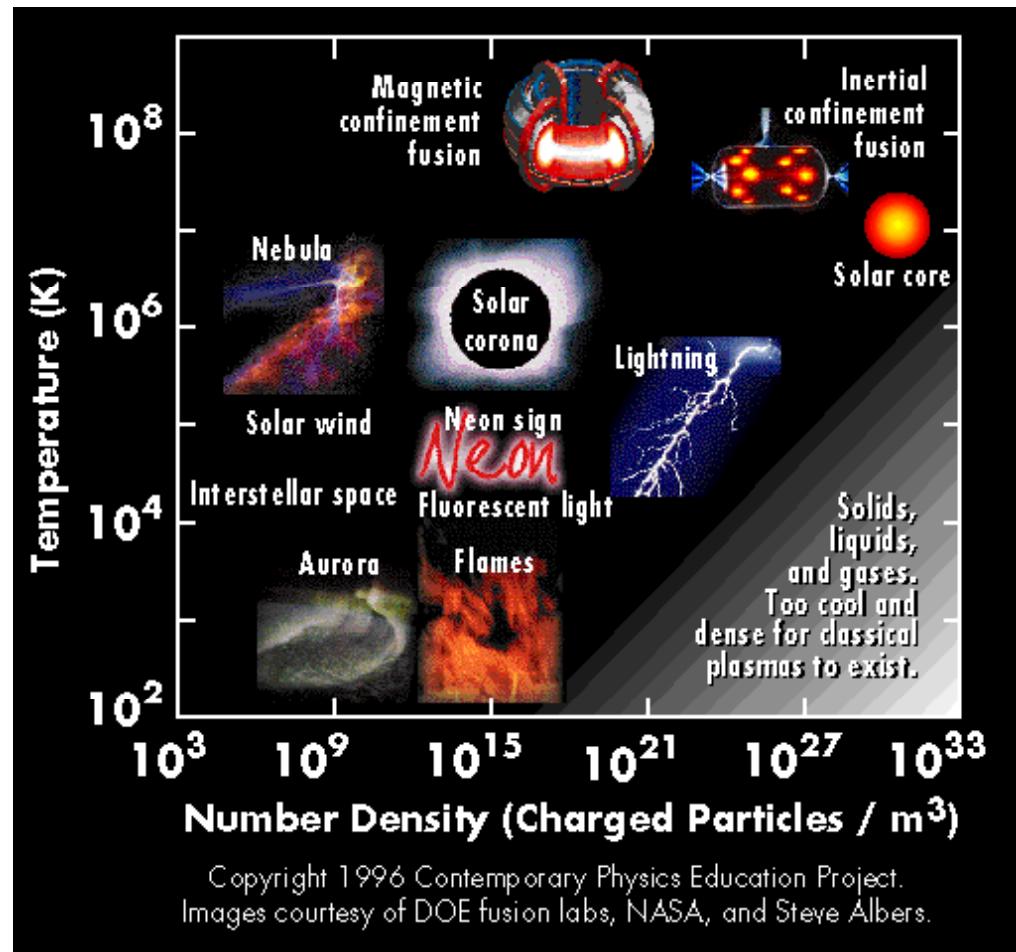


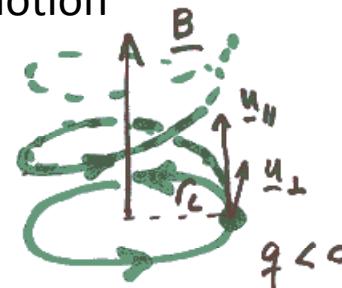
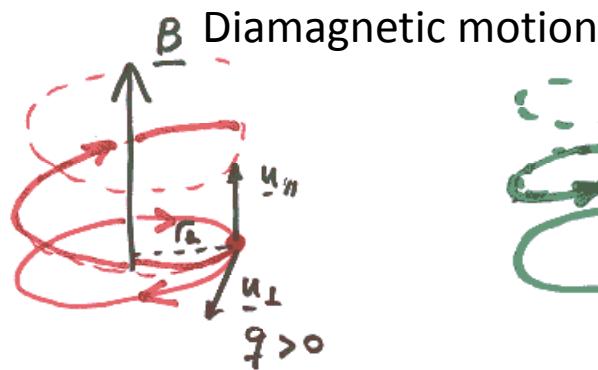
# Fundamentals of plasma physics in a nutshell

# Plasmas on earth and in the universe



# Motion of charges in magnetic fields

$$m \frac{d\vec{u}}{dt} = q \vec{u} \times \vec{B}$$



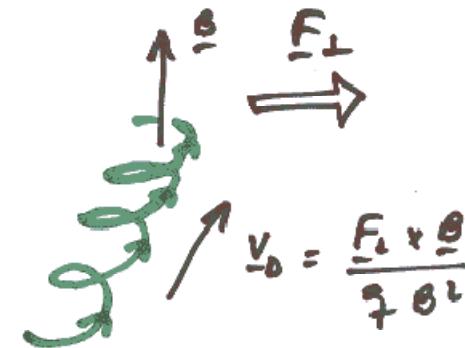
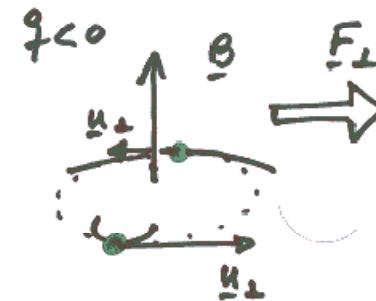
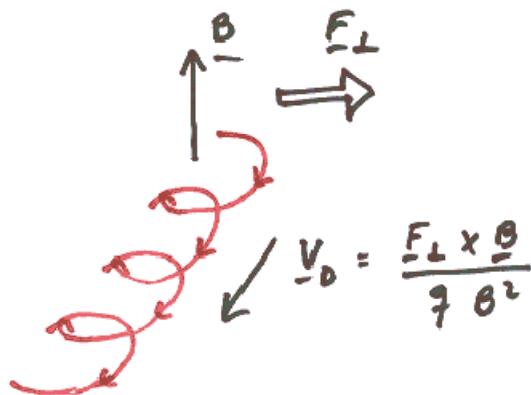
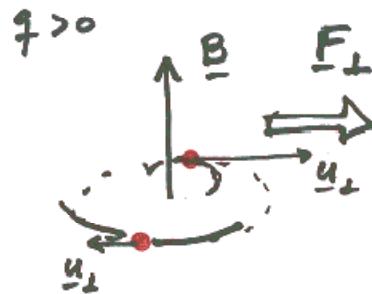
Larmor radius  $r_L = \frac{|q| v_\perp m}{|q| B}$

