

Waves in non-magnetized plasmas

Plasma oscillations with $\underline{B}_0 = 0$ (base field)

- Small amplitude waves (linear theory)
- Very high frequency (only electrons participate)
- Cold electrons ($T_e \rightarrow 0$)

$$n_e = n_{eo} + \delta n_e, \quad n_i = n_{io} = n_{eo}/Z$$

$$\frac{\partial \delta n_e}{\partial t} + \nabla \cdot (n_{eo} \underline{v}_e) = 0$$

$$m_e \frac{\partial \underline{v}_e}{\partial t} = -e \underline{E}$$

$$\nabla \cdot \underline{E} = \frac{e}{\epsilon_0} (Zn_i - n_e) = -\frac{e \delta n_e}{\epsilon_0}$$

$$\nabla \cdot \underline{B} = 0, \quad \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{E} = \mu_0 (-e) n_{eo} \underline{v}_e + \frac{1}{c^2} \frac{\partial \underline{B}}{\partial t}$$

$$a(\underline{z}, t) \rightarrow a e^{i(\underline{k} \cdot \underline{z} - \omega t)}$$

$$\left. \begin{array}{l} -i\omega \delta n_e + i n_{eo} \underline{k} \cdot \underline{u}_e = 0 \\ i\omega n_e \underline{u}_e = e \underline{E} \end{array} \right\} \rightarrow \left. \begin{array}{l} \delta n_e = -\frac{i e n_{eo}}{m_e \omega^2} \underline{k} \cdot \underline{E} \\ \underline{u}_e = -\frac{i e}{m_e \omega} \underline{E} \end{array} \right.$$

$$i \underline{k} \cdot \underline{E} = -\frac{e}{\omega} \delta n_e$$

$$\underline{k} \cdot \underline{\theta} = 0, \quad \underline{k} \times \underline{E} = \omega \underline{\theta}$$

$$i \underline{k} \times \underline{\theta} = -\mu_0 e n_{eo} \underline{u}_e - i \frac{\omega}{c^2} \underline{E}$$

Electron plasma frequency

$$\omega_{pe}^2 = \frac{e^2 n_{eo}}{m_e E_0}$$

$$\hookrightarrow \left\{ \begin{array}{l} \underline{k} \cdot \underline{E} = \frac{\omega_{pe}^2}{\omega^2} \underline{k} \cdot \underline{E}, \quad \underline{k} \cdot \underline{\theta} = 0 \\ \underline{k} \times \underline{E} = \omega \underline{\theta} \quad , \quad \underline{k} \times \underline{\theta} = \frac{1}{\omega c^2} (\omega_{pe}^2 - \omega^2) \underline{E} \end{array} \right.$$

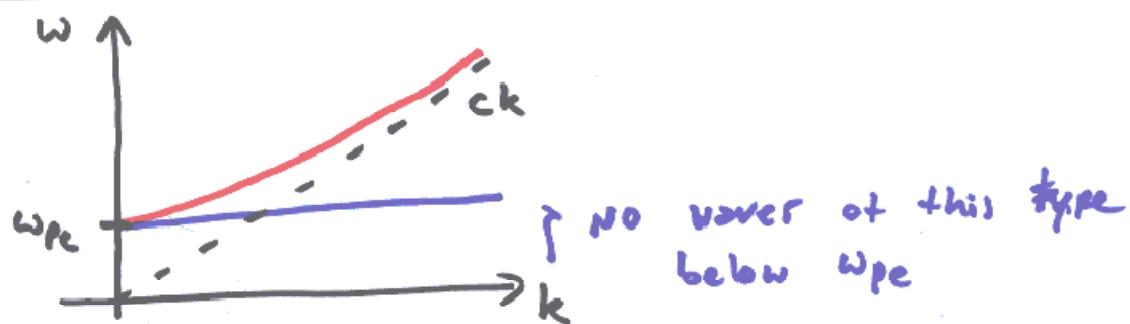
Possible solutions

i) Longitudinal wave : $\underline{k} \cdot \underline{E} \neq 0$

$$\Rightarrow \omega = \omega_{pe}, \quad \underline{B} = 0 \quad (\text{Plasma, Langmuir wave})$$

ii) Transversal wave : $\underline{k} \cdot \underline{E} = 0$

$$\Rightarrow \omega^2 = \omega_{pe}^2 + k^2 c^2 \quad (\text{Electromagnetic Wave})$$



Lower frequencies require ion dynamics and pressure effects:

With general barotropic relation: $p \sim n^{\gamma}$

$$\begin{aligned} p &= p_0 + \delta p \\ n &= n_0 + \delta n \end{aligned} \quad \delta p = \gamma \frac{p_0}{n_0} \delta n = m \underset{\text{sound speed}}{\overset{\uparrow}{c_0^2}} \delta n$$

Derivation analogous to previous one given:

