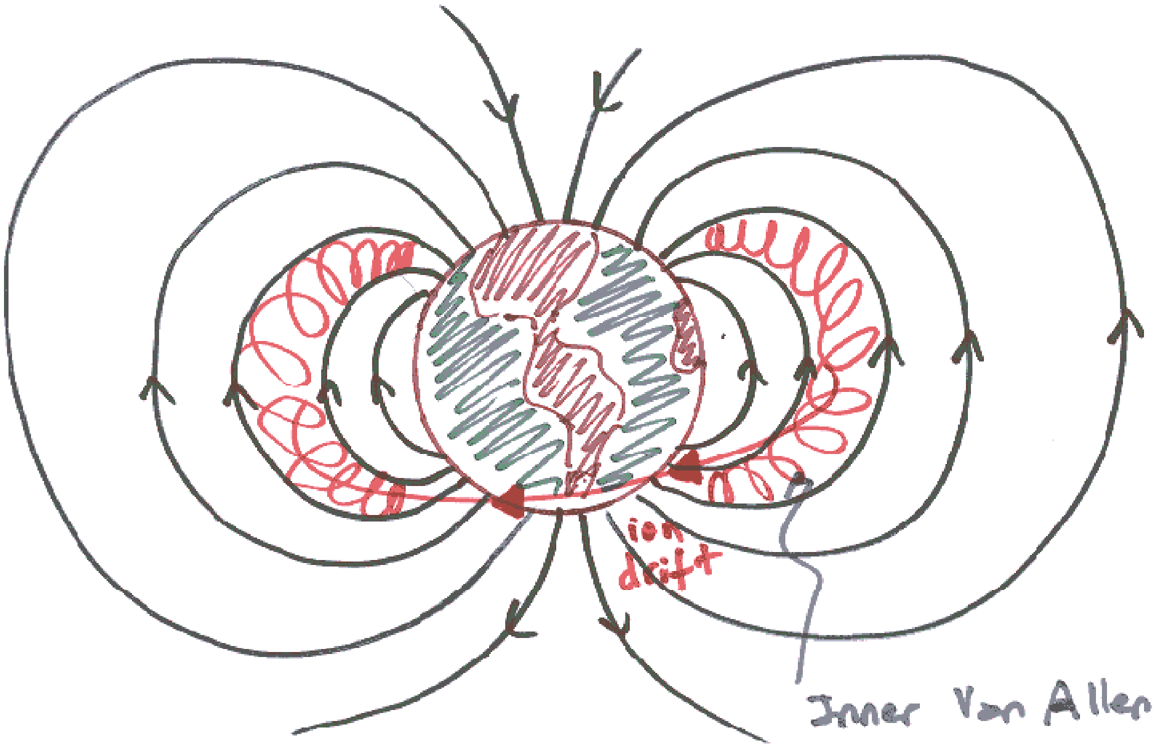
$$\Rightarrow \frac{dW_{\perp}}{dz} = \frac{W_{\perp}}{B}\frac{dB}{dz} \Rightarrow \frac{W_{\perp}}{B} = \text{constant}$$

$$\begin{cases} W_{\parallel} + W_{\perp} = W_{\parallel o} + W_{\perp o} \\ \dfrac{W_{\perp}}{B} = \dfrac{W_{\perp o}}{B_o} \end{cases}$$

Particles escape if $W_{\parallel M} > 0$

Confinement so requires

$$W_{\parallel o} \leq W_{\perp o}\left(\frac{B_M}{B_o} - 1\right)$$

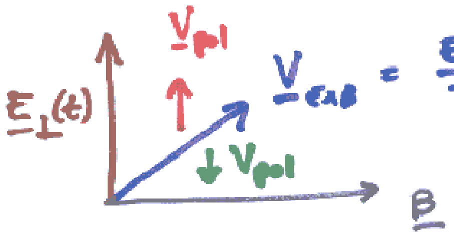


Inner Van Allen belt
Mirror trapping of
h.e. ions $\gtrsim$ 100 MeV
& electrons ~ 0,5 MeV

Polarization drift

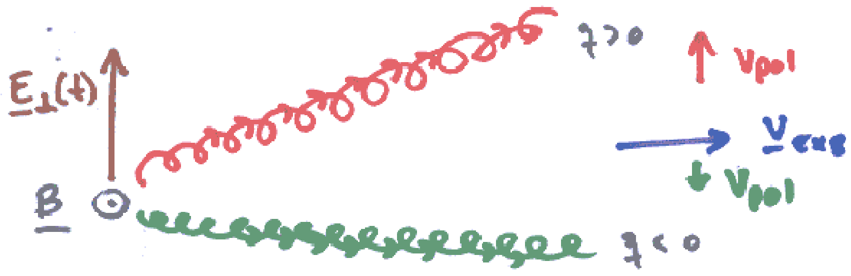
Polarization drift:

For time dependent $E_\perp$; $\frac{\partial}{\partial t} << \omega_{ci}$



$$\underline{V}_{E\times B} = \frac{\underline{E} \times \underline{B}}{B^2}$$

$$\underline{V}_{pol} = \frac{m}{qB^2} \frac{\partial \underline{E}_\perp}{\partial t}$$



Debye length

Electron and ion gases in equilibrium:

$$\begin{aligned} -T\nabla n_e + e n_e \nabla\phi &= 0, \\ -T\nabla n_i - Zen_i\nabla\phi &= 0, \end{aligned}$$

$$\nabla^2\phi = -\frac{e\left(Zn_i - n_e\right)}{\varepsilon_0}.$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\frac{Q}{4\pi\varepsilon_0 r^2}\delta\left(r\right) + \frac{e^2 n_{e0}}{\varepsilon_0 T}\left(1+Z\right)\phi,$$ Linearized Poisson equation:

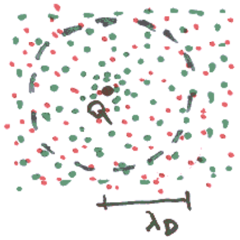
with solution:

$$\phi = \frac{Q}{4\pi\varepsilon_0 r}\exp\left(-r/\lambda_D\right),$$

$$\lambda_D \equiv \sqrt{\frac{\varepsilon_0 T}{e^2 n_{e0}\left(1+Z\right)}}$$

$$\lambda_D\left[m\right] = 7.4\times 10^3\sqrt{\frac{T\left[eV\right]}{n_{e0}\left[m^{-3}\right]\left(1+Z\right)}}$$

$$1\ eV = 1.6\times 10^{-19} J = 11600^\circ K$$



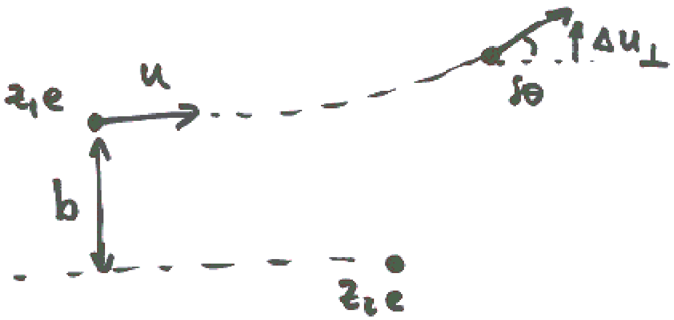
$$N_D = \frac{4\pi}{3}\lambda_D^3 n_{e0}\left(1+\frac{1}{Z}\right) >> 1$$

$$d \sim n_{e0}^{-1/3} \qquad U \sim \frac{e^2}{4\pi\varepsilon_0 d}$$

$$W \sim T$$

$$\frac{W}{U} \sim N_D^{2/3} >> 1$$

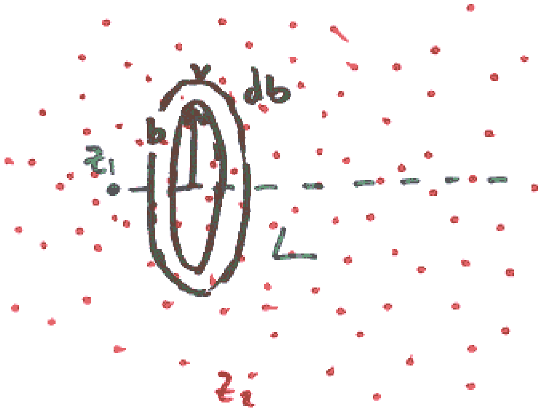
Collisions



$$\Delta u_\perp \sim \frac{F\,\Delta t}{m_1} \sim \frac{z_1 z_2 e^2}{4\pi\varepsilon_0 b^2 m_1}\,\frac{b}{u}$$

$$\delta\theta \sim \frac{\Delta u_\perp}{u} \sim \frac{b_0}{b}$$

$$b_0 \equiv \frac{z_1 z_2 e^2}{4\pi\varepsilon_0 m_1 u^2}$$



$$\langle(\delta\theta)^2\rangle \sim \int_{b_{min}}^{b_{max}} (\delta\theta)^2\, n_2\, 2\pi b\, db\; L$$

$$\langle(\delta\theta)^2\rangle \sim 2\pi n_2 b_0^2 \ln\left(\frac{b_{max}}{b_{min}}\right) L$$

$b_{max} \sim \lambda_D$; $b_{min} \Rightarrow \delta\theta \sim 1$ for $m_1 u^2 \sim T \Rightarrow$

$$b_{min} \sim \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 T}$$

$$\frac{b_{max}}{b_{min}} \sim \frac{4\pi\epsilon_0 T \lambda_D}{Z_1 Z_2 e^2} \equiv \Lambda \sim N_D$$

$$\langle(\delta\theta)^2\rangle \sim 2\pi n_2 b_0^2 \ln\Lambda \; L$$

$\langle(\delta\theta)^2\rangle \sim 1$ if $L \sim \lambda_\perp \equiv \dfrac{1}{2\pi n_2 b_0^2 \ln\Lambda} \equiv \dfrac{1}{n_2 \sigma_\perp}$

$$\sigma_\perp \sim \frac{(Z_1 Z_2)^2 e^4}{8\pi\epsilon_0^2 (m_1 u)^2} \ln\Lambda$$

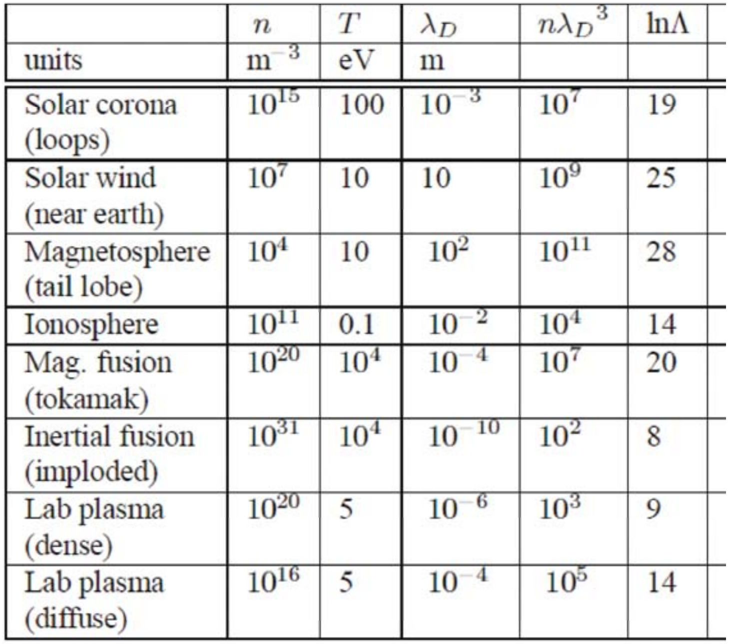
Large deviation in single (Coulomb) collision if

$$\delta\theta \sim \frac{b_0}{b} \sim 1 \Rightarrow b \sim b_0 = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 m_1 u^2}$$

Corresponding cross-section:

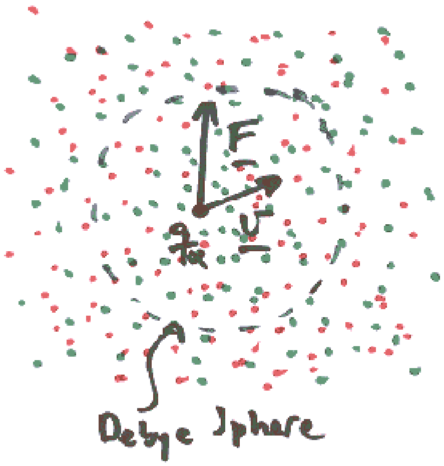
$$\sigma_c \sim 4\pi b^2 \sim \frac{(Z_1 Z_2)^2 e^4}{4\pi \epsilon_0^2 (m_1 u^2)^2} \sim \frac{\sigma_\perp}{\ln\Lambda} \ll \sigma_\perp$$

$$\left(\ln\Lambda \sim \ln N_D \sim 10-30 \right)$$



	n	T	λ_D	$n\lambda_D{}^3$	$\ln\Lambda$
units	m^{-3}	eV	m		
Solar corona (loops)	10^{15}	100	10^{-3}	10^7	19
Solar wind (near earth)	10^7	10	10	10^9	25
Magnetosphere (tail lobe)	10^4	10	10^2	10^{11}	28
Ionosphere	10^{11}	0.1	10^{-2}	10^4	14
Mag. fusion (tokamak)	10^{20}	10^4	10^{-4}	10^7	20
Inertial fusion (imploded)	10^{31}	10^4	10^{-10}	10^2	8
Lab plasma (dense)	10^{20}	5	10^{-6}	10^3	9
Lab plasma (diffuse)	10^{16}	5	10^{-4}	10^5	14