$$|v_\perp| = \omega_c r_L$$

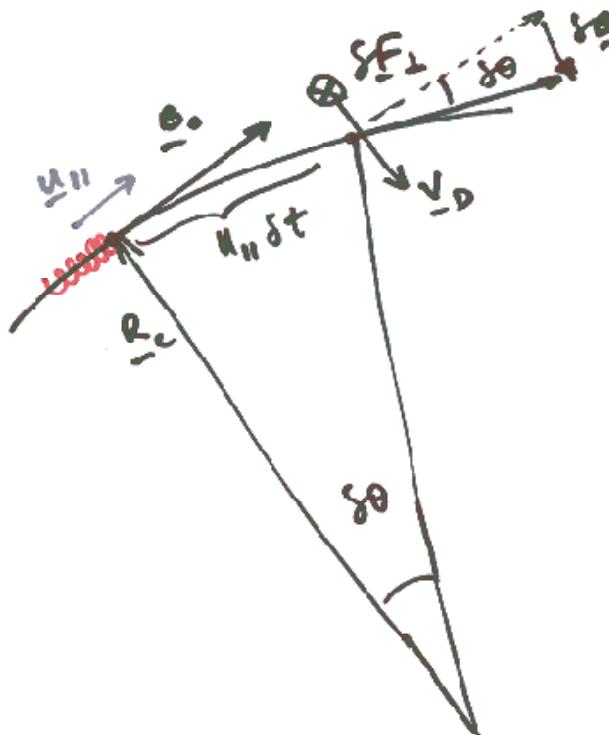
Cyclotron frequency  $\omega_c = \frac{|q| B}{m}$

$$\begin{cases} W_\perp = \frac{1}{2} m v_\perp^2 \\ W_\parallel = \frac{1}{2} m v_\parallel^2 \end{cases}$$

# Drift due to perpendicular force



# Gyration center follows curved B lines

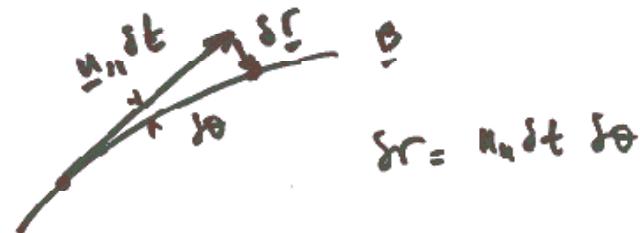


$$\underline{s}_0 = \frac{\underline{u}_{\perp} \delta t}{R_c} ; \underline{s}_B = \underline{s}_0 \underline{B}_0$$

$$\underline{\delta F}_\perp = q \underline{u}_{\perp} \times \underline{B}_0$$

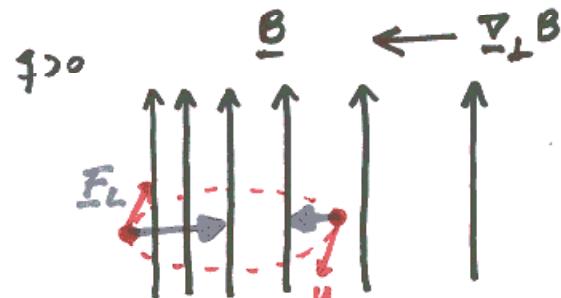
$$\underline{v}_D = \frac{\underline{\delta F}_\perp \times \underline{B}_0}{q B_0^2}$$

$$\underline{\delta r} = \underline{v}_D \delta t ; \delta r = \frac{\underline{u}_{\perp}^2 \delta t^2}{R_c}$$



$$\underline{s}_r = \underline{u}_{\perp} \delta t \underline{s}_\theta$$

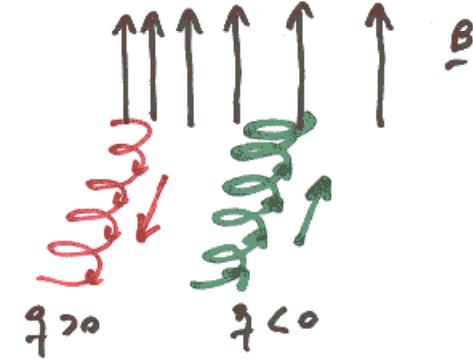
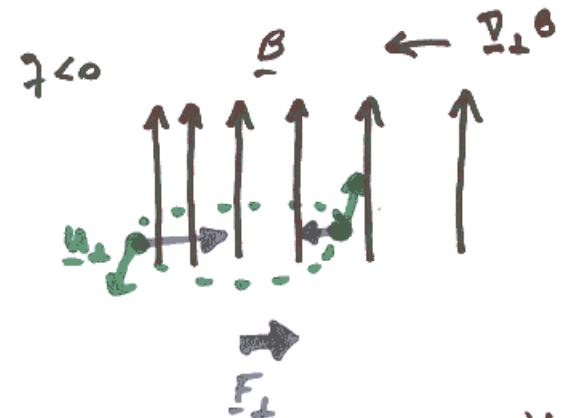
# Drift due to varying $B$ intensity



$$F_L = q \underline{u}_\perp \times \underline{B}$$

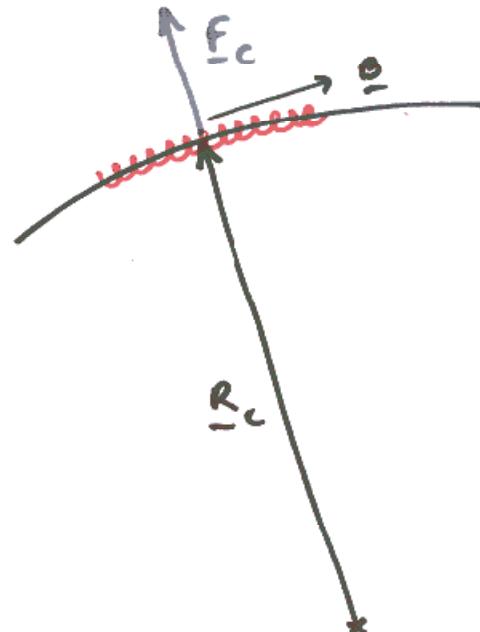
$$\underline{F}_\perp = \langle q \underline{u}_\perp \times \underline{B} \rangle =$$

$$= -\frac{1}{2} 191 C_e^2 W_c \nabla_\perp B$$



$$V_0 = \frac{F_\perp \times B}{q B^2} = -W_\perp \cdot \frac{\nabla_\perp B \times B}{q B^3} \equiv V_{GRAD}$$