1) Longitudinal waves: $k_z \cdot \xi \neq 0$

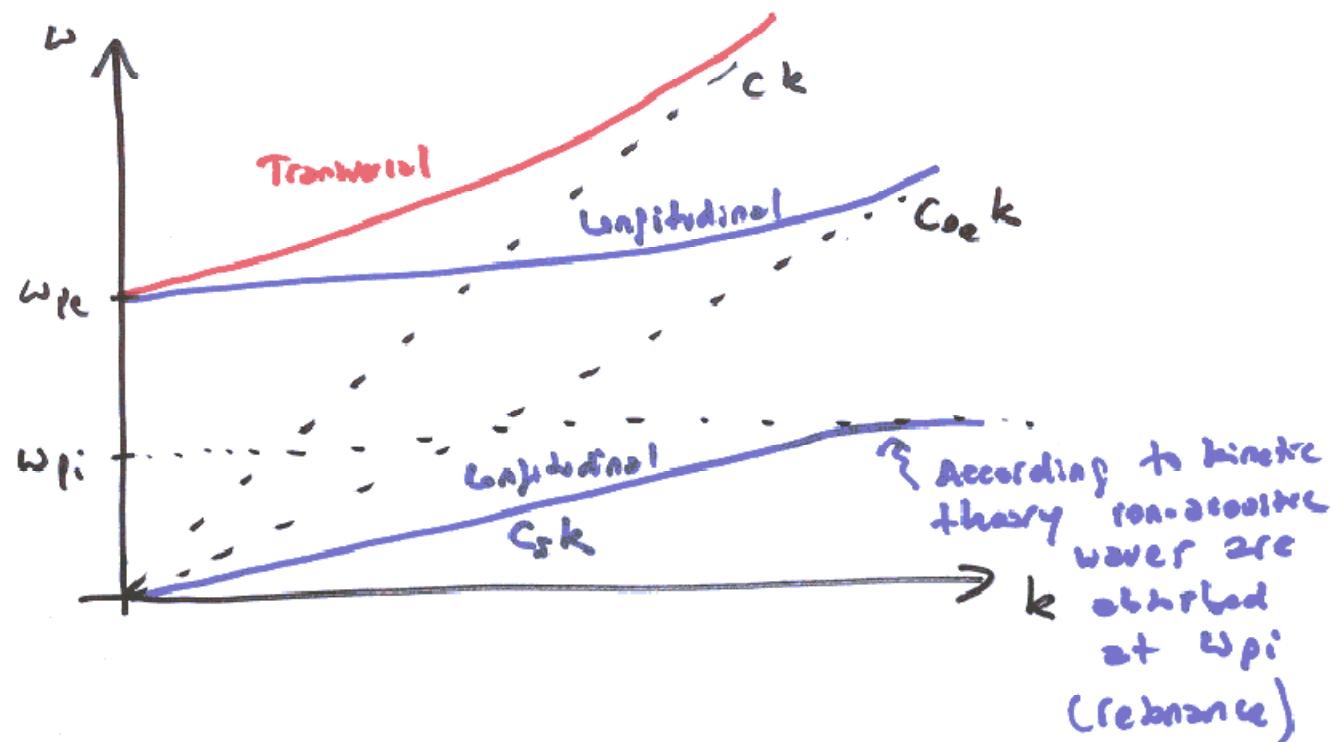
$$\omega^2 = \omega_{pe}^2 + k^2 c_s^2 \quad (\text{Langmuir wave with } T_e \neq 0)$$

$$\omega^2 = c_s^2 k^2, \quad c_s^2 \equiv \frac{\gamma_e e^2 T_e + \gamma_i T_i}{m_i} \quad (\text{ion-acoustic wave})$$

\uparrow
ion sound velocity

2) Transversal waves: $k_z \cdot \xi = 0$, same as above.

Dispersion relation $B=0$

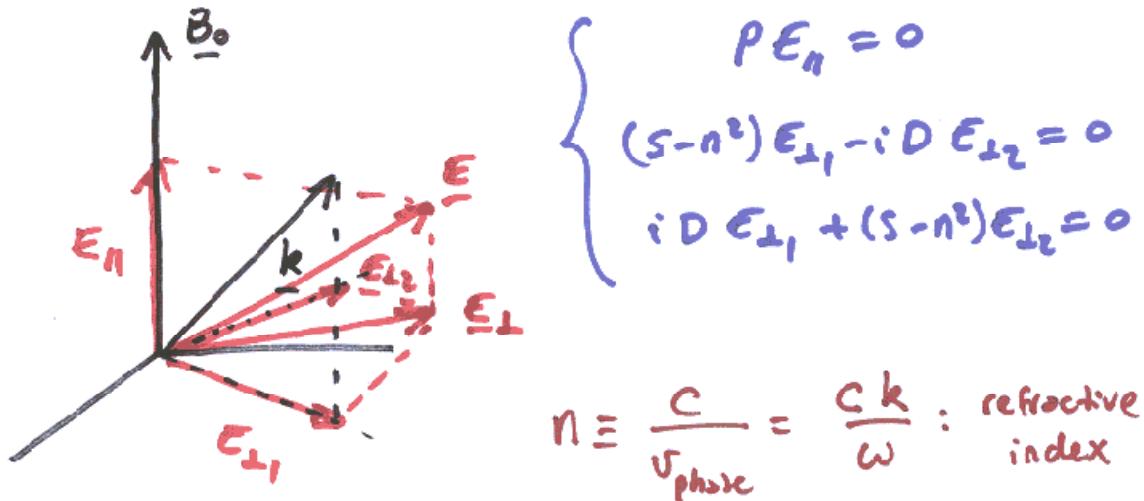


Waves in magnetized plasmas

Waves in a magnetized plasma :