Kinetic description



$$\underline{F} = \underline{F}_c + \underline{F}_{NC}$$

due to particles inside the Debye sphere (collisions)

$\sigma \sim \sigma_\perp$

due to particles outside the Debye sphere

$q_\alpha(\underline{E} + \underline{v} \times \underline{B})$

"Smooth" Fields

$$\frac{\partial f_\alpha}{\partial t} + \underline{v} \cdot \underline{\nabla} f_\alpha + \frac{\underline{F}}{m_\alpha} \cdot \frac{\partial f_\alpha}{\partial \underline{v}} = 0$$

Transition to fluid description

$$n_\alpha = \int f_\alpha(\underline{x}, \underline{v}, t)\, d^3v$$

$$\underline{u}_\alpha = \frac{1}{n_\alpha}\int \underline{v}\; f_\alpha(\underline{x}, \underline{v}, t)\, d^3v \equiv \langle \underline{v} \rangle_\alpha \quad ; \quad \delta\underline{v} = \underline{v} - \underline{u}_\alpha$$

$$\left\{ \begin{aligned} &\frac{\partial n_\alpha}{\partial t} + \underline{\nabla}\cdot(n_\alpha \underline{u}_\alpha) = 0 \\ &\frac{\partial \underline{u}_\alpha}{\partial t} + (\underline{u}_\alpha \cdot \underline{\nabla})\,\underline{u}_\alpha = -\frac{1}{n_\alpha}\underline{\nabla}\cdot\Big[n_\alpha \overbrace{\langle \delta\underline{v}\,\delta\underline{v} \rangle_\alpha}^{\text{modelled or } \frac{T_\alpha}{m_\alpha}\underline{\underline{1}}}\Big] \\ &\qquad + \frac{q_\alpha}{m_\alpha}(\underline{E} + \underline{u}_\alpha \times \underline{B}) + \sum_{\beta \neq \alpha} (u_\beta - u_\alpha)\, \underset{\uparrow}{\nu_{\alpha\beta}} \end{aligned} \right.$$

collision frequency
with β species

Two-fluid model

Two fluids (electrons & ions)

$$\begin{cases} \dfrac{\partial n_{e,i}}{\partial t} + \underline{\nabla}\cdot(n_{e,i}\,\underline{u}_{e,i}) = 0 \\ m_e n_e \dfrac{d\underline{u}_e}{dt} = -\underline{\nabla} p_e - e n_e(\underline{E} + \underline{u}_e \times \underline{B}) + \underline{R}_{ei} \\ m_i n_i \dfrac{d\underline{u}_i}{dt} = -\underline{\nabla} p_i + z e n_i(\underline{E} + \underline{u}_i \times \underline{B}) - \underline{R}_{ei} \end{cases}$$

$\underline{R}_{ei} \equiv m_e n_e(\underline{u}_i - \underline{u}_e)\nu_{ei}$; $p_{e,i} = n_{e,i} T_{e,i}$

polytropic model for pressures: $p n^{-\gamma} = \text{const.}$

$+$ Maxwell Eqs.

$$\begin{cases} \underline{\nabla}\cdot\underline{E} = \dfrac{e}{\epsilon_0}(z n_i - n_e) \;;\; \underline{\nabla}\times\underline{E} = -\dfrac{\partial \underline{B}}{\partial t} \\ \underline{\nabla}\cdot\underline{B} = 0 \;;\; \underline{\nabla}\times\underline{B} = \mu_0 e(z n_i \underline{u}_i - n_e \underline{u}_e) \\ \qquad\qquad + \dfrac{1}{c^2}\dfrac{\partial \underline{E}}{\partial t} \end{cases}$$

Single fluid MHD

Single fluid MHD

$$\left|\frac{\partial}{\partial \underline{x}}\right| \ll \frac{1}{\lambda_D} \Rightarrow Zn_i \simeq n_e \Rightarrow \underline{j} = e n_e (\underline{u}_i - \underline{u}_e)$$

$$\underline{u} \equiv \frac{m_i n_i \underline{u}_i + m_e n_e \underline{u}_e}{m_i n_i + m_e n_e} \quad , \quad \rho = m_i n_i + m_e n_e$$

$$p = p_e + p_i$$

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0 \\ \rho \dfrac{d\underline{u}}{dt} = - \underline{\nabla} p + \underline{j} \times \underline{B} + \overbrace{\cancelto{\simeq 0}{\rho_e} \underline{E} - \frac{m_e}{e^2} \cancelto{\simeq 0}{\underline{\nabla}} \cdot \left(\frac{\underline{j}\,\underline{j}}{n_e} \right)}^{\text{very small as compared with } \rho (\underline{u} \cdot \underline{\nabla}) \underline{u}} \end{cases}$$

$$\text{M.E.} : \begin{cases} \underline{\nabla} \cdot \underline{B} = 0 \quad ; \quad \underline{\nabla} \times \underline{B} = \mu_0 \underline{j} \\ \underline{\nabla} \times \underline{E} = - \dfrac{\partial \underline{B}}{\partial t} \end{cases}$$

Ohm law : electron momentum eq. with $m_e \to 0$

$$\underline{E} + \underline{u} \times \underline{B} = \eta \, \underline{j} + \underbrace{\frac{1}{e n_e}\left(\cancelto{\simeq 0}{\underline{j} \times \underline{B}} - \cancelto{\simeq 0}{\nabla p_e} \right)}_{\text{small as compared with } \underline{u} \times \underline{B}}$$

if $\left| \frac{\partial}{\partial \underline{x}} \right| \ll \frac{1}{r_{Li}}$

$$\eta \equiv \frac{m_e \nu_{ei}}{e^2 n_e} \quad : \text{resistivity}$$

$$\nu_{ei} \sim n_i \langle \sigma_\perp u_e \rangle \sim \frac{n_i Z^2 e^4 \ln \Lambda}{8 \pi \varepsilon_0^2 m_e^2} \underbrace{\left\langle \frac{1}{u_e^3} \right\rangle}_{\sim \left(\frac{m_e}{T_e} \right)^{3/2}}$$

$$\Rightarrow \eta \sim \frac{Z e^2 \sqrt{m_e}}{8 \pi \varepsilon_0^2 T_e^{3/2}} \ln \Lambda$$

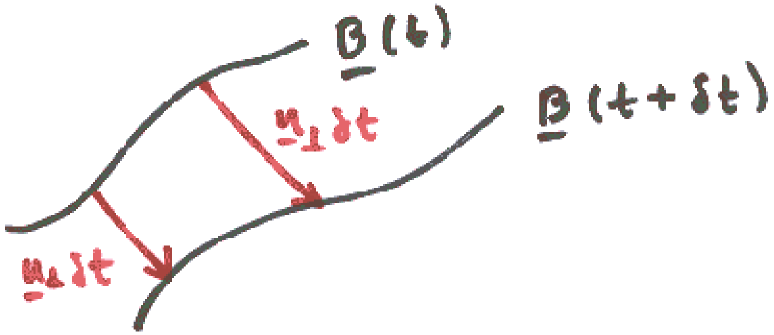
: small, comparable to metals

Magnetic flux freezing

With $\eta \to 0$

$\underline{E} + \underline{u} \times \underline{B} = 0$ + $\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ $\Rightarrow$

$\Rightarrow$ $\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{u} \times \underline{B})$: Lines of $\underline{B}$ frozen in the plasma



Magnetic Reynolds number

In general:

$$\underline{E} + \underline{u} \times \underline{B} = \eta \underline{j} \quad + \quad \underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t} \quad + \quad \underline{\nabla} \times \underline{B} = \mu_0 \underline{j} \quad \Rightarrow$$

$$\Rightarrow \quad \frac{\partial \underline{B}}{\partial t} = \underbrace{\underline{\nabla} \times (\underline{u} \times \underline{B})}_{\sim \frac{uB}{L}} + \underbrace{\frac{\eta}{\mu_0} \nabla^2 \underline{B}}_{\sim \frac{\eta B}{\mu_0 L^2}}$$

$$R_M = \frac{uB/L}{\eta B/\mu_0 L^2} = \frac{\mu_0 u L}{\eta}$$

Frozen $\underline{B}$ lines for $R_M \gg 1$

Magnetic pressure and tension

$$\rho \frac{d\underline{u}}{dt} = -\underline{\nabla} p + \underline{j} \times \underline{B} \qquad + \qquad \underline{\nabla} \times \underline{B} = \mu_0 \underline{j} \qquad \Longrightarrow$$

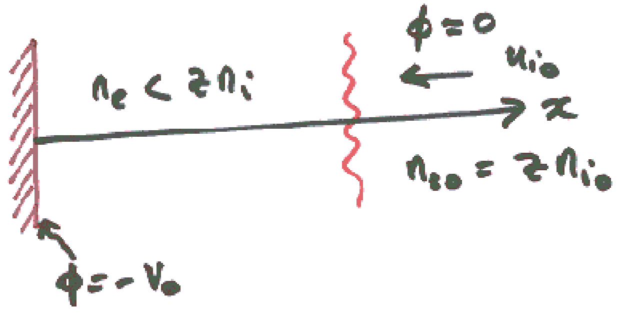
$$\Longrightarrow \rho \frac{d\underline{u}}{dt} = -\underline{\nabla} p - \underline{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \underbrace{\frac{1}{\mu_0}(\underline{B}\cdot\underline{\nabla})\underline{B}}_{\text{magnetic tension: acts to straighten bent } \underline{B} \text{ lines}}$$

magnetic pressure

$$\beta \equiv \frac{p}{B^2/2\mu_0} \quad : \quad \beta \ll 1 \; : \text{Plasma driven by } \underline{B}$$

Cathode sheath

Example two-fluid model : cathode layer (sheath)



$$\left\{ \begin{aligned} &\frac{d}{dx}(n_i u_i) = 0 \\ &m_i u_i \frac{du_i}{dx} = -Ze\frac{d\phi}{dx} \\ &0 = -\frac{T_e}{n_e}\frac{dn_e}{dx} + e\frac{d\phi}{dx} \end{aligned} \right.$$

$$+ \quad \frac{d^2\phi}{dx^2} = \frac{e}{\varepsilon_0}(n_e - Z n_i)$$

$$\left. \begin{aligned} &n_i u_i = n_{i0} u_{i0} \\ &m_i \frac{u_i^2}{2} + Ze\phi = \frac{m_i u_{i0}^2}{2} \\ &n_e = n_{e0} \exp\left(\frac{e\phi}{T_e}\right) \end{aligned} \right\} \Leftarrow$$

$$+ \quad \frac{d^2\phi}{dx^2} = \frac{e n_{e0}}{\varepsilon_0}\left[\exp\left(\frac{e\phi}{T_e}\right) - \left(1 - \frac{2Ze\phi}{m_i u_{i0}^2}\right)^{-1/2}\right]$$

multiplying by $\frac{d\phi}{dx}$ a first integral is obtained:

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{2 T_e n_{e0}}{\varepsilon_0}\left[\exp\left(\frac{e\phi}{T_e}\right) + \frac{m_i u_{i0}^2}{Z T_e}\left(1 - \frac{2Ze\phi}{m_i u_{i0}^2}\right)^{1/2} - 1 - \frac{m_i u_{i0}^2}{Z T_e}\right]$$

Taylor developing about $\phi = 0$:

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{n_{e0} e^2}{\varepsilon_0 T_e}\left(1 - \frac{Z T_e}{m_i u_{i0}^2}\right)\phi^2 + O(\phi^3) \geq 0$$

$$\Longrightarrow \quad u_{i0}^2 \geq \frac{Z T_e}{m_i} = v_{Bohm}^2$$

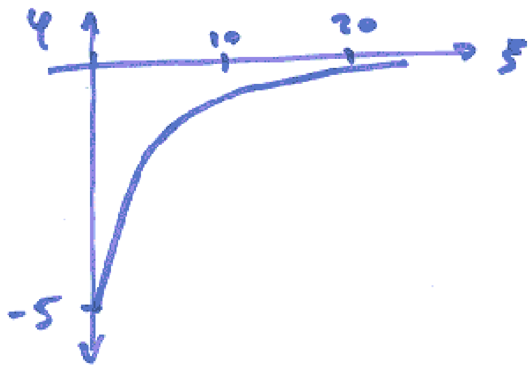
: Ions must be accelerated in pre-sheath up to (at least) v_{Bohm}

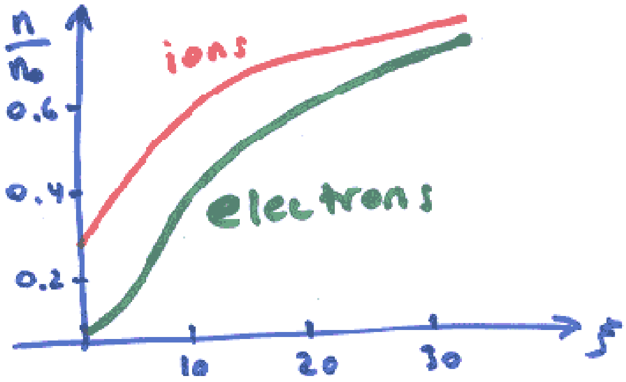
Bohm conjecture : $u_{i0}^2 = v_{Bohm}^2$.