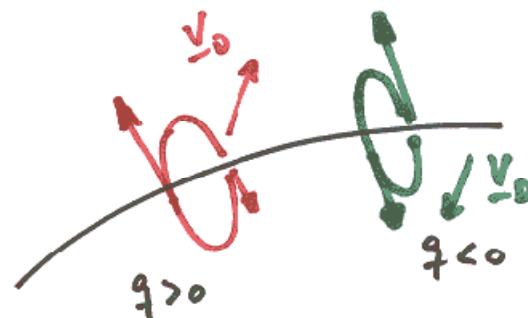
# Drift due to curved B lines



$F_c$  in system of moving center

$$F_c = m \frac{u_{||}^2}{R_c} = \frac{2 W_{||}}{R_c}; F_c = F_c \frac{R_c}{R_c}$$

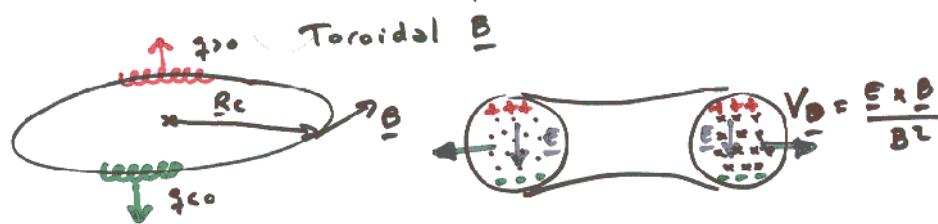
$$V_0 = \frac{F_c \times B}{q B^2} = \frac{2 W_{||}}{q} \frac{R_c \times B}{R_c^2 B^2} \equiv \\ \equiv V_{\text{CURV.}}$$



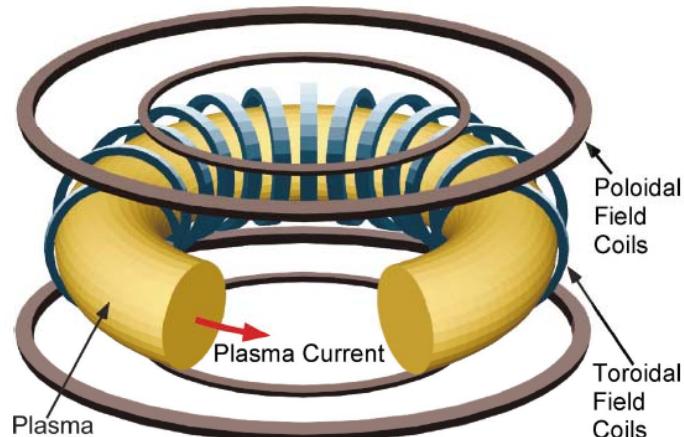
# Combined drifts

$$\text{If } \nabla \times \underline{B} = 0 \Rightarrow \frac{\nabla_{\perp} B \times \underline{B}}{B} = - \frac{\underline{R}_c \times \underline{B}}{R_c^2}$$

$$\Rightarrow V_{Gyro} + V_{cycl.} = (W_{\perp} + 2W_{||}) \frac{\underline{R}_c \times \underline{B}}{q R_c^2 B^2}$$



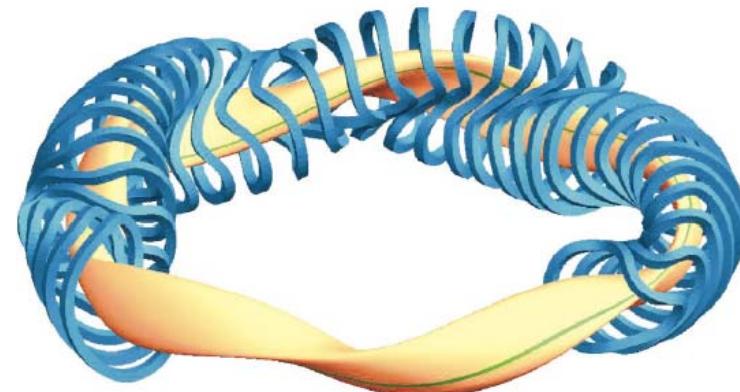
tokamak



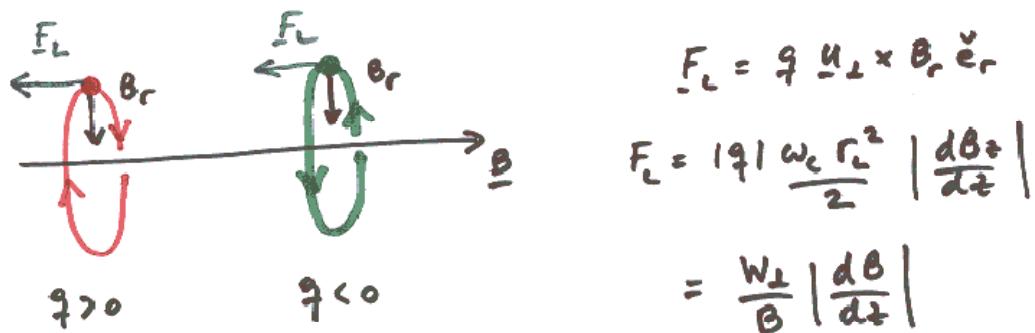
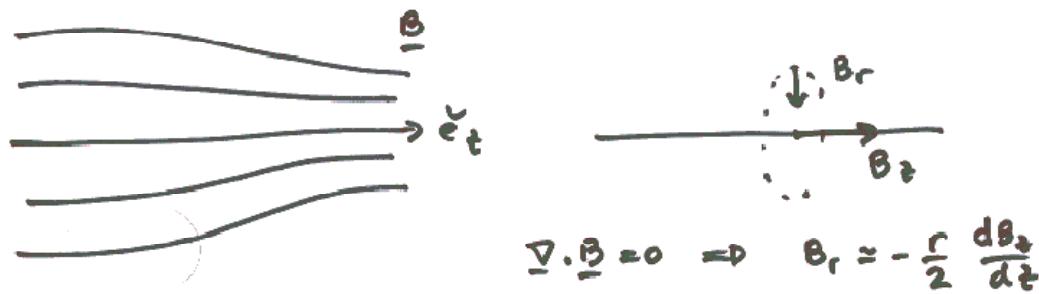
Solution :