- Thermal effects can be neglected if $\frac{\omega}{k} \gg C_{e,i}$
- Collisional " " " " " " $\omega \gg \nu_{ci}$

Linearized system with conventions :



$$\left\{ \begin{array}{l} P \equiv 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2} \\ S \equiv 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2} \\ D \equiv - \frac{\omega_{pe} \omega_{ce}}{\omega(\omega^2 - \omega_{ce}^2)} + \frac{\omega_{pi} \omega_{ci}}{\omega(\omega^2 - \omega_{ci}^2)} \end{array} \right.$$

$$\omega_{pe}^2 = \frac{e^2 n_e}{m_e E_0}, \quad \omega_{pi}^2 = \frac{z^2 e^2 n_{i0}}{m_i E_0} = \frac{z m_e}{m_i} \omega_{pe}^2 \ll \omega_{pe}^2$$

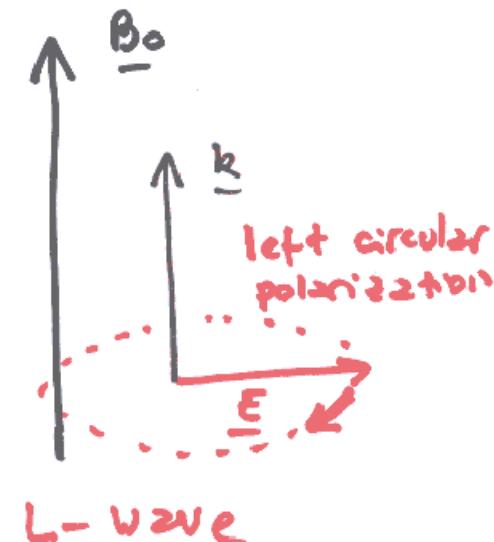
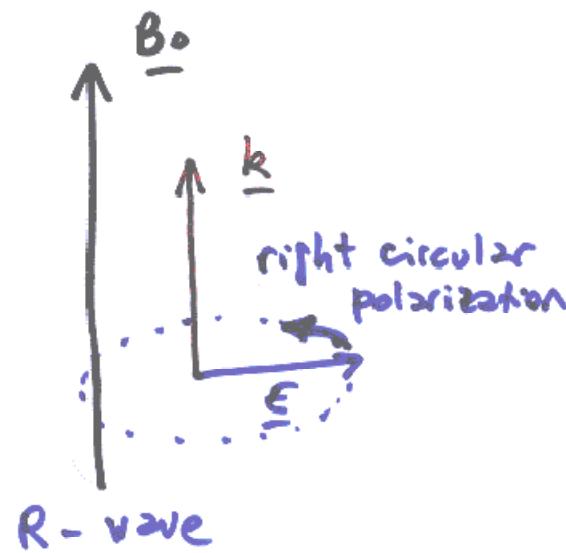
$$\omega_{ce} = \frac{e B_0}{m_e}, \quad \omega_{ci} = \frac{z e B_0}{m_i} = \frac{z m_e}{m_i} \omega_{ce} \ll \omega_{ce}$$