Using $\varphi \equiv \frac{e\phi}{T_e}$, $\xi \equiv \frac{x}{\lambda_D}$, $\lambda_D = \left(\frac{\epsilon T_e^2}{2 n_{e0} e^2}\right)^{1/2}$

$$\frac{d\varphi}{d\xi} = \left[e^{\varphi} + (1-2\varphi)^{1/2} - 2\right]^{1/2} \qquad \varphi(\xi=0) = -\frac{eV_0}{T_e}$$

Numerically , for $\varphi(0) = -5$:





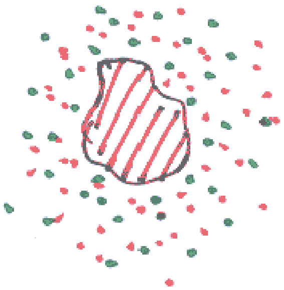
Floating potential

Electric current density to the wall :

$$j = Ze\, n_{i0}\, v_{Bohm} - e\, \underbrace{n_e(x=0)}_{n_{e0}\exp\left(-\frac{eV_0}{T_e}\right)}\, \underbrace{\frac{1}{4} v_{Te}}$$

from average velocity directed to the wall

for f_{MB} : $v_{Te} = \left(\frac{8T_e}{\pi m_e}\right)^{1/2}$



Floating object accumulates charges up to the condition $j = 0$

$$j=0 \Rightarrow V_0 = -\frac{T_e}{e} \ln\left(\frac{4 v_{Bohm}}{v_{Te}}\right) =$$

$$= -\frac{T_e}{2e} \ln\left(\frac{m_i}{2\pi Z m_e}\right) \equiv V_F$$

Floating potential (relative to the plasma)

High-voltage sheath

For a high-voltage sheath, $\frac{eV_0}{T_e} >> 1$, there are practically no electrons present

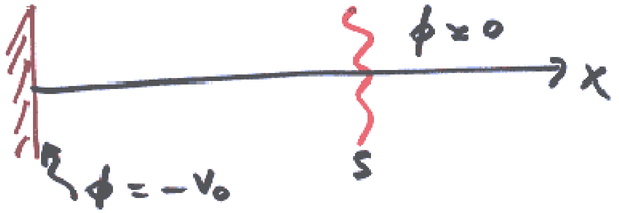
Previous results with $\left|\frac{e\phi}{T_e}\right|, \left|\frac{e\phi}{m_i u_{i0}^2}\right| >> 1$ gives

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{2 n_{i0}}{\varepsilon_0} m_i u_{i0}^2 \left(-\frac{2 Z e \phi}{m_i u_{i0}^2}\right)^{1/2} \qquad \left(\phi \simeq \frac{d\phi}{dx} \simeq 0 \text{ in the plasma}\right)$$

Calling $j_0 = Z e n_{i0} |u_{i0}|$ (current density to the wall)

one obtains on integration

$$(-\phi)^{3/4} = (s-x)\,\frac{3}{4}\left(\frac{j_0}{\varepsilon_0}\right)^{1/2}\left(\frac{8 m_i}{Z e}\right)^{1/4} \qquad s: \text{sheath thickness}$$



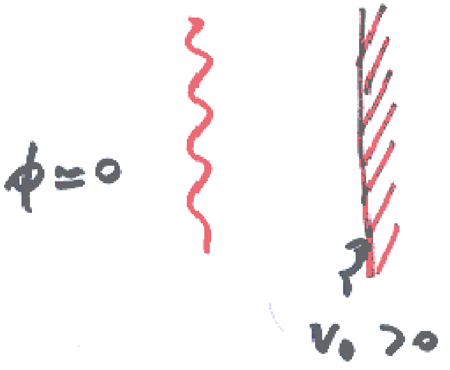
Child law:

$$j_0 \propto \frac{V_0^{3/2}}{s^2}$$

Using $u_{io} = v_{Bohm} = \left(\frac{Z T_e}{m_i}\right)^{1/2}$ one further has:

$$s = \frac{\sqrt{2}}{3} \lambda_D \left(\frac{2V_0}{T_e}\right)^{3/4}$$

Anode layer ?

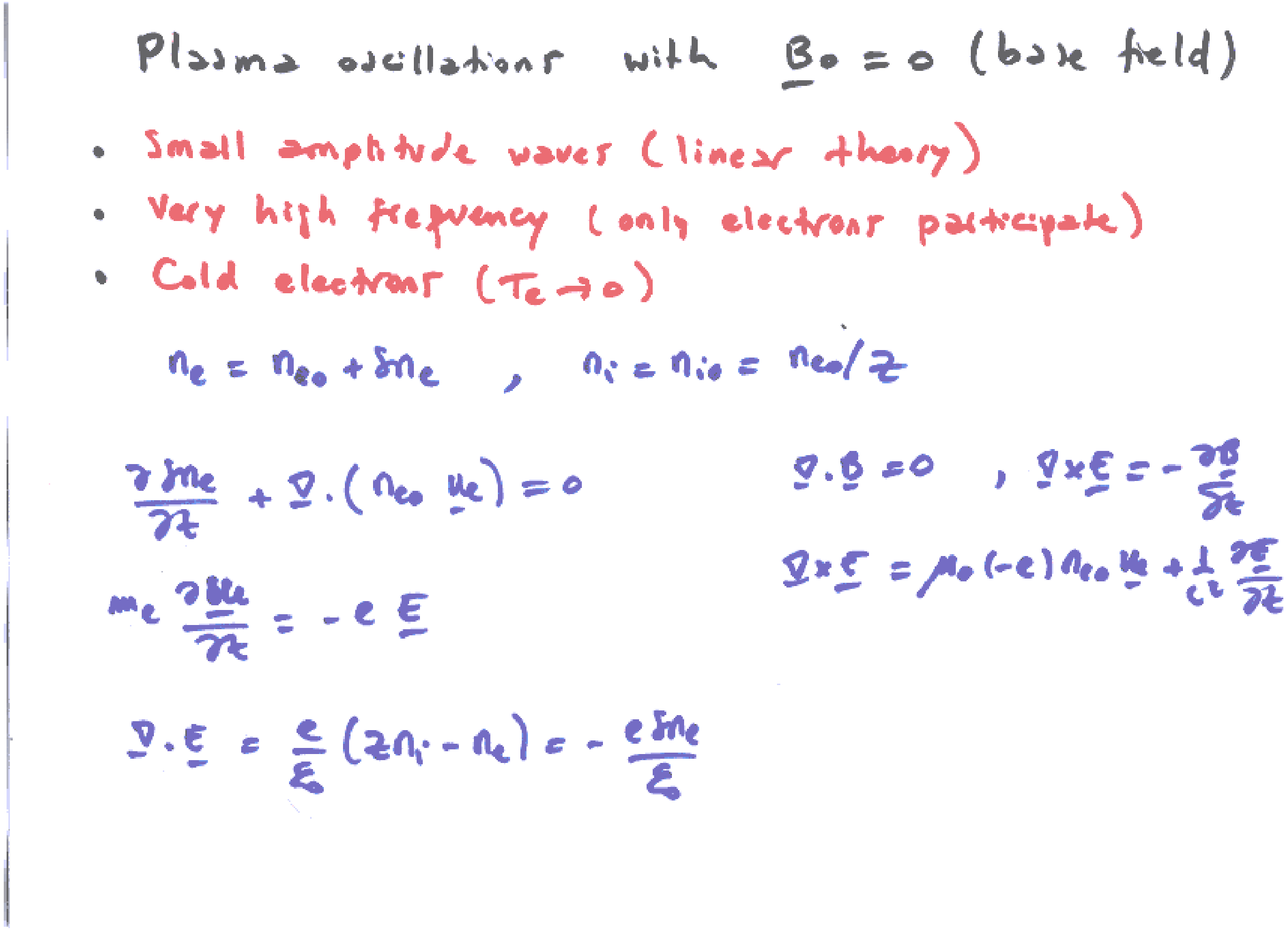


If $\frac{eV_0}{T_e} \gg 1$ enormous currents:

electrons leaving the plasma increases plasma potential to be close to V_0

Only small potential differences, $\frac{e\Delta\phi}{T_e} \sim 1$, can be sustained in anode layers.

Waves in non-magnetized plasmas



Plasma oscillations with $\underline{B}_0 = 0$ (base field)

- Small amplitude waves (linear theory)
- Very high frequency (only electrons participate)
- Cold electrons ($T_e \to 0$)

$$n_e = n_{e0} + \delta n_e \quad , \quad n_i = n_{i0} = n_{e0}/Z$$

$$\frac{\partial \delta n_e}{\partial t} + \underline{\nabla} \cdot (n_{e0} \, \underline{u}_e) = 0$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad , \quad \underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\delta t}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 (-e) n_{e0} \underline{u}_e + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t}$$

$$m_e \frac{\partial \underline{u}_e}{\partial t} = -e \, \underline{E}$$

$$\underline{\nabla} \cdot \underline{E} = \frac{e}{\varepsilon_0} (Z n_i - n_e) = -\frac{e \, \delta n_e}{\varepsilon_0}$$

$$a(\underline{x},t) \rightarrow a\, e^{i(\underline{k}.\underline{x}-\omega t)}$$

$$\left.\begin{aligned} & -i\omega\,\delta n_e + i\, n_{e0}\,\underline{k}.\underline{u}_e = 0 \\ & i\omega\, m_e\, \underline{u}_e = e\,\underline{E} \end{aligned}\right\} \rightarrow \left\{\begin{aligned} & \delta n_e = -\frac{i\,e\,n_{e0}}{m_e\,\omega^2}\,\underline{k}.\underline{E} \\ & \underline{u}_e = -\frac{i\,e}{m_e\,\omega}\,\underline{E} \end{aligned}\right.$$

$$\left.\begin{aligned} & i\underline{k}.\underline{E} = -\frac{e}{\varepsilon_0}\,\delta n_e \\ & \underline{k}.\underline{B} = 0 \quad , \quad \underline{k}\times\underline{E} = \omega\,\underline{B} \\ & i\underline{k}\times\underline{B} = -\mu_0\, e\, n_{e0}\,\underline{u}_e - \frac{i\omega}{c^2}\,\underline{E} \end{aligned}\right\}$$

Electron plasma frequency

$$\boxed{\omega_{pe}^2 \equiv \frac{e^2\, n_{e0}}{m_e\,\varepsilon_0}}$$

$$\Rightarrow \left\{\begin{aligned} & \underline{k}.\underline{E} = \frac{\omega_{pe}^2}{\omega^2}\,\underline{k}.\underline{E} \quad , \quad \underline{k}.\underline{B} = 0 \\ & \underline{k}\times\underline{E} = \omega\,\underline{B} \quad , \quad \underline{k}\times\underline{B} = \frac{1}{\omega c^2}\left(\omega_{pe}^2 - \omega^2\right)\underline{E} \end{aligned}\right.$$

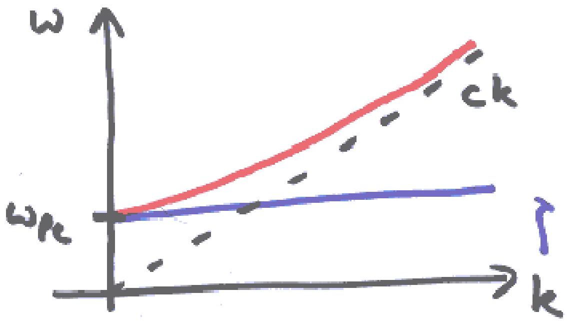
Possible solutions

1) Longitudinal wave : $\underline{k}\cdot\underline{E} \neq 0$

$\Rightarrow \omega = \omega_{pe}$, $\underline{B} = 0$ (Plasma, Langmuir wave)