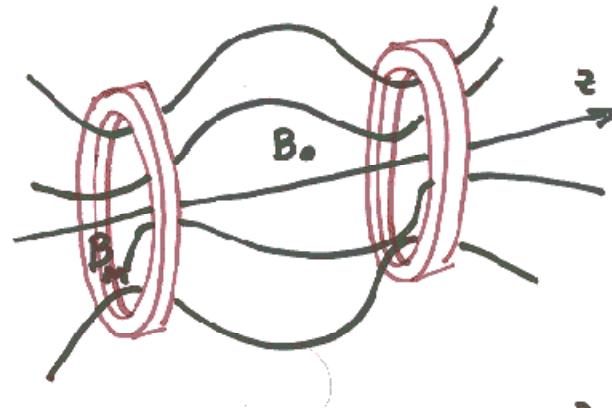


stellarator



# Magnetic mirror





$$m \frac{du_{||}}{dt} = m u_{||} \frac{du_{||}}{dz} = - \frac{w_L}{B} \frac{dB}{dz}$$

$\underbrace{\frac{dN_n}{dt}}$

Also  $w_{||} + w_{\perp} = \text{constant} \Rightarrow$

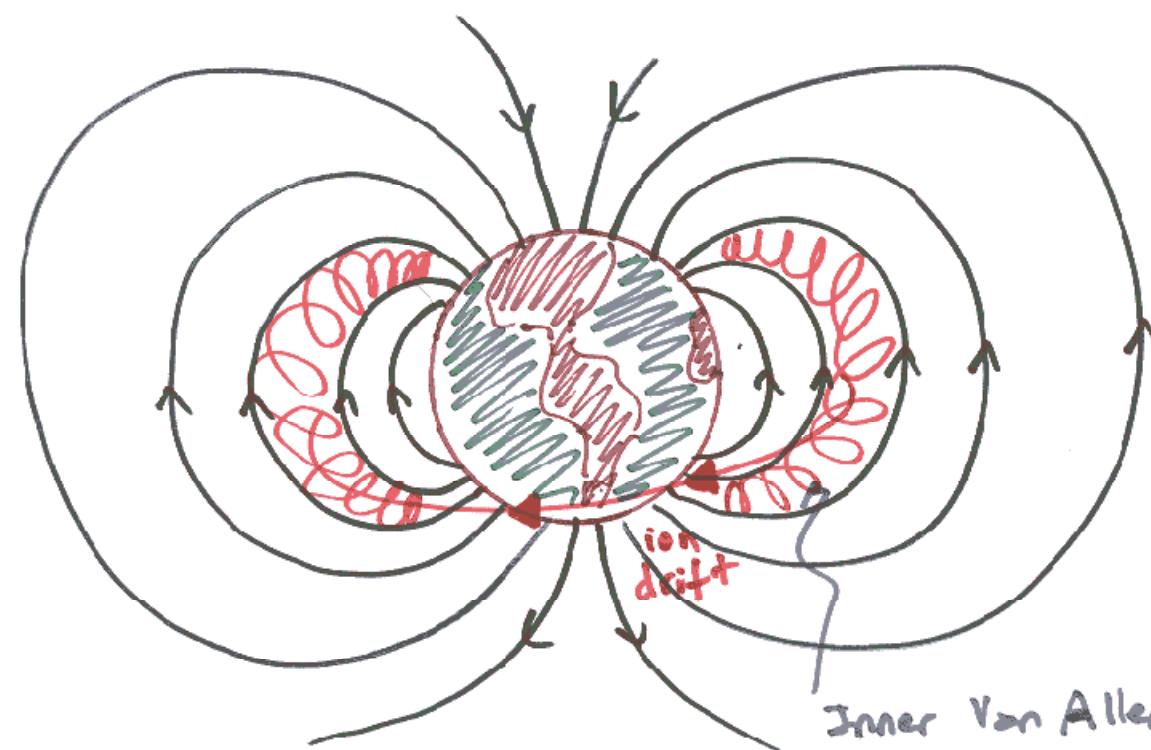
$$\Rightarrow \frac{dw_{\perp}}{dt} = \frac{w_L}{B} \frac{dB}{dz} \Rightarrow \frac{w_L}{B} = \text{constant}$$

$$\left\{ \begin{array}{l} w_{||} + w_{\perp} = w_{||0} + w_{\perp 0} \\ \frac{w_L}{B} = \frac{w_{L0}}{B_0} \end{array} \right.$$