Longitudinal propagation

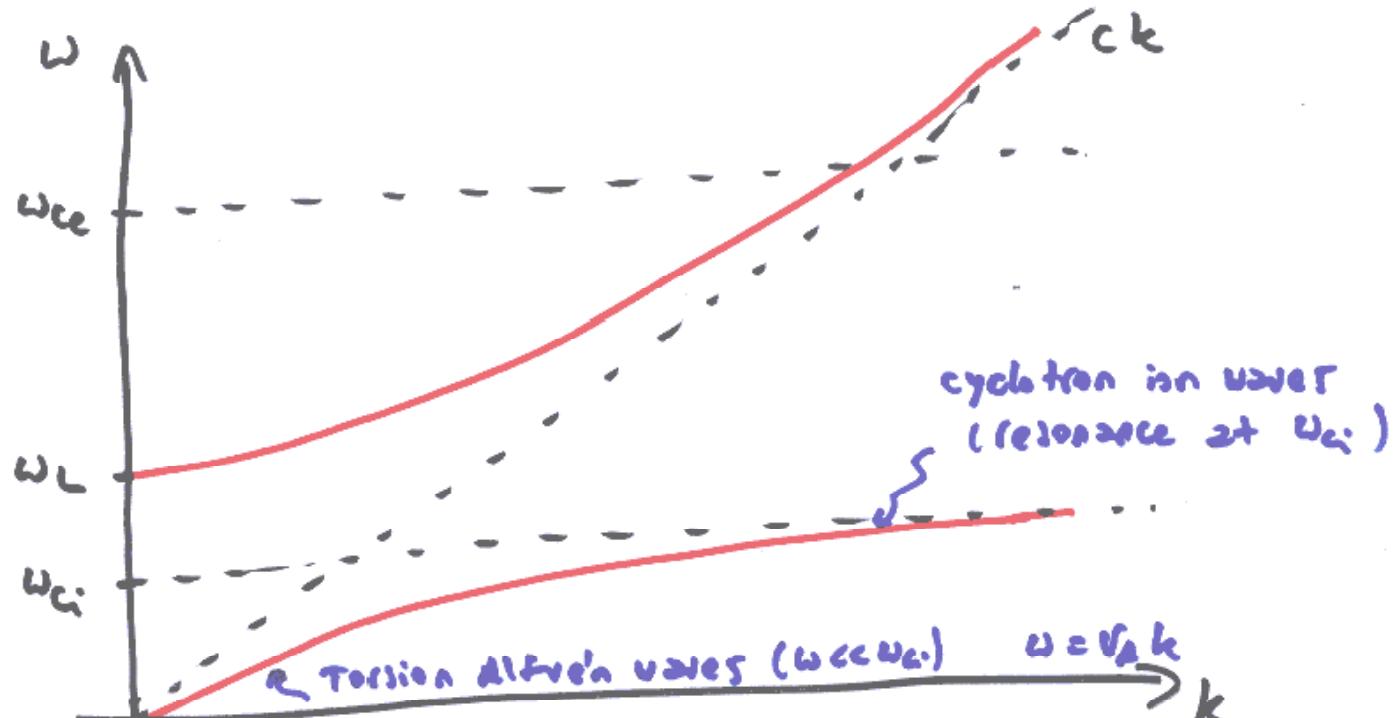
Propagation along the magnetic field : $\underline{k} \parallel \underline{B_0}$

1) Longitudinal wave: $\underline{E} \parallel \underline{k}$: same as with $B_0 = 0$

2) Transversal wave: $\underline{E} \perp \underline{k}$:



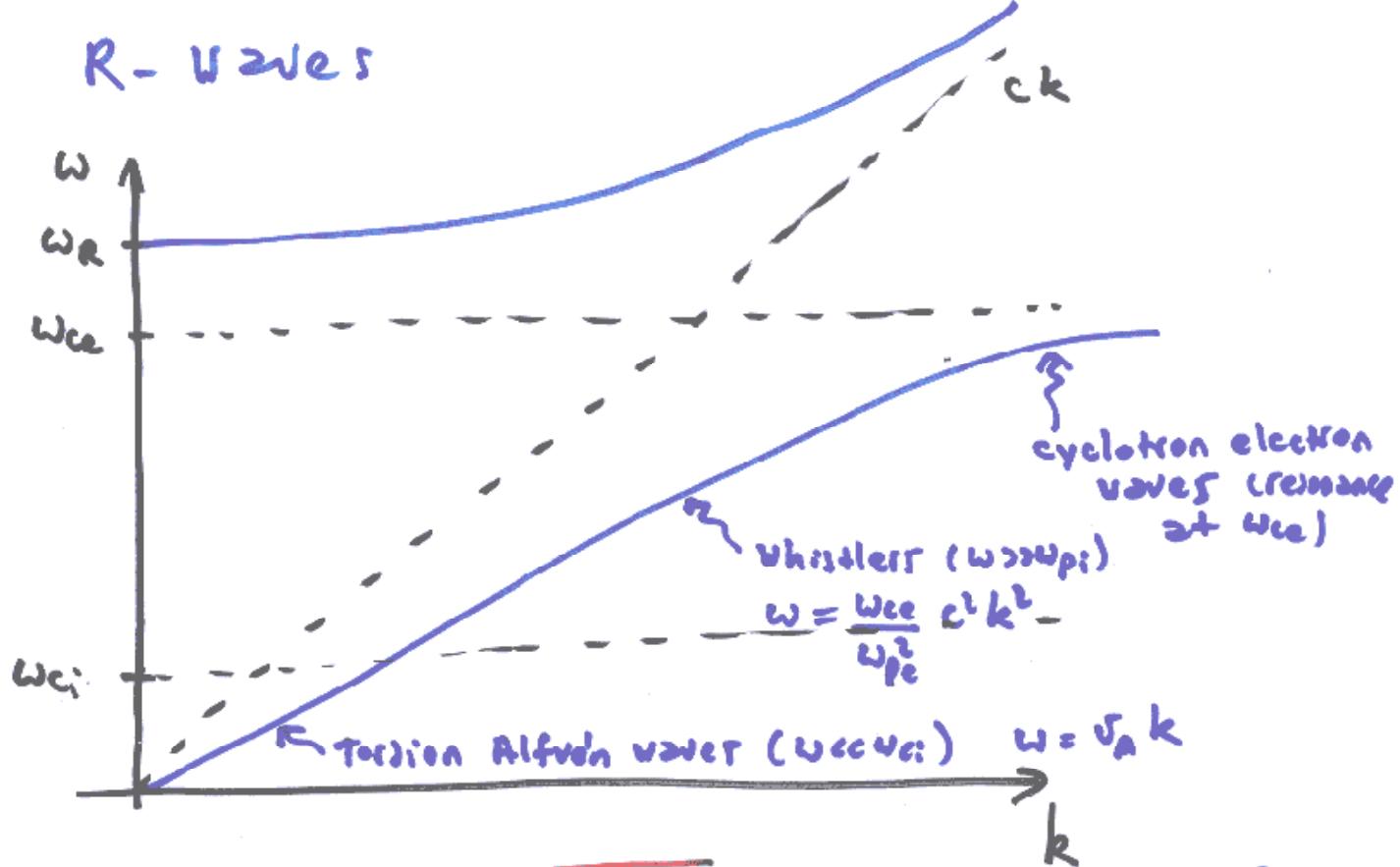
L - waves



$$\omega_L = \frac{1}{2} \left(-\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_{pe}^2} \right)$$

left - wave cut-off

R-waves

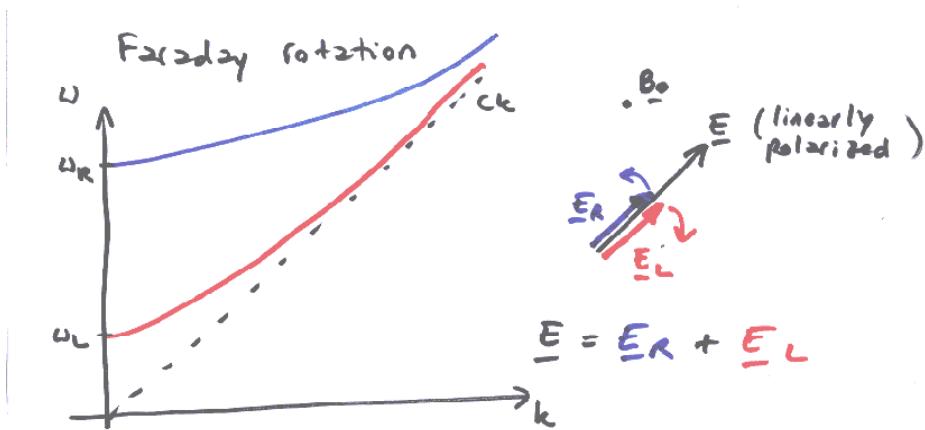


$$\omega_R = \frac{1}{2} (\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_{pe}^2})$$

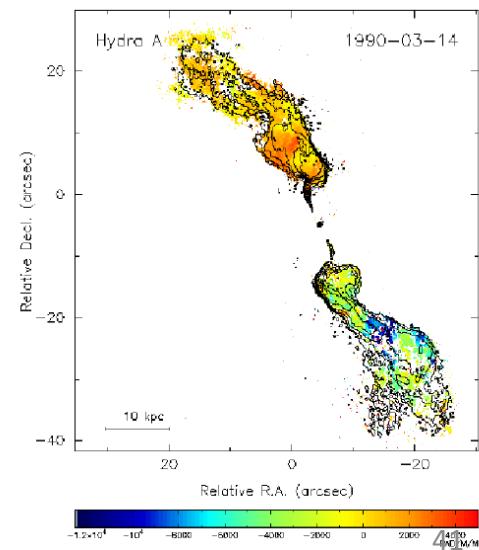
right wave cut-off.

$$v_A = c \frac{\omega_R}{\omega_{pi}} = \frac{B_0}{\sqrt{\mu_0 \rho_0 m_i}}$$

Faraday rotation



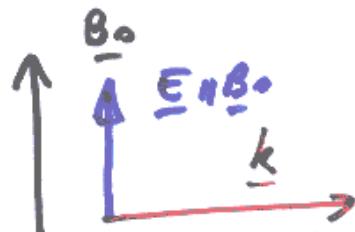
As the rotation is k dependent the effect is measurable, which allows galactic magnetic fields to be determined.