2) Transversal wave : $\underline{k}\cdot\underline{E} = 0$

$\Rightarrow \omega^2 = \omega_{pe}^2 + k^2c^2$ (Electromagnetic wave)



No waves of this type below ω_{pe}

Lower frequencies require ion dynamics and pressure effects :

With general barotropic relation: $p \sim n^{\gamma}$

$$\left.\begin{array}{l} p = p_0 + \delta p \\ n = n_0 + \delta n \end{array}\right\} \quad \delta p = \gamma \frac{p_0}{n_0} \delta n = m \underset{\substack{\uparrow \\ \text{sound speed}}}{c_0^2} \delta n$$

Derivation analogous to previous one gives :

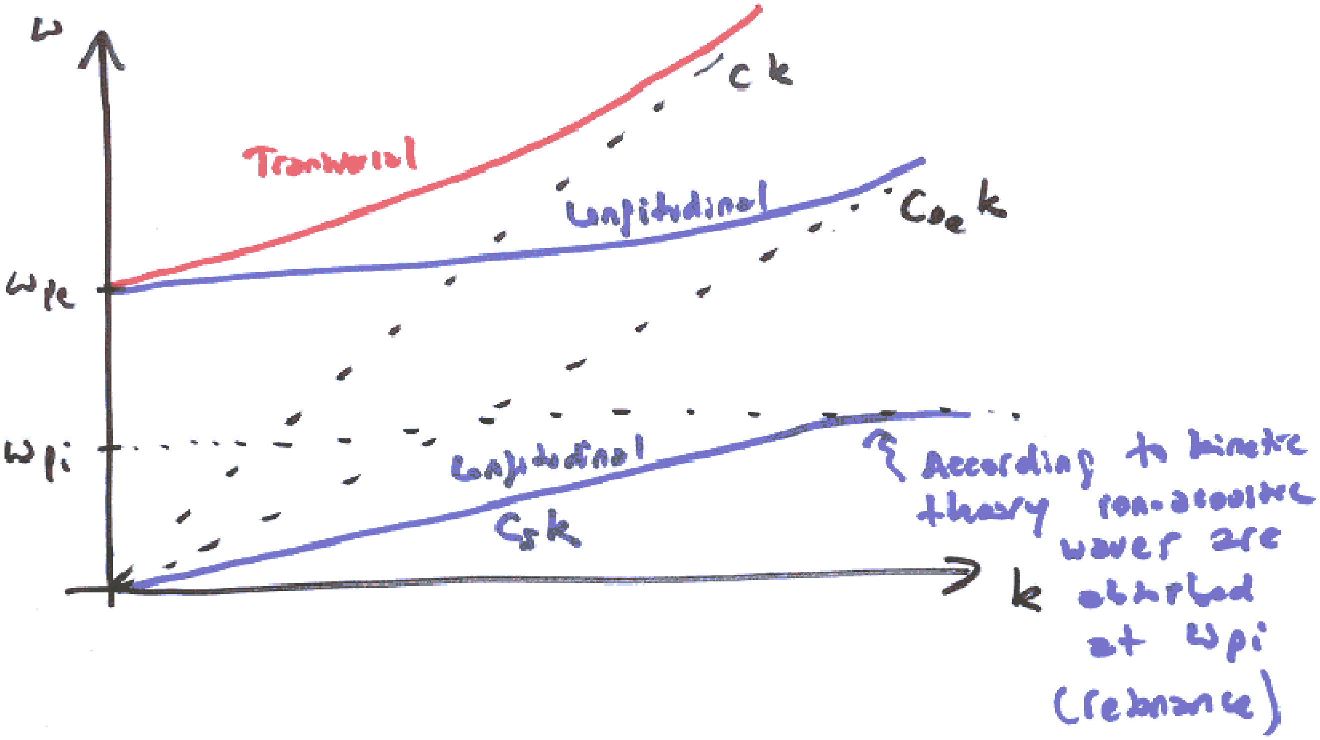
1) Longitudinal waves: $\underline{k} \cdot \underline{E} \neq 0$

$$\omega^2 = \omega_{pe}^2 + k^2 c_{0e}^2 \qquad (\text{Langmuir wave with } T_e \neq 0)$$

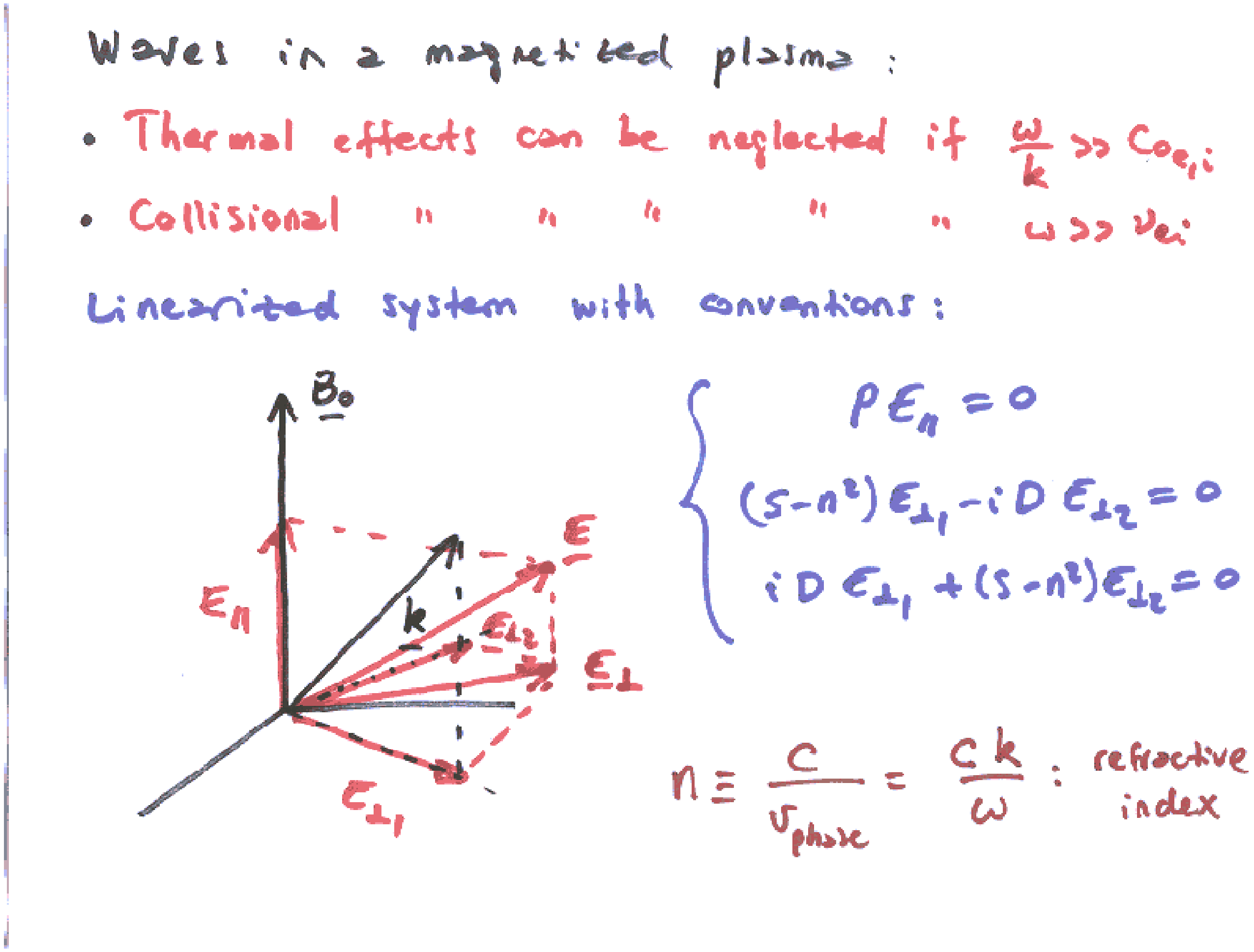
$$\omega^2 = c_s^2 k^2 \;, \quad \underset{\substack{\uparrow \\ \text{ion sound velocity}}}{c_s^2} \equiv \frac{\gamma_e Z T_e + \gamma_i T_i}{m_i} \qquad (\text{ion-acoustic wave})$$

2) Transversal waves: $\underline{k} \cdot \underline{E} = 0$, same as above .

Dispersion relation **B**=0



Waves in magnetized plasmas



Waves in a magnetized plasma:

- Thermal effects can be neglected if $\frac{\omega}{k} \gg C_{0e,i}$
- Collisional " " " " " $\omega \gg \nu_{ei}$

Linearized system with conventions:

$$\begin{cases} P E_\parallel = 0 \\ (S-n^2) E_{\perp 1} - iD E_{\perp 2} = 0 \\ iD E_{\perp 1} + (S-n^2) E_{\perp 2} = 0 \end{cases}$$

$n \equiv \frac{c}{v_{phase}} = \frac{ck}{\omega}$: refractive index

$$\left\{\begin{aligned} P &\equiv 1-\frac{\omega_{pe}^2}{\omega^2}-\frac{\omega_{pi}^2}{\omega^2} \\ S &\equiv 1-\frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2}-\frac{\omega_{pi}^2}{\omega^2-\omega_{ci}^2} \\ D &\equiv -\frac{\omega_{pe}^2\omega_{ce}}{\omega(\omega^2-\omega_{ce}^2)}+\frac{\omega_{pi}^2\omega_{ci}}{\omega(\omega^2-\omega_{ci}^2)} \end{aligned}\right.$$

$$\omega_{pe}^2=\frac{e^2 n_{eo}}{m_e \varepsilon_o}\quad,\quad \omega_{pi}^2=\frac{Z^2e^2n_{io}}{m_i\varepsilon_o}=\frac{Z m_e}{m_i}\,\omega_{pe}^2 \ll \omega_{pe}^2$$

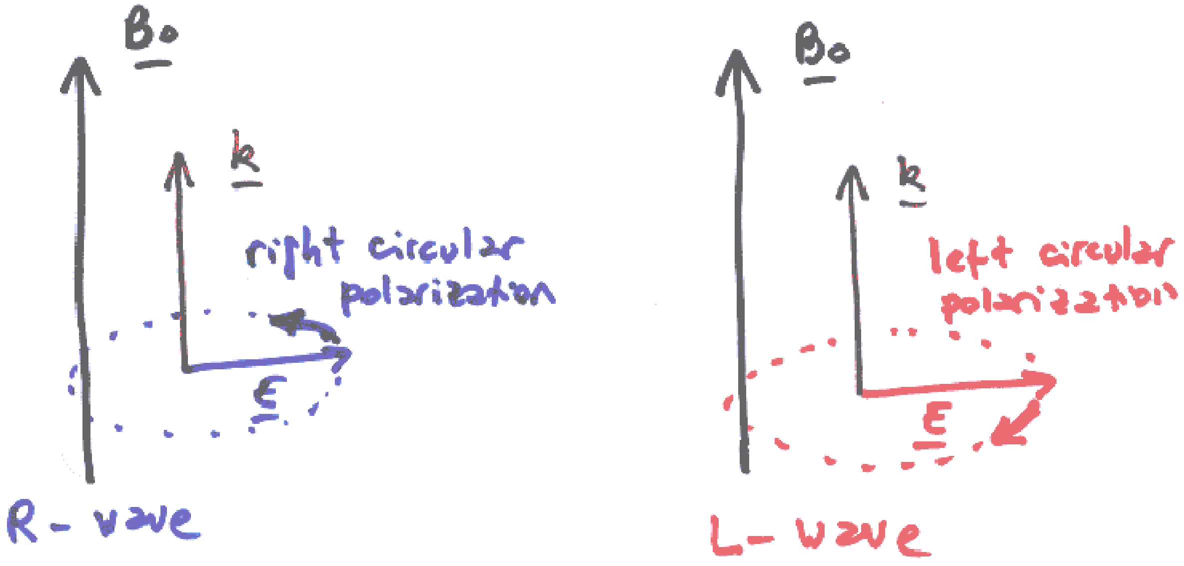
$$\omega_{ce}=\frac{e B_o}{m_e}\quad,\quad \omega_{ci}=\frac{Z e B_o}{m_i}=\frac{Z m_e}{m_i}\,\omega_{ce}\ll\omega_{ce}$$