Particles escape if  $w_{||M} > 0$

Confinement so requires

$$w_{||0} \leq w_{\perp 0} \left( \frac{B_M}{B_0} - 1 \right)$$

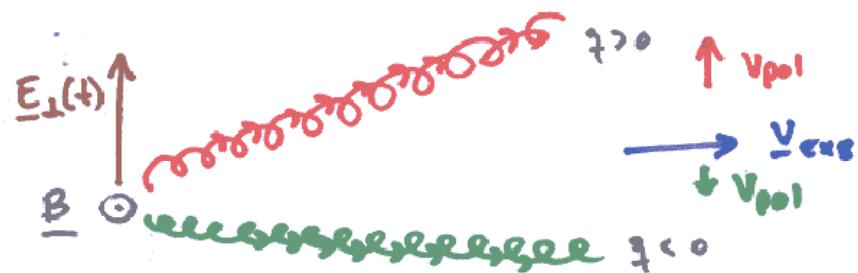
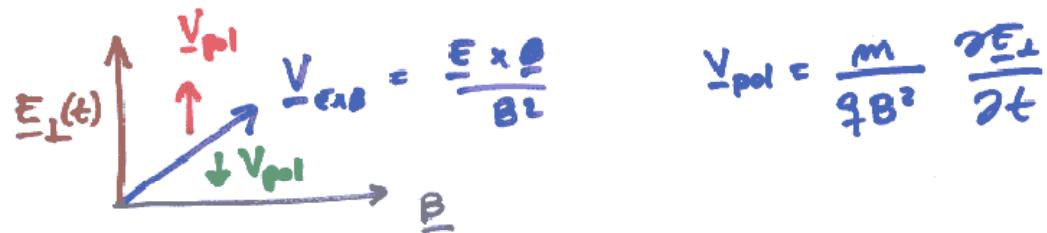


Inner Van Allen belt  
Mirror trapping of  
h.e. ions  $\gtrsim 100$  MeV  
& electrons  $\sim 0,5$  MeV

# Polarization drift

Polarization drift :

For time dependent  $\underline{E}_\perp$  ;  $\frac{\partial}{\partial t} \ll \omega_c$ :



# Debye length

Electron and ion gases in equilibrium:

$$\begin{aligned} -T\nabla n_e + en_e \nabla \phi &= 0, \\ -T\nabla n_i - Zen_i \nabla \phi &= 0, \end{aligned} \quad \nabla^2 \phi = -\frac{e(Zn_i - n_e)}{\epsilon_0}.$$

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = -\frac{Q}{4\pi\epsilon_0 r^2} \delta(r) + \frac{e^2 n_{e0}}{\epsilon_0 T} (1+Z) \phi, \text{ Linearized Poisson equation:}$$

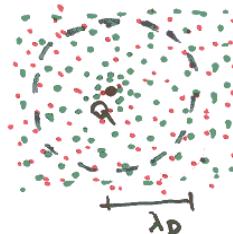
with solution:

$$\phi = \frac{Q}{4\pi\epsilon_0 r} \exp(-r/\lambda_D),$$

$$\lambda_D \equiv \sqrt{\frac{\epsilon_0 T}{e^2 n_{e0} (1+Z)}}$$

$$\lambda_D [m] = 7.4 \times 10^3 \sqrt{\frac{T [eV]}{n_{e0} [m^{-3}] (1+Z)}}$$

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ J} = 11600^\circ \text{ K}$$



$$N_D = \frac{4\pi}{3} \lambda_D^3 n_{e0} (1 + \frac{1}{\epsilon}) \gg 1$$

$$d \sim n_{e0}^{-1/3} \quad U \sim \frac{e^2}{4\pi\epsilon d}$$

$$\frac{W}{U} \sim N_0^{2/3} \gg 1$$

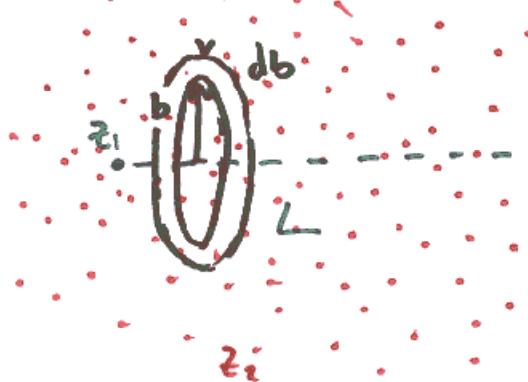
# Collisions



$$\Delta u_{\perp} \sim \frac{F \Delta t}{m_1} \sim \frac{z_1 z_2 e^2}{4\pi \epsilon_0 b^2 m_1} \frac{b}{u}$$

$$\delta\theta \sim \frac{\Delta u_{\perp}}{u} \sim \frac{b_0}{b}$$

$$b_0 = \frac{z_1 z_2 e^2}{4\pi \epsilon_0 m_1 u^2}$$



$$\langle (s_0)^2 \rangle \sim \int_{b_{\min}}^{b_{\max}} (s_0)^2 n_2 2\pi b \, db \, L$$

$$\langle (\delta\theta)^2 \rangle \sim 2\pi n_2 b_0^2 \ln\left(\frac{b_{\max}}{b_{\min}}\right) L$$

$$b_{\max} \sim \lambda_0 \quad ; \quad b_{\min} \Rightarrow \delta_0 \sim 1 \quad \text{for } m_i u^i \sim T \Rightarrow$$

$$b_{\min} \sim \frac{z_1 z_2 e^2}{4\pi \epsilon T}$$

$$\frac{b_{\max}}{b_{\min}} \sim \frac{4\pi \epsilon T \lambda_0}{z_1 z_2 e^2} \equiv \Lambda \sim N_0$$

$$\langle (\delta_0)^2 \rangle \sim 2\pi n_z b_0^2 \ln \Lambda L$$

$$\langle (\delta_0)^2 \rangle \sim 1 \quad \text{if} \quad L \sim \lambda_\perp \equiv \frac{1}{2\pi n_z b_0^2 \ln \Lambda} = \frac{1}{n_z \sigma_\perp}$$

$$\sigma_\perp \sim \frac{(z_1 z_2)^2 e^4}{8\pi \epsilon^2 (m_i u)^2} \ln \Lambda$$

Large deviation in single (Coulomb) collision if

$$s_0 \sim \frac{b_0}{b} \sim 1 \Rightarrow b \sim b_0 = \frac{z_1 z_2 e^2}{4\pi \epsilon_0 m_1 u^2}$$