Perpendicular propagation

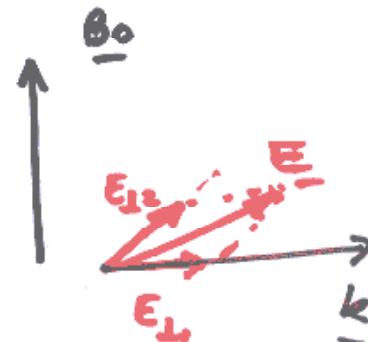
Propagation $\perp \underline{B_0}$ ($\underline{k} \perp \underline{B_0}$)

1)

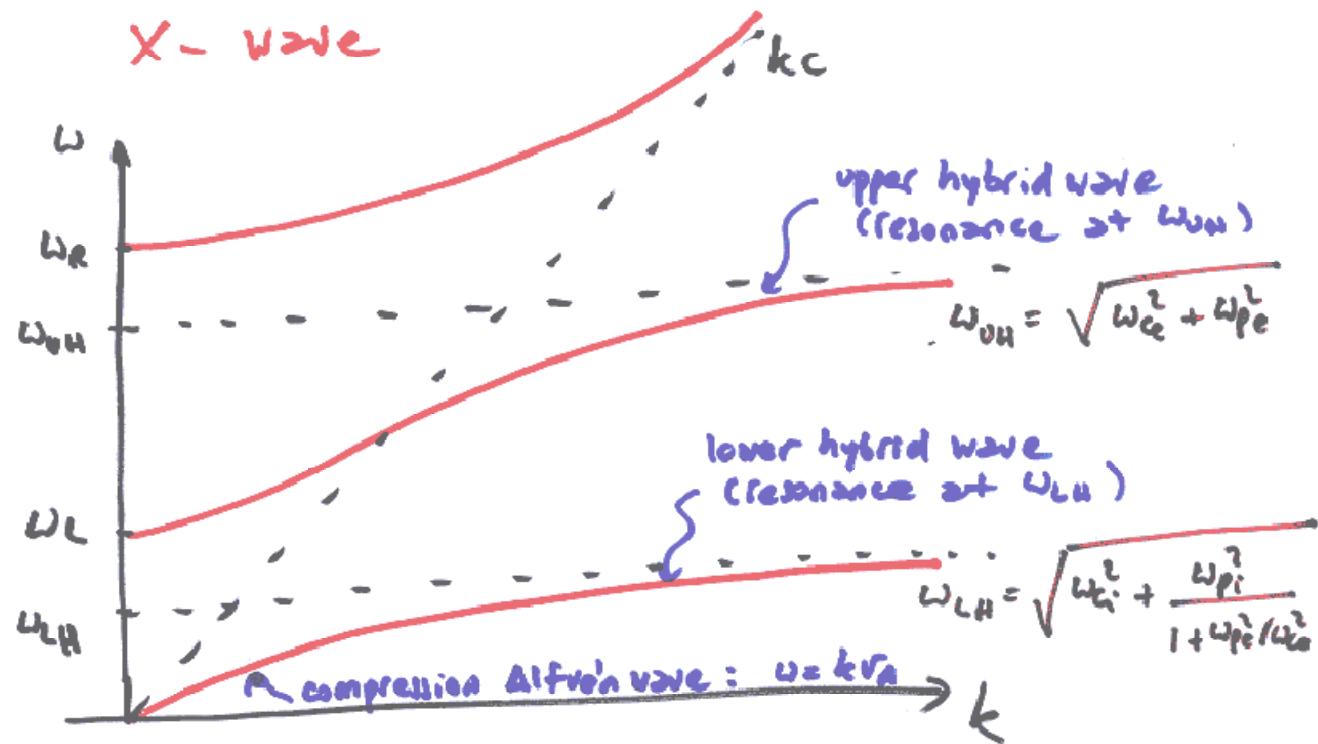


O-mode (ordinary)
zr without $\underline{B_0}$:
 $\omega^L = \omega_{pe}^2 + k^2 c^2$

2)

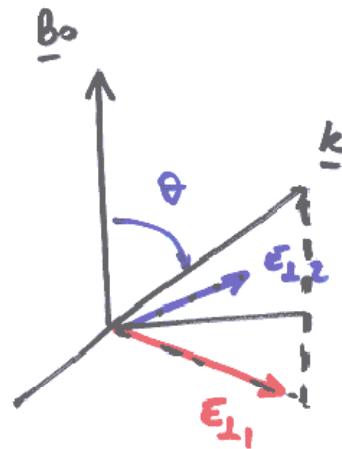


X-mode (Extraordinary)



Low-frequency oblique waves

General propagation with $\omega \ll \omega_A$: Alfvén waves



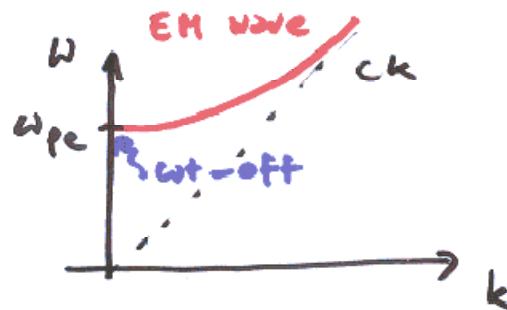
One has $E_{\parallel} = 0$

$$E_{L1} : \omega = kV_A \cos \theta \\ (\text{slow Alfvén mode})$$

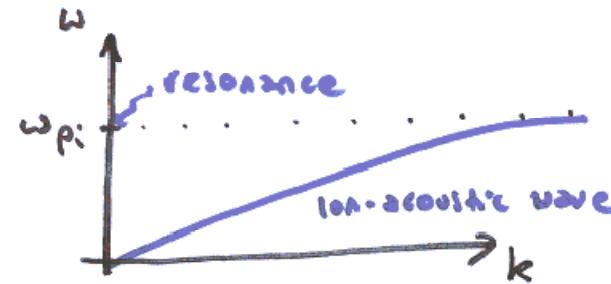
$$E_{L2} : \omega = kV_A \\ (\text{fast Alfvén mode})$$