Longitudinal propagation

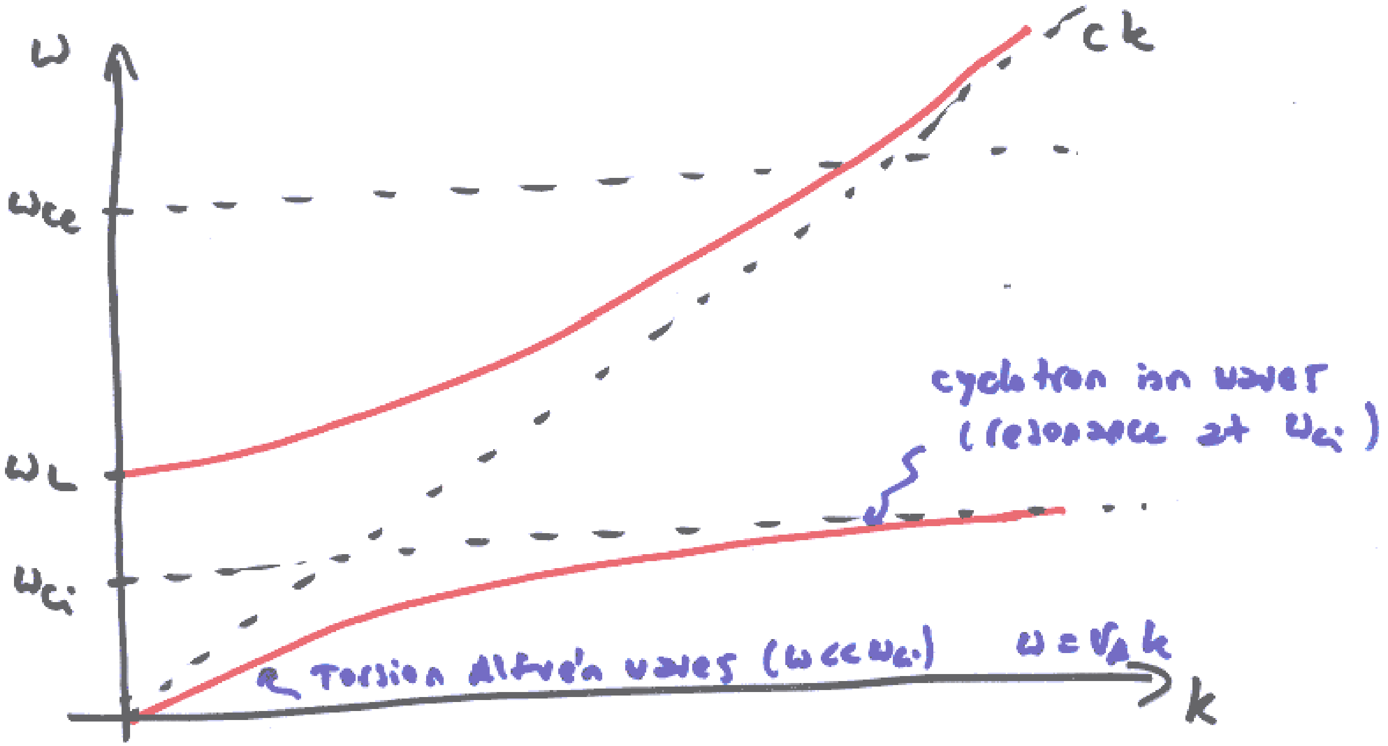
Propagation along the magnetic field : $\underline{k} \parallel \underline{B}_0$

1) Longitudinal waves : $\underline{E} \parallel \underline{k}$: same as with $\underline{B}_0 = 0$

2) Transversal waves : $\underline{E} \perp \underline{k}$:



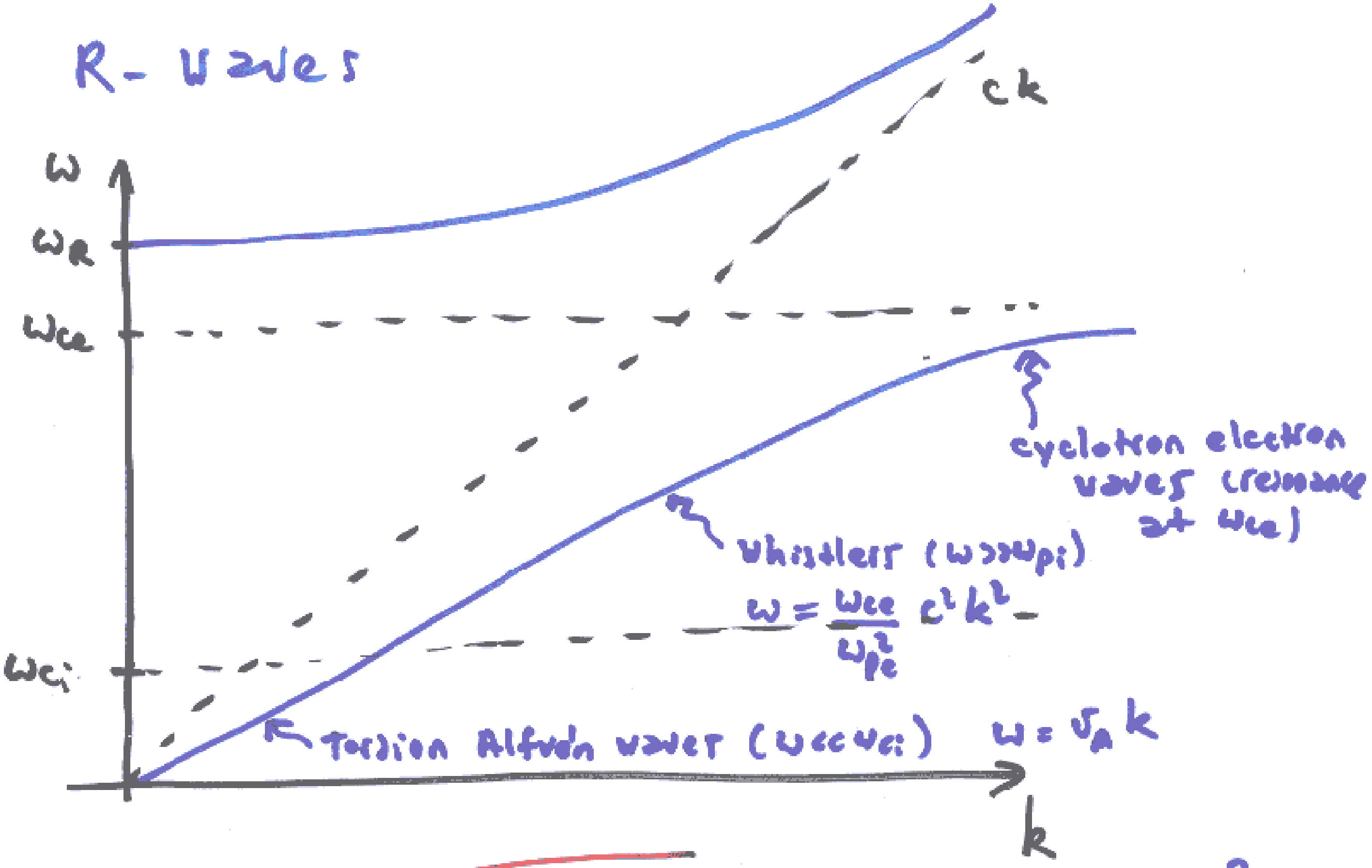
L - waves



$$\omega_L = \frac{1}{2}\left(-\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_{pe}^2}\right)$$

left - wave cut-off

R- waves

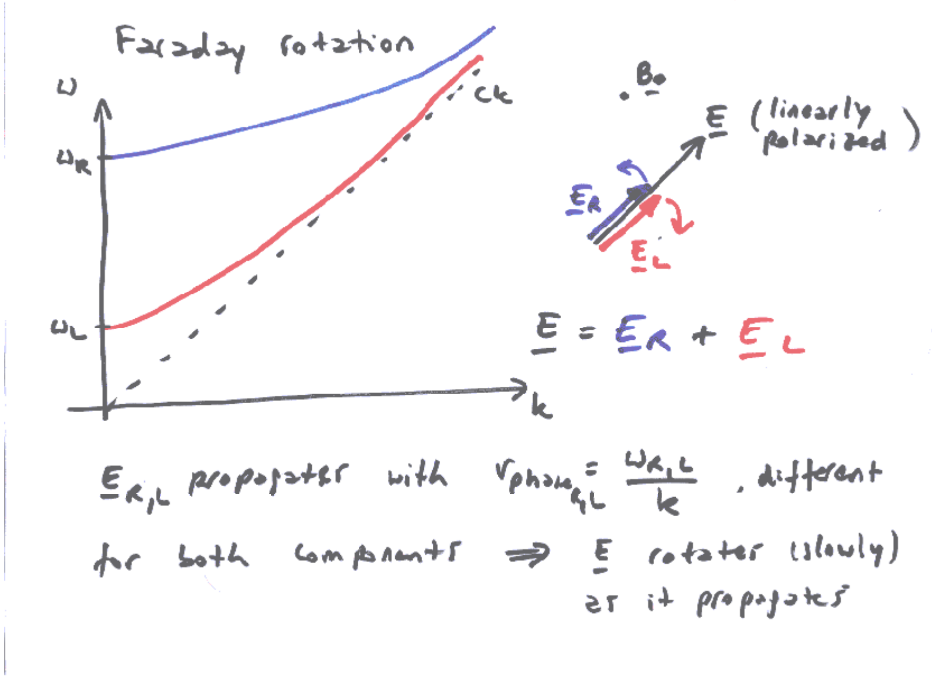


$$\omega_R = \frac{1}{2}\left(\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\,\omega_{pe}^2}\right)$$

right wave cut-off.

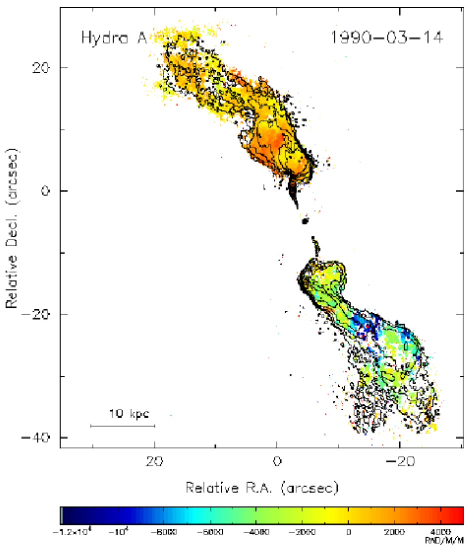
$$v_A = c\,\frac{\omega_{ci}}{\omega_{pi}} = \frac{B_0}{\sqrt{\mu_0 n_{i0} m_i}}$$

Faraday rotation



$E_{R,L}$ propagates with $v_{phase_{R,L}} = \frac{\omega_{R,L}}{k}$, different for both components $\Rightarrow$ E rotates (slowly) as it propagates

As the rotation is *k* dependent the effect is measurable, which allows galactic magnetic fields to be determined.



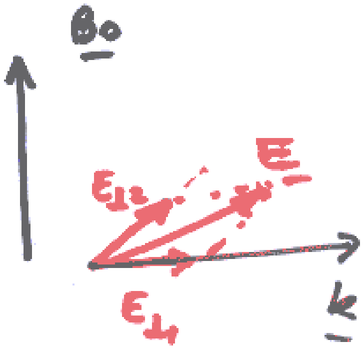
Perpendicular propagation

Propagation $\perp$ $\underline{B}_0$ $(\underline{k} \perp \underline{B}_0)$

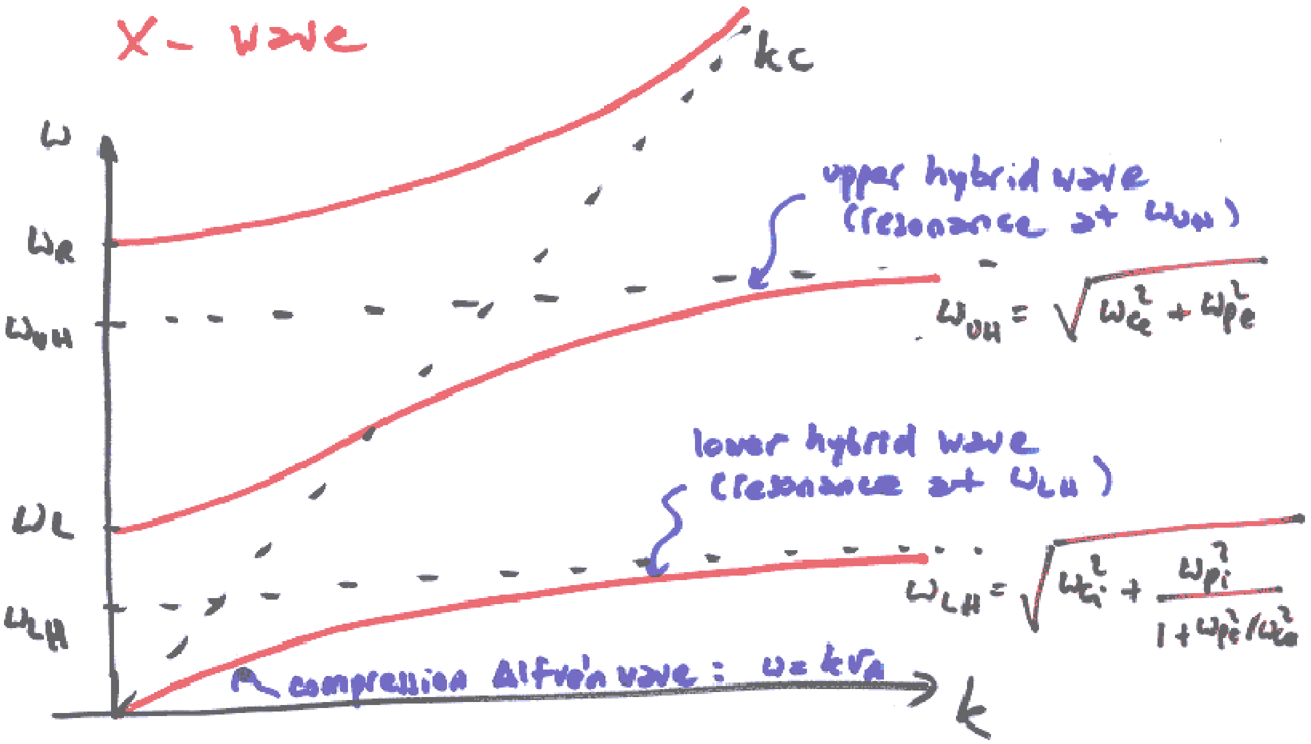
1)