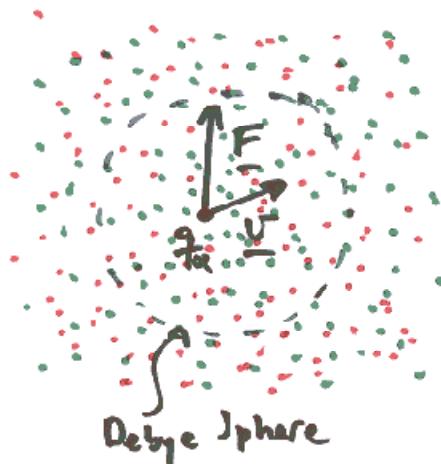
Corresponding cross-section:

$$\sigma_c \sim 4\pi b^2 \sim \frac{(z_1 z_2)^2 e^4}{4\pi \epsilon_0^2 (m_1 u^2)^2} \sim \frac{\sigma_\perp}{\ln \Lambda} \ll \sigma_\perp$$

$$(\ln \Lambda \sim \ln N_0 \sim 10 - 30)$$

units	$n$ $\text{m}^{-3}$	$T$ eV	$\lambda_D$ m	$n\lambda_D^3$	$\ln \Lambda$
Solar corona (loops)	$10^{15}$	100	$10^{-3}$	$10^7$	19
Solar wind (near earth)	$10^7$	10	10	$10^9$	25
Magnetosphere (tail lobe)	$10^4$	10	$10^2$	$10^{11}$	28
Ionosphere	$10^{11}$	0.1	$10^{-2}$	$10^4$	14
Mag. fusion (tokamak)	$10^{20}$	$10^4$	$10^{-4}$	$10^7$	20
Inertial fusion (imploded)	$10^{31}$	$10^4$	$10^{-10}$	$10^2$	8
Lab plasma (dense)	$10^{20}$	5	$10^{-6}$	$10^3$	9
Lab plasma (diffuse)	$10^{16}$	5	$10^{-4}$	$10^5$	14

# Kinetic description



$$\underline{F} = \underline{F}_c + \underline{F}_{nc}$$

due to particles  
inside the Debye  
sphere (collisions)

$$\sigma \sim \sigma_{\perp}$$

due to  
particles  
outside the  
Debye sphere

$$g_n(\underline{E} + \underline{v} \times \underline{B})$$

"Smooth"  
Fields

$$\frac{\partial f_n}{\partial t} + \underline{v} \cdot \nabla f_n + \frac{\underline{F}}{m_n} \cdot \frac{\partial f_n}{\partial \underline{v}} = 0$$

# Transition to fluid description

$$n_\alpha = \int f_\alpha(\underline{x}, \underline{v}, t) d^3 v$$

$$\underline{u}_\alpha = \frac{1}{n_\alpha} \int \underline{v} f_\alpha(\underline{x}, \underline{v}, t) d^3 v \equiv \langle \underline{v} \rangle_\alpha ; \delta \underline{v} = \underline{v} - \underline{u}_\alpha$$

$$\begin{cases} \frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \underline{u}_\alpha) = 0 & \text{modelled as } \frac{T_\alpha}{m_\alpha} \equiv \\ \frac{\partial \underline{u}_\alpha}{\partial t} + (\underline{u}_\alpha \cdot \nabla) \underline{u}_\alpha = -\frac{1}{n_\alpha} \nabla \cdot [n_\alpha \underbrace{\langle \delta \underline{v} \delta \underline{v} \rangle_\alpha}] \\ + \frac{q_\alpha}{m_\alpha} (\underline{E} + \underline{u}_\alpha \times \underline{B}) + \sum_{\beta \neq \alpha} (u_\beta - u_\alpha) \nu_{\alpha\beta} \end{cases}$$

↑  
Collision frequency  
with β species

# Two-fluid model

Two fluids (electrons & ions)

$$\left\{ \begin{array}{l} \frac{\partial n_{e,i}}{\partial t} + \nabla \cdot (n_{e,i} \underline{u}_{e,i}) = 0 \\ m_e n_e \frac{d\underline{u}_e}{dt} = - \nabla p_e - e n_e (\underline{E} + \underline{u}_e \times \underline{B}) + \underline{R}_{ei} \\ m_i n_i \frac{d\underline{u}_i}{dt} = - \nabla p_i + z e n_i (\underline{E} + \underline{u}_i \times \underline{B}) - \underline{R}_{ei} \end{array} \right.$$

$$\underline{R}_{ei} \in m_e n_e (\underline{u}_i - \underline{u}_e) v_{ei} ; \quad p_{e,i} = n_{e,i} T_{e,i}$$

polytropic model for pressures:  $p n^{-\gamma} = \text{const.}$

+ Maxwell Eq's.  $\left\{ \begin{array}{l} \nabla \cdot \underline{E} = \frac{e}{\epsilon_0} (z n_i - n_e) ; \quad \nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} \\ \nabla \cdot \underline{B} = 0 ; \quad \nabla \times \underline{B} = \mu_0 e (z n_i \underline{u}_i - n_e \underline{u}_e) \\ \quad + \frac{1}{c} i \frac{\partial \underline{E}}{\partial t} \end{array} \right.$

# Single fluid MHD

Single fluid MHD

$$\left| \frac{\partial \underline{B}}{\partial x} \right| \ll \frac{1}{\lambda_B} \Rightarrow e n_i \approx n_e \Rightarrow \underline{j} = e n_e (\underline{u}_i - \underline{u}_e)$$

$$\underline{u} = \frac{m_i n_i \underline{u}_i + m_e n_e \underline{u}_e}{m_i n_i + m_e n_e}, \quad \rho = m_i n_i + m_e n_e$$

$$p = p_e + p_i$$

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0 \\ \rho \frac{d\underline{u}}{dt} = - \nabla p + \underline{j} \times \underline{B} + \rho_e \underline{E} - \frac{m_e}{e} \nabla \cdot \left( \frac{\underline{j} \underline{j}}{n_e} \right) \end{cases}$$

very small as compared with  
 $\rho(\underline{u}, \underline{B}) \underline{u}$

M.E. :  $\begin{cases} \nabla \cdot \underline{B} = 0 ; \nabla \times \underline{B} = \mu_0 \underline{j} \\ \nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} \end{cases}$