Slow mode: linear polarization, superposition of *R* and *L* (torsional) waves.
Fast mode: compressional wave.

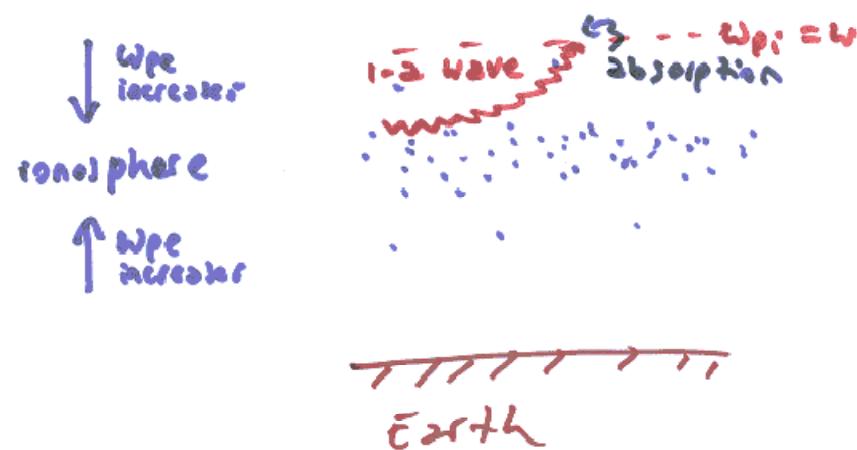
Cut-off and resonances



cut-off frequencies:
 $\omega_{pe}, \omega_R, \omega_L$

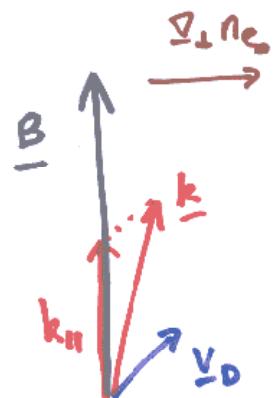


resonance frequencies:
 $\omega_{pi}, \omega_{ce}, \omega_{ci}, \omega_{uh}, \omega_{lh}$



Drift waves

Waves in inhomogeneous plasmas : Drift waves



Along \underline{B} electrons in mechanical equilibrium:

$$-\frac{T_e}{n_e} \nabla_n n_e + e \nabla_{||} \phi = 0$$

$$\Rightarrow \phi = \frac{T_e}{e} \ln n_e$$

$$\Rightarrow \underline{v}_{exB} = -\frac{T_e}{eB^2} \nabla_{\perp} n_e \times \underline{B} \equiv \underline{v}_D$$

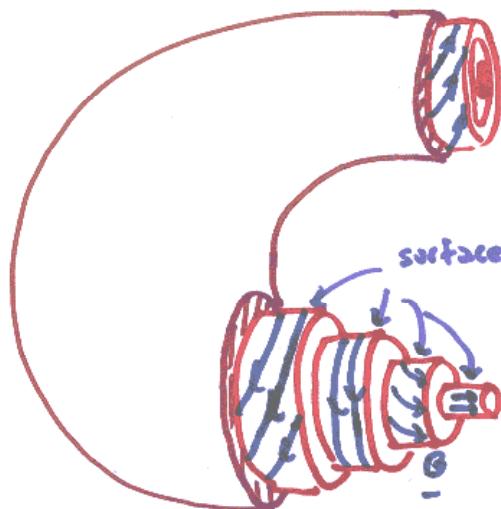
Perturbations of density generates oscillations with

$$\omega (\omega - \omega_*) = k_{||}^2 C_s^2$$

$\omega_* = k_{||} v_D$: electron drift wave frequency.