O- mode (ordinary)
as without $\underline{B}_0$:
$\omega^2 = \omega_{pe}^2 + k^2c^2$

2)

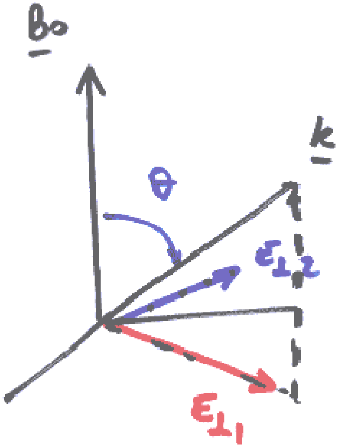


X- mode (Extraordinary)


X - wave
kc
ω
ω_R
ω_{UH}
ω_L
ω_{LH}
k
upper hybrid wave
(resonance at ω_{UH})
$\omega_{UH} = \sqrt{\omega_{ce}^2 + \omega_{pe}^2}$
lower hybrid wave
(resonance at ω_{LH})
$\omega_{LH} = \sqrt{\omega_{ci}^2 + \frac{\omega_{pi}^2}{1+\omega_{pi}^2/\omega_{ce}^2}}$
compression Alfvén wave : $\omega = kv_A$

Low-frequency oblique waves

General propagation with $\omega << \omega_{ci}$: Alfvén waves



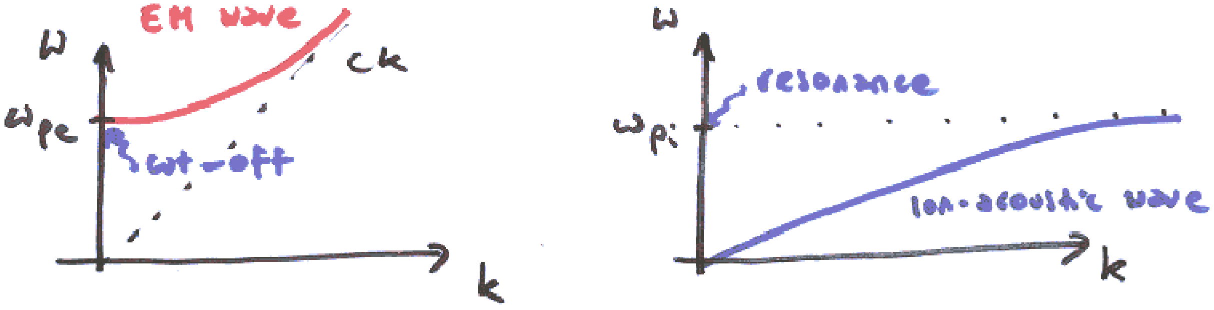
One has $E_{\parallel} = 0$

$E_{\perp 1}$: $\omega = k v_A \cos\theta$
(slow Alfvén mode)

$E_{\perp 2}$: $\omega = k v_A$
(fast Alfvén mode)

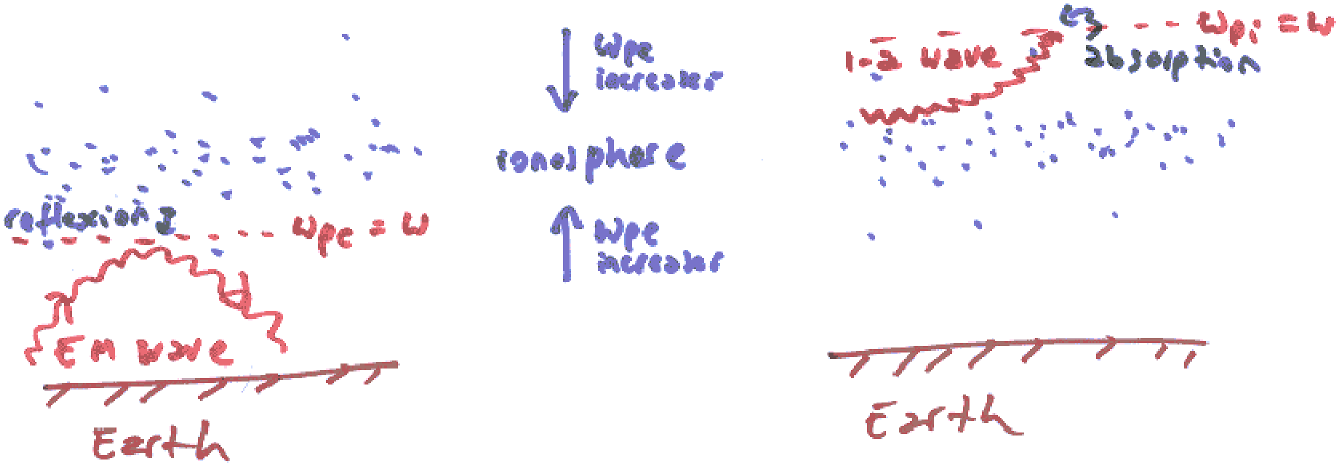
Slow mode: linear polarization, superposition of *R* and *L* (torsional) waves.
Fast mode: compressional wave.

Cut-off and resonances



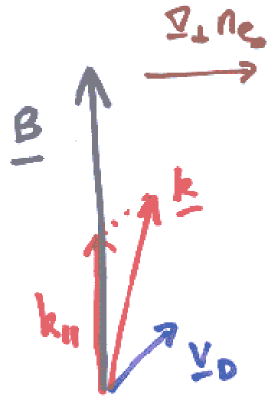
cut-off frequencies:
ω_{pe}, ω_R, ω_L

resonance frequencies:
ω_{pi}, ω_{ce}, ω_{ci}, ω_{UH}, ω_{LH}



Drift waves

Waves in inhomogeneous plasmas : Drift waves



Along $\underline{B}$ electrons in mechanical equilibrium:

$$-\frac{T_e}{n_e}\nabla_\parallel n_e + e\nabla_\parallel \phi = 0$$

$$\Rightarrow \phi = \frac{T_e}{e}\ln n_e$$

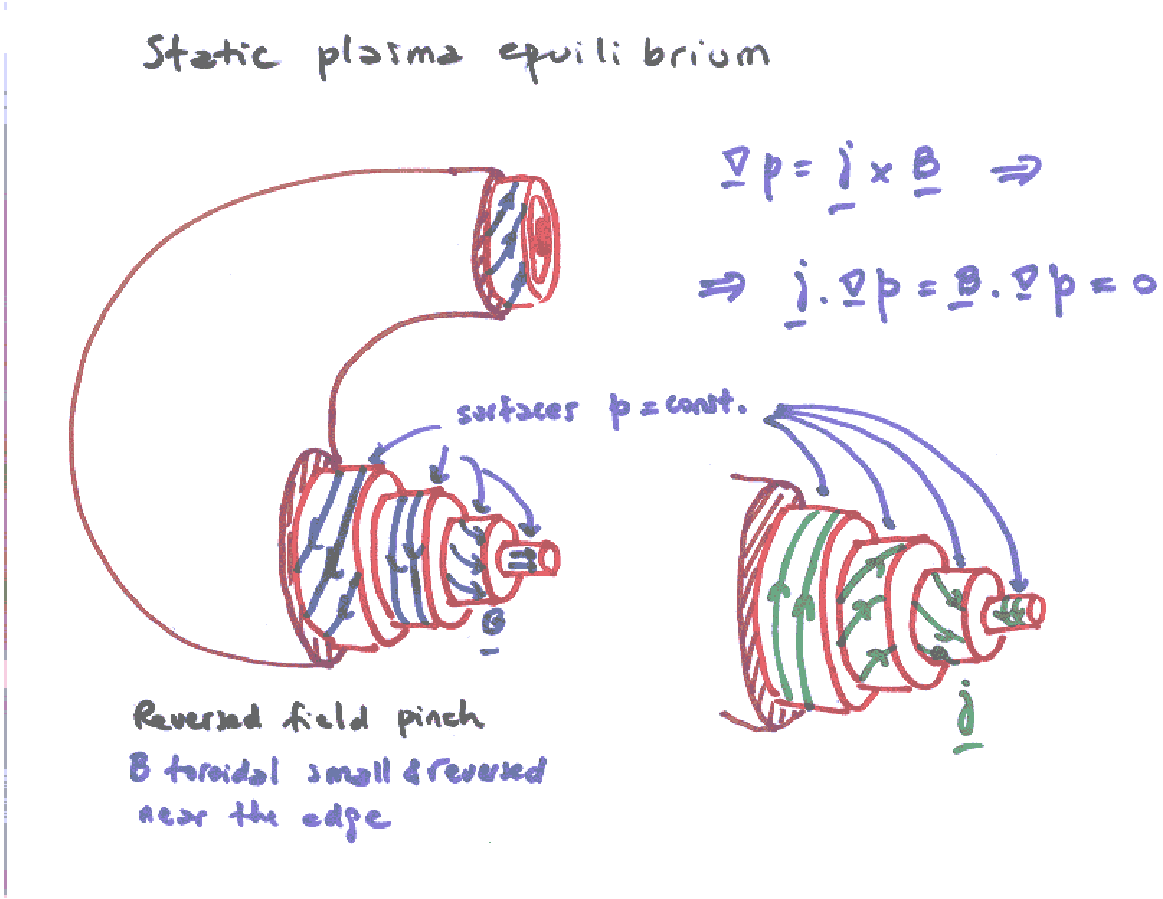
$$\Rightarrow \underline{V}_{E\times B} = -\frac{T_e}{eB^2}\underline{\nabla}_\perp \ln n_{e0} \times \underline{B} \equiv \underline{V}_D$$

Perturbations of density generates oscillations with

$$\omega(\omega-\omega_*) = k_\parallel^2 C_s^2$$

$\omega_* = \underline{k}\cdot\underline{V}_D$: electron drift wave frequency.

Plasma equilibrium



Static plasma equilibrium

$$\underline{\nabla} p = \underline{j} \times \underline{B} \Rightarrow$$

$$\Rightarrow \underline{j} \cdot \underline{\nabla} p = \underline{B} \cdot \underline{\nabla} p = 0$$

surfaces p = const.