Ohm law : electron momentum eq. with  $m_e \rightarrow 0$

$$\underline{E} + \underline{u} \times \underline{\Omega} = \gamma \underline{j} + \underbrace{\frac{1}{en_e} (\underline{j} \times \underline{\Omega} - \nabla p_e)}_{\text{small as compared with } \underline{u} \times \underline{\Omega}}$$

$$\gamma = \frac{m_e v_{ei}}{e^2 n_e} : \text{resistivity} \quad \text{if } |\frac{\partial \underline{v}_e}{\partial \underline{x}}| \ll \frac{1}{\tau_{ei}}$$

$$v_{ei} \sim n_i \langle \sigma_v u_e \rangle \sim \frac{n_i e^2 e^4 \ln \Lambda}{8\pi \epsilon_0^2 m_e} \underbrace{\langle \frac{1}{u_e^2} \rangle}_{\sim \left(\frac{m_e}{\tau_e}\right)^{3/2}}$$

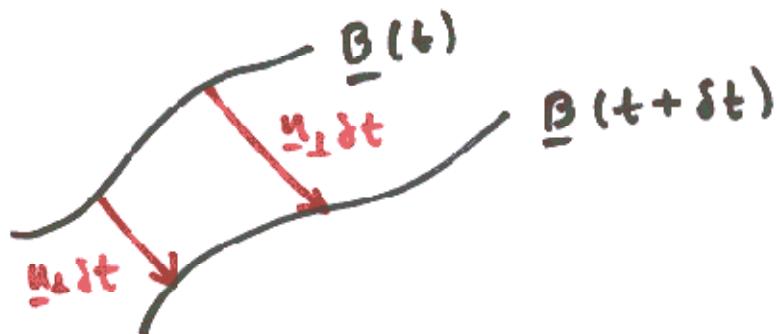
$$\Rightarrow \gamma \sim \frac{2 e^2 \sqrt{m_e}}{8\pi \epsilon_0^2 \tau_e^{3/2}} \ln \Lambda : \text{small, comparable to metals}$$

# Magnetic flux freezing

With  $\eta \rightarrow 0$

$$\underline{\underline{\epsilon}} + \underline{\underline{u}} \times \underline{\underline{B}} = 0 \quad + \quad \nabla \times \underline{\underline{\epsilon}} = - \frac{\partial \underline{\underline{B}}}{\partial t} \Rightarrow$$

$$\Rightarrow \frac{\partial \underline{\underline{B}}}{\partial t} = \nabla \times (\underline{\underline{u}} \times \underline{\underline{B}}) \quad : \text{Lines of } \underline{\underline{B}} \text{ frozen in the plasma}$$



# Magnetic Reynolds number

In general:

$$\underline{\underline{E}} + \underline{\underline{u}} \times \underline{\underline{B}} = \underline{\underline{j}} + \nabla \times \underline{\underline{E}} = - \frac{\nabla B}{\mu_0} + \nabla \times \underline{\underline{B}} = \mu \underline{\underline{j}} \Rightarrow$$

$$\Rightarrow \frac{\nabla B}{\mu_0} = \underbrace{\nabla \times (\underline{\underline{u}} \times \underline{\underline{B}})}_{\sim \frac{uB}{L}} + \underbrace{\frac{\eta}{\mu_0} \nabla^2 \underline{\underline{B}}}_{\sim \frac{\eta B}{\mu_0 L^2}}$$

$$R_M = \frac{uB/L}{\eta B / \mu_0 L^2} = \frac{\mu_0 u L}{\eta}$$

Frozen  $\underline{\underline{B}}$  lines for  $R_M \gg 1$

# Magnetic pressure and tension

$$\rho \frac{du}{dt} = -\nabla p + \underline{j} \times \underline{B} + \underline{\nabla} \times \underline{B} = \mu_0 \underline{j} \Rightarrow$$

$$\Rightarrow \rho \frac{du}{dt} = -\nabla p - \nabla \left( \frac{B^2}{2\mu_0} \right) + \underbrace{\frac{1}{\mu_0} (\underline{B} \cdot \nabla) \underline{B}}_{\text{magnetic tension: acts to straighten bent } \underline{B} \text{ lines}}$$

magnetic pressure

$$\beta \equiv \frac{p}{B^2/2\mu_0} : \beta \ll 1 : \text{Plasma driven by } \underline{B}$$

# Cathode sheath

Example two-fluid model : cathode layer (sheath)

$$\begin{aligned}
 & n_e < z n_i \quad \left. \begin{array}{l} \phi = 0 \\ u_{i0} \end{array} \right\} \quad \left. \begin{array}{l} \phi = -V_0 \\ n_{e0} = z n_{i0} \end{array} \right\} \\
 & \left. \begin{array}{l} \frac{d}{dx}(n_i u_i) = 0 \\ m_i u_i \frac{du_i}{dx} = -ze \frac{d\phi}{dx} \\ 0 = -\frac{T_e}{n_e} \frac{dn_e}{dx} + e \frac{d\phi}{dx} \\ + \frac{d^2\phi}{dx^2} = \frac{e}{\epsilon_0} (n_e - z n_i) \end{array} \right\} \\
 & n_i u_i = n_{i0} u_{i0} \\
 & m_i \frac{u_i^2}{2} + ze\phi = \frac{m_i u_{i0}^2}{2} \\
 & n_e = n_{e0} \exp\left(\frac{e\phi}{T_e}\right) \\
 & + \frac{d^2\phi}{dx^2} = \frac{e n_{e0}}{\epsilon_0} \left[ \exp\left(\frac{e\phi}{T_e}\right) - \left(1 - \frac{ze\phi}{m_i u_{i0}^2}\right)^{-1/2} \right]
 \end{aligned}$$

Multiplying by  $\frac{d\phi}{dx}$  a first integral is obtained:

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{zTe n_{eo}}{\epsilon_0} \left[ \exp\left(\frac{e\phi}{Te}\right) + \frac{m_i u_{io}^2}{zTe} \left(1 - \frac{zTe}{m_i u_{io}^2}\right)^{1/2} - 1 - \frac{m_i u_{io}^2}{zTe} \right]$$

Taylor developing about  $\phi = 0$ :

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{n_{eo} e^2}{\epsilon_0 Te} \left(1 - \frac{zTe}{m_i u_{io}^2}\right) \phi^2 + O(\phi^3) \geq 0$$