Reversed field pinch

B toroidal small & reversed near the edge

Force-free (Spheromak)

For $\beta = \frac{p}{B^2/2\mu_0} \ll 1$, force-free equilibrium

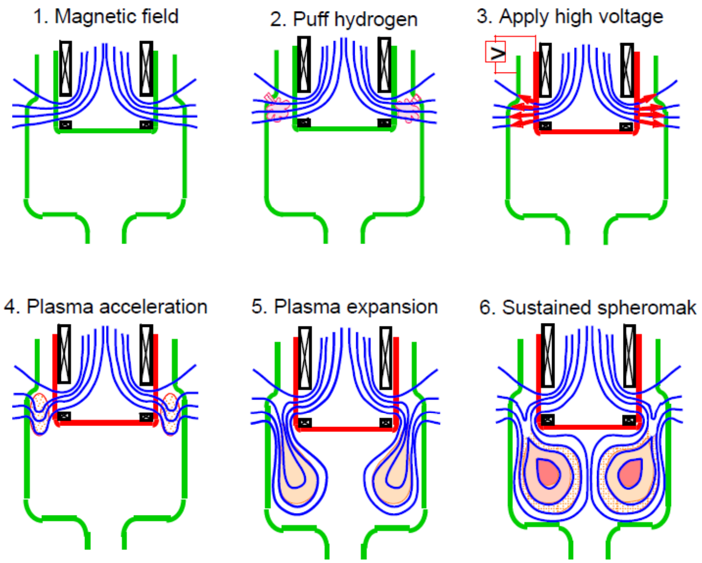
$$\underline{j} \times \underline{B} = 0 \Rightarrow (\underline{\nabla} \times \underline{B}) \times \underline{B} = 0 \Rightarrow$$

$$\Rightarrow \underline{\nabla} \times \underline{B} = \gamma \underline{B}$$

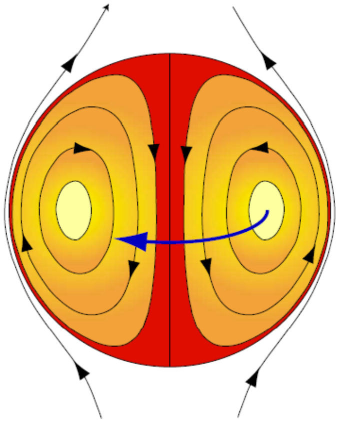
↑ scalar function

Also : $\underline{B} \cdot \underline{\nabla} \gamma = 0$ (γ = const : magnetic surface)

Typical spheromak formation sequence



Essential Characteristics of a Spheromak Plasma

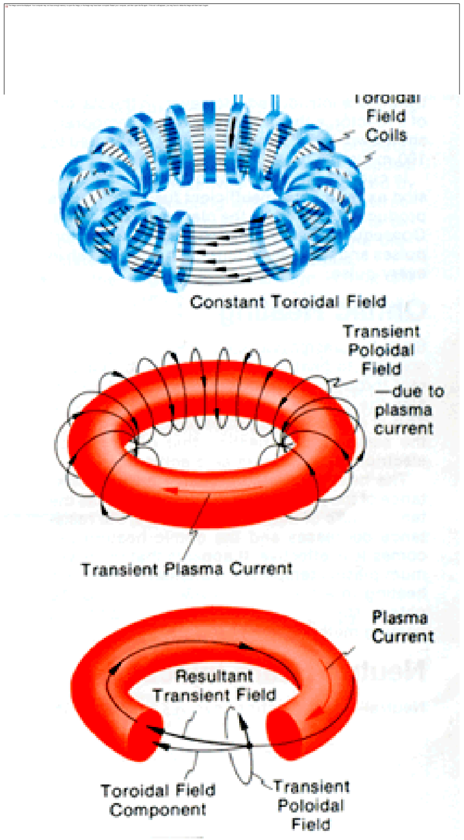


- Low-aspect-ratio (R/a) toroidal magnetic configuration.
- Confining magnetic fields produced by currents in the plasma itself.
- Nearly force-free field aligned currents:

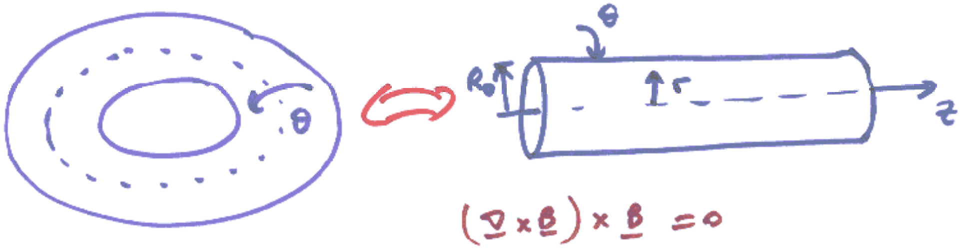
$$\lambda = \frac{\mu_0 j}{B} \qquad \nabla \times \vec{B} = \lambda \vec{B}$$

- Magnetic topology:
  - — edge: Poloidal fields & currents
  - — core: Toroidal fields & currents

Tokamak



Tokamak $(\beta \ll 1)$



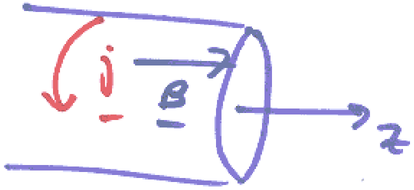
$$(\underline{\nabla} \times \underline{B}) \times \underline{B} = 0$$

$$B_z \Leftarrow \begin{cases} \frac{d}{dr}(B_z^2 + B_\theta^2) + \frac{2}{r} B_\theta^2 = 0 \\ B_\theta(r) = \frac{\mu_0 I(r)}{2\pi r} \end{cases}$$

Uniform current : $I(r) = \frac{I_0 r^2}{R_0^2} \Rightarrow B_z = B_{z\,ext} + \frac{\mu_0^2 I_0^2}{2\pi^2 R_0^4}\left(1 - \frac{r^2}{R_0^2}\right)$

Pinch

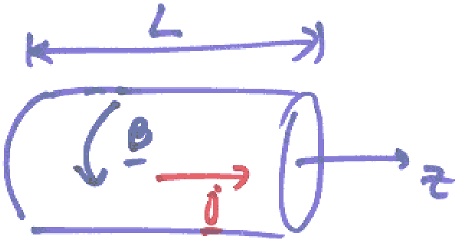
θ - pinch



$$\left\{\begin{aligned} \frac{dB}{dr} &= -\mu_0 j(r) \\ \frac{dp}{dr} &= j B \end{aligned}\right.$$

$$\hookrightarrow \quad p + \frac{B^2}{2\mu_0} = \text{const.}$$

z - pinch



$$\left\{\begin{aligned} & B(r) = \frac{\mu_0 I(r)}{2\pi r}, \quad j = \frac{1}{2\pi r}\frac{dI}{dr} \\ & \frac{dp}{dr} = -j B \end{aligned}\right.$$

Bennett :

$$I_0^2 = \frac{8\pi}{\mu_0}(T_i + Z T_e)\frac{N_i}{L}$$

Plasma stability

For ideal MHD :

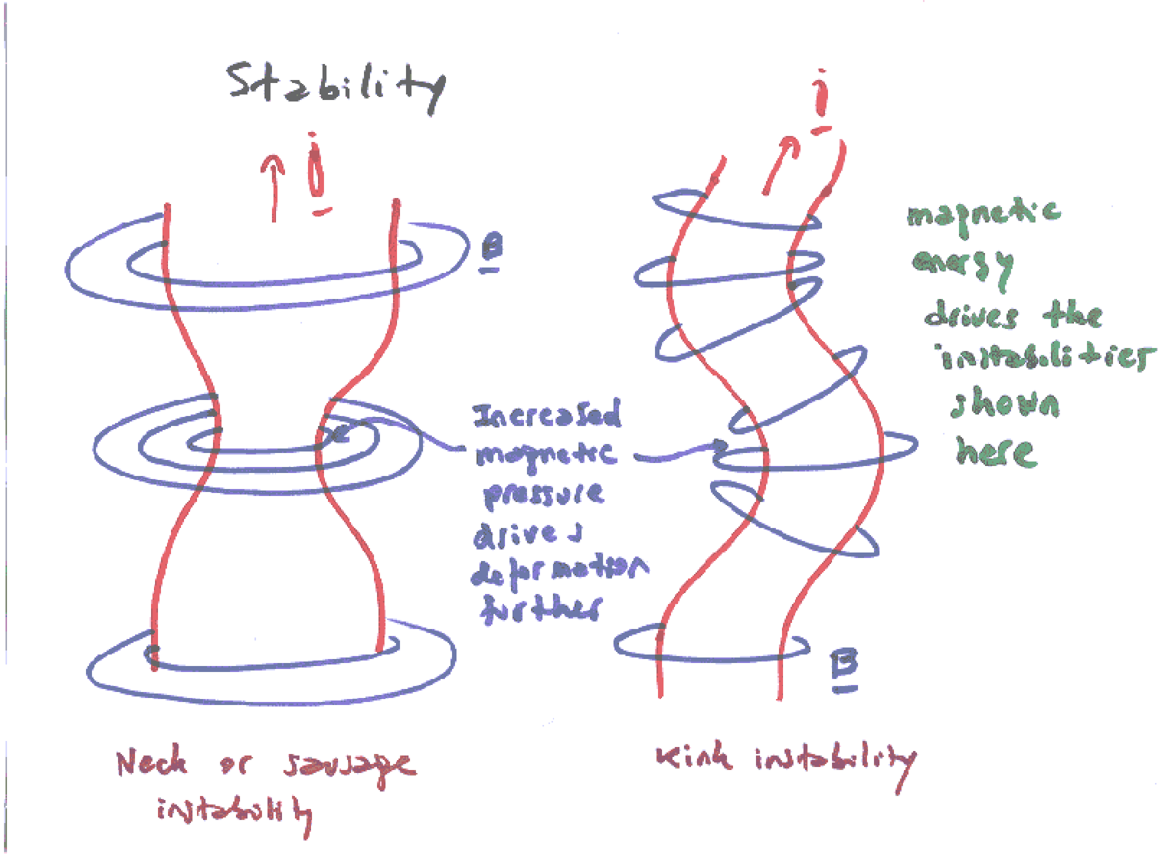
$$\underbrace{\int \frac{1}{2}\rho u^2 \, dV}_{\text{kinetic energy}} + \underbrace{\int \frac{p}{\gamma-1} \, dV}_{\text{internal energy}} + \underbrace{\int \frac{B^2}{2\mu_0} \, dV}_{\text{magnetic energy}} = \text{const.}$$

All terms are positive definite.

Motion can develop in a static equilibrium $(\underline{u} = 0)$ driven by magnetic and/or internal energy.

Equilibrium is (linearly) stable if internal + magnetic energy increases for every possible small perturbation of the equilibrium configuration .

Electromagnetic instabilities

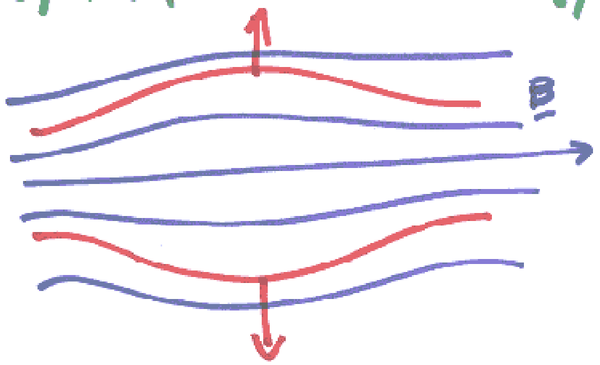


Neck or sausage instability

Kink instability

Flute modes

Instabilities can be driven by the plasma internal energy



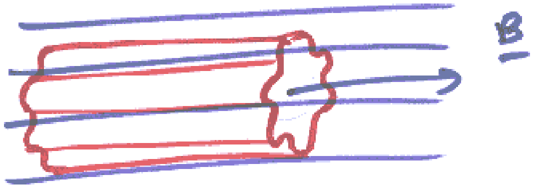
B frozen in the plasma.

Magnetic tension results line bending.

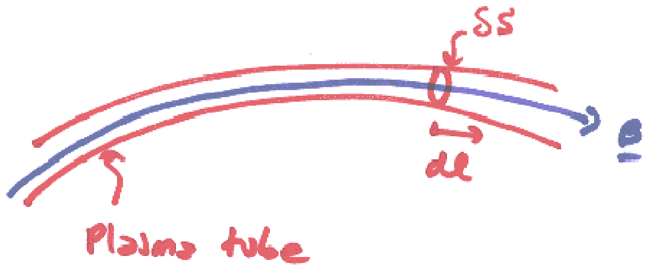
For $\beta \ll 1$ plasma internal energy is small compared to magnetic energy.

More unstable modes do not bend magnetic lines:

"Flute" modes:



Interchange instability



$$U_M = \frac{1}{2\mu_0}\int B^2 dV$$

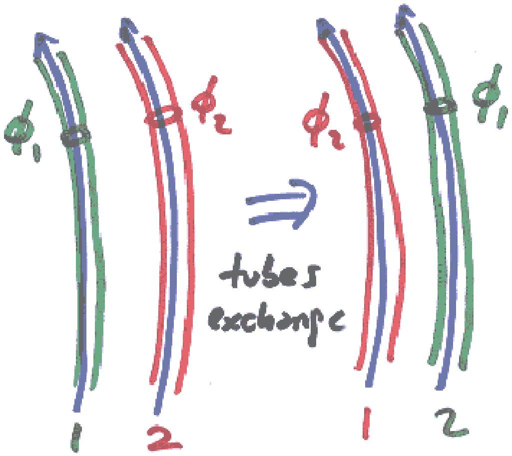
$$= \frac{1}{2\mu_0}\int B^2 \delta S\, dl$$

$$\phi = B\,\delta S = \text{const} \Rightarrow B = \frac{\phi}{\delta S}$$

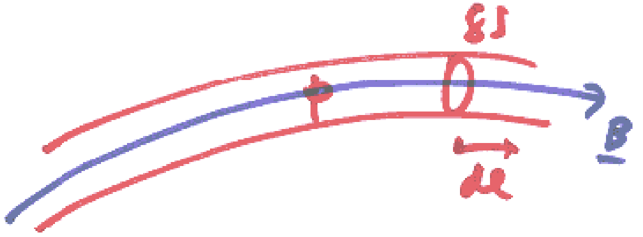
$$U_M = \frac{\phi^2}{2\mu_0}\int \frac{dl}{\delta S}$$

$$2\mu_0 \Delta U_M = \phi_1^2\int_2 \frac{dl}{\delta S} + \phi_2^2\int_1 \frac{dl}{\delta S}$$

$$- \phi_1^2\int_1 \frac{dl}{\delta S} - \phi_2^2\int_2 \frac{dl}{\delta S}$$



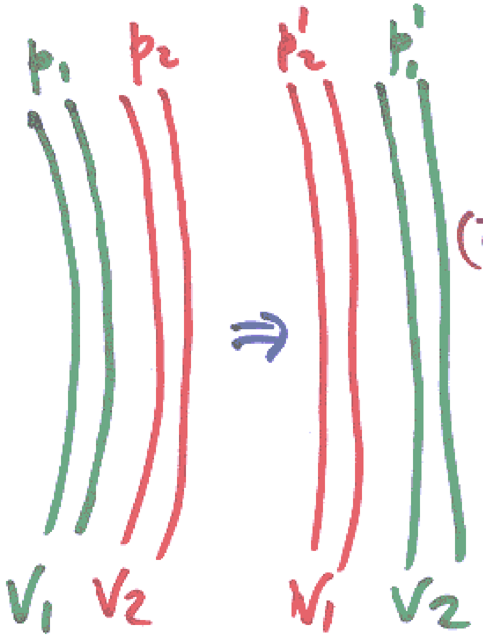
$$\Delta U_M = 0$$
$$\Updownarrow$$
$$\phi_1 = \phi_2$$



$$U_I = \frac{1}{\gamma-1}\int p\,dV$$

$$= \frac{pV}{\gamma-1}$$

lines of $\underline{B}$
on $p = \text{const.}$
surfaces.



$$(\gamma-1)\,\Delta U_I = p_1' V_2 + p_2' V_1 - p_1 V_1 - p_2 V_2$$

$$pV^\gamma = \text{const} \Rightarrow \begin{cases} p_1 \equiv p & p_2 \equiv p + \delta p \\ V_1 \equiv V & V_2 \equiv V + \delta V \end{cases}$$

$$\Delta U_I = \gamma p \frac{\delta V^2}{V} + \delta p\,\delta V$$

With $\phi_1 = \phi_2$ $\Delta U_M = 0$ $\Rightarrow$

Stability if $\Delta U_I > 0$

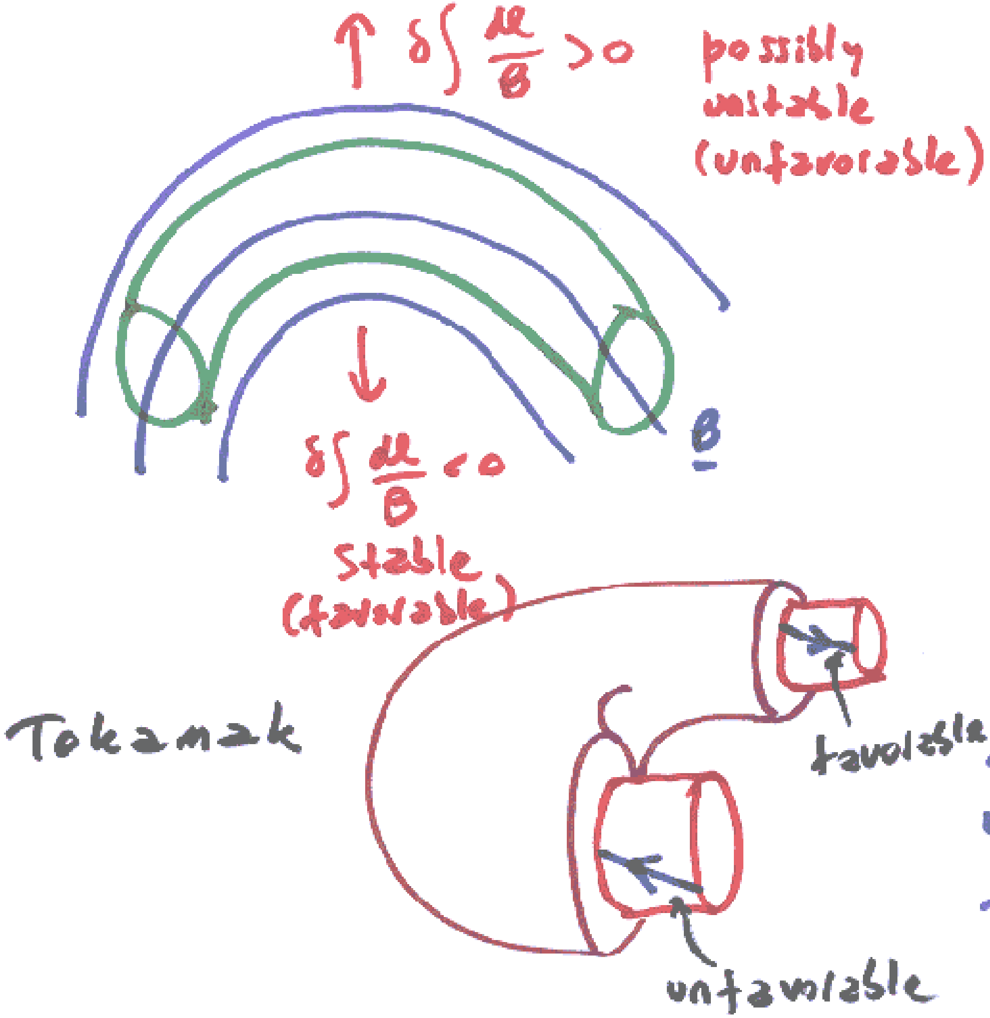
Sufficient condition: $\delta p \, \delta V > 0$

$$\left.\begin{aligned} V_1 &= \int_1 \delta S \, dl = \phi \int_1 \frac{dl}{B} \\ V_2 &= \int_2 \delta S \, dl = \phi \int_2 \frac{dl}{B} \end{aligned}\right\} \quad \delta V = \phi \, \delta \int \frac{dl}{B}$$

pressure decreases outward $\Rightarrow$ stability condition

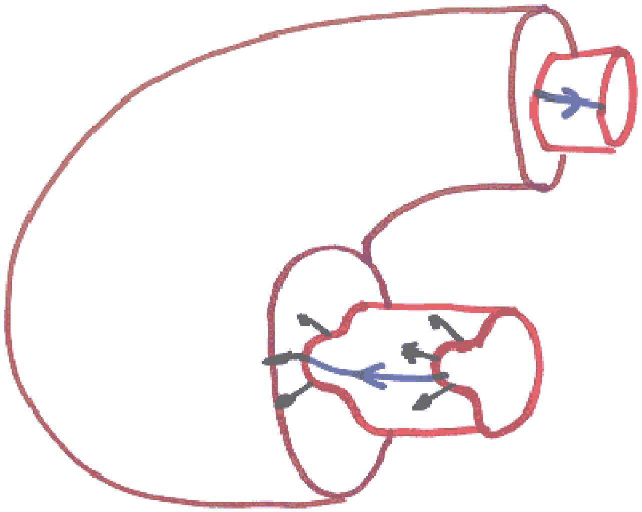
$$\delta \int \frac{dl}{B} < 0$$

outward



Same $\underline{B}$ line goes through favorable and unfavorable regions for stability

Ballooning instability



If instability develops
in unfavorable region
it must bend the magnetic
line (ballooning instability)

⇓

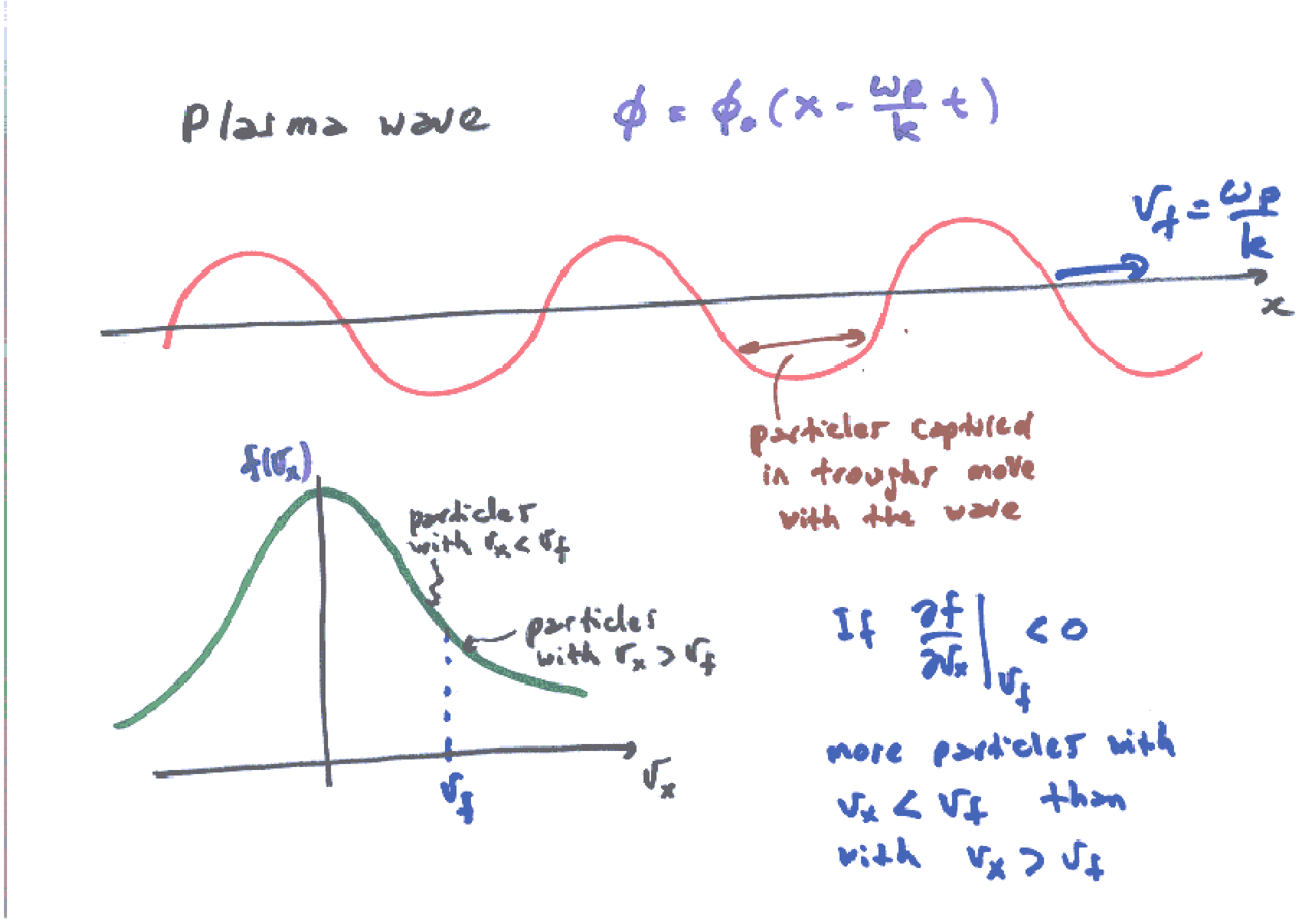
Internal energy must
be sufficient

⇓

stable for low enough β

tipically $\beta < 0{,}06$

Kinetic effects



Landau damping

According to Landau, wave amplitude decays as

$$\phi = \phi_0 \left(x - \frac{\omega_p}{k} t\right) e^{-\gamma t}$$

Fourier transform of f in (x, t)

$$\gamma = -\frac{\pi}{2} \frac{\omega_p^3}{k^2} \frac{\partial \hat{f}}{\partial v_x}\Big|_{v_x = \frac{\omega_p}{k}}$$

Landau damping if $\frac{\partial f}{\partial v_x}\Big|_{\omega/k} < 0$

WAVE ENERGY
TO
PARTICLES

Instability if $\frac{\partial f}{\partial v_x}\Big|_{\omega_p/k} > 0$

PARTICLES KINETIC
ENERGY TO WAVE

Two-stream instability