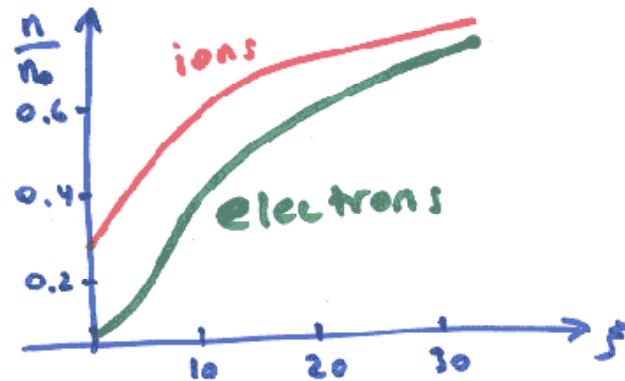
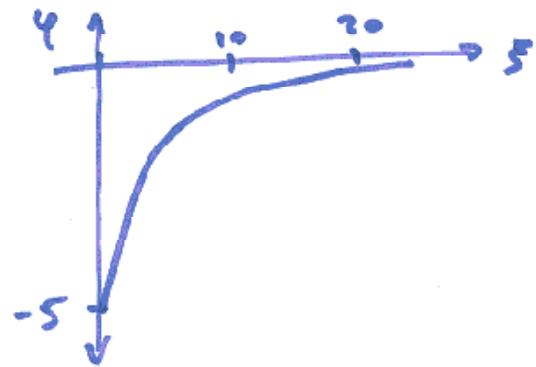
$$\Rightarrow u_{io}^2 \geq \frac{zTe}{m_i} = V_{Bohm}^2 : \text{Ions must be accelerated in pre-sheath up to (at least) } V_{Bohm}$$

Bohm conjecture :  $v_{i_0}^2 = v_{\text{Bohm}}^2$ .

Using  $\varphi \equiv \frac{e\phi}{T_e}$ ,  $\xi \equiv \frac{x}{2D}$ ,  $2D = \left( \frac{e T_e}{2 n_{e_0} e^2} \right)^{1/2}$

$$\frac{d\varphi}{d\xi} = [e^{\varphi} + (1 - 2\varphi)^{1/2} - 2]^{1/2} \quad \varphi(\xi=0) = -\frac{eV_0}{T_e}$$

Numerically, for  $\varphi(0) = -5$  :

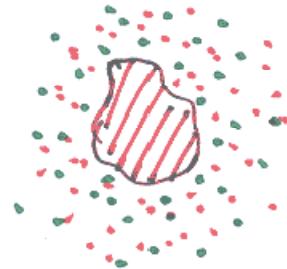


# Floating potential

Electric current density to the wall :

$$j = ze n_{\text{eo}} V_{\text{Bath}} - e n_e(x=0) \underbrace{\frac{1}{4} V_{T_e}}_{\text{from average velocity directed to the wall}}$$

$n_{\text{eo}} \exp\left(-\frac{eV_0}{T_e}\right)$  for  $f_{\text{MB}} : V_{T_e} = \left(\frac{8T_e}{\pi m_e}\right)^{1/2}$



Floating object  
accumulates charges  
up to the condition  
 $j = 0$

$$\begin{aligned} j = 0 \Rightarrow V_0 &= - \frac{T_e}{e} \ln\left(\frac{4V_{\text{Bath}}}{V_{T_e}}\right) = \\ &= - \frac{T_e}{ze} \ln\left(\frac{m_i}{2\pi e m_e}\right) \approx V_F \end{aligned}$$

Floating potential  
(relative to the plasma)

# High-voltage sheath

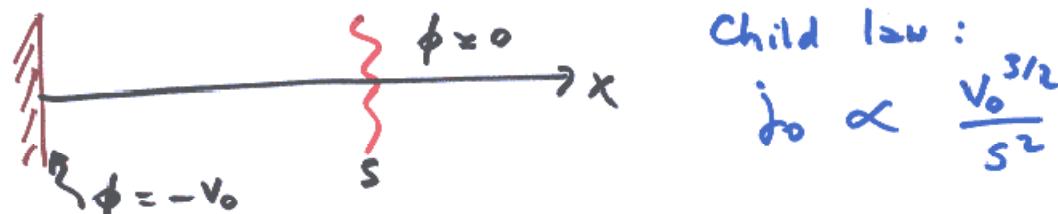
For a high-voltage sheath,  $\frac{eV_0}{T_e} \gg 1$ , there are practically no electrons present

Previous results with  $|\frac{e\phi}{T_e}|, |\frac{e\phi}{m_i u_{i0}^2}| \gg 1$  gives

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{2n_{i0} m_i u_{i0}^2}{\epsilon_0} \left(-\frac{e\phi}{m_i u_{i0}^2}\right)^{1/2} \quad (\phi \approx \frac{d\phi}{dx} \approx 0 \text{ in the plasma})$$

Calling  $j_0 = 2en_{i0}u_{i0}$  (current density to the wall)  
one obtains on integration

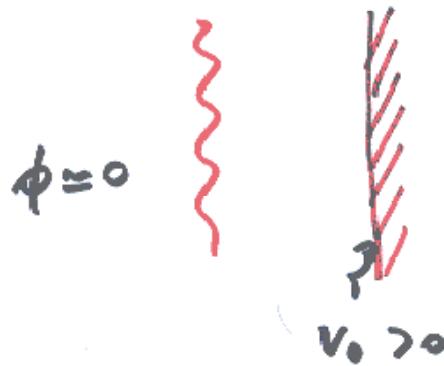
$$(-\phi)^{3/4} = (s-x) \frac{3}{4} \left(\frac{j_0}{\epsilon_0}\right)^{1/2} \left(\frac{8m_i}{ze}\right)^{1/4} \quad s: \text{sheath thickness}$$



Using  $v_{th} = \sqrt{8kT_e/m_e} = \left(\frac{2T_e}{m_e}\right)^{1/2}$  one further has:

$$S = \frac{\sqrt{2}}{3} \lambda_D \left(\frac{2V_0}{T_e}\right)^{3/4}$$

Anode layer? If  $\frac{eV_0}{T_e} \gg 1$  enormous currents:



electrons leaving the plasma increase plasma potential to be close to  $V_0$

Only small potential difference,  $\frac{e\Delta\phi}{T_e} \sim 1$ , can be sustained in anode layers.