Plasma equilibrium

Static plasma equilibrium



Reversed field pinch
 B_{toroidal} small & reversed
near the edge

$$\nabla p = \underline{j} \times \underline{B} \Rightarrow$$
$$\Rightarrow \underline{j} \cdot \nabla p = \underline{B} \cdot \nabla p = 0$$



Force-free (Spheromak)

For $\beta = \frac{B}{8\pi\rho c} \ll 1$, force-free equilibrium

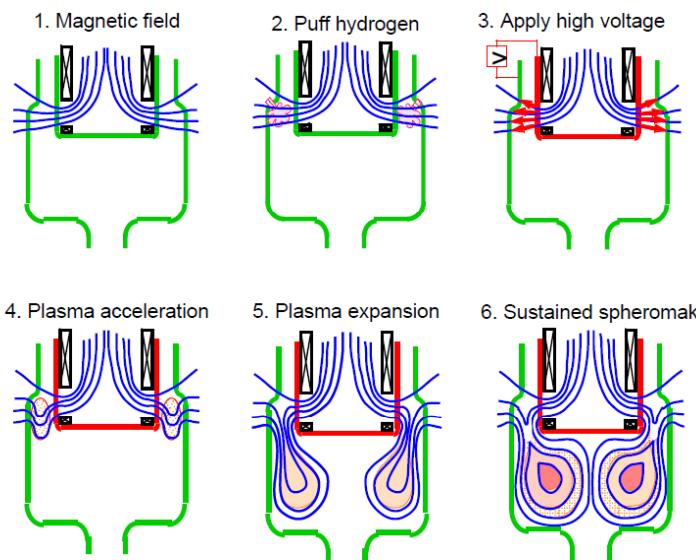
$$\underline{j} \times \underline{B} = 0 \Rightarrow (\nabla \times \underline{B}) \times \underline{B} = 0 \Rightarrow$$

$$\rightarrow \nabla \times \underline{B} = \gamma \underline{B}$$

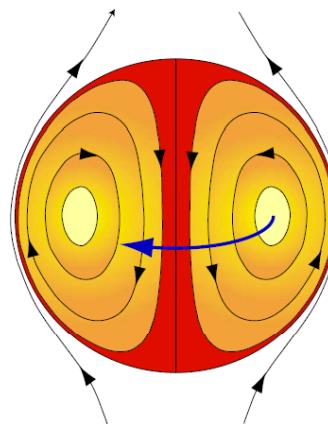
γ scalar function

$$\text{Also: } \underline{B} \cdot \nabla \gamma = 0 \quad (\gamma = \text{const : magnetic surface})$$

Typical spheromak formation sequence



Essential Characteristics of a Spheromak Plasma

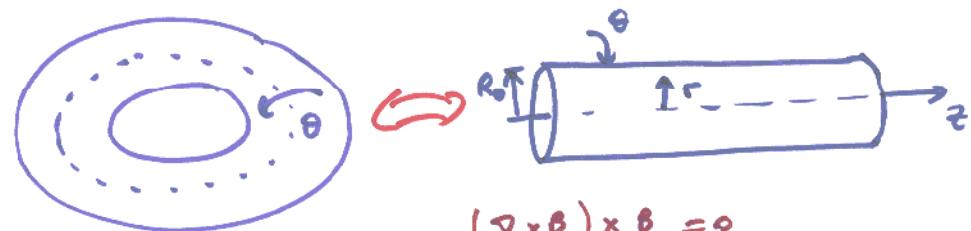


- Low-aspect-ratio (R/a) toroidal magnetic configuration.
- Confining magnetic fields produced by currents in the plasma itself.
- Nearly force-free field aligned currents:

$$\lambda = \frac{\mu_0 j}{B} \quad \nabla \times \vec{B} = \lambda \vec{B}$$
- Magnetic topology:
 - edge: Poloidal fields & currents
 - core: Toroidal fields & currents

Tokamak

Tokamak ($\beta \ll 1$)



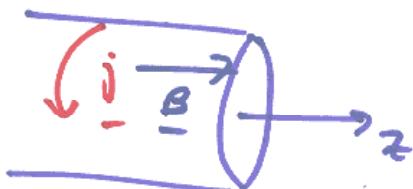
$$(\nabla \times \underline{B}) \times \underline{B} = 0$$

$$B_z \leftarrow \begin{cases} \frac{d}{dr}(B_z^2 + B_\theta^2) + \frac{2}{r} B_\theta^2 = 0 \\ B_\theta(r) = \frac{\mu_0 I(r)}{2\pi r} \end{cases}$$

Uniform current : $I(r) = \frac{I_0 r^2}{R_0^2} \Rightarrow B_z = B_{z\text{ext}} + \frac{\mu_0 I^2}{2\pi^2 R_0^4} \left(1 - \frac{r^2}{R_0^2}\right)$

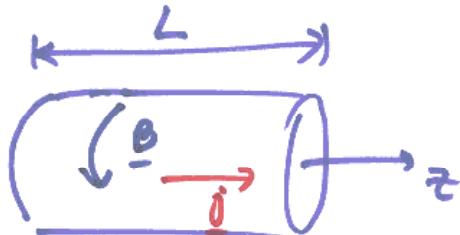
Pinch

θ -pinch



$$\left[\begin{array}{l} \frac{dB}{dr} = -\mu_0 j(r) \\ \frac{dp}{dr} = j B \end{array} \right] \Rightarrow p + \frac{B^2}{2\mu_0} = \text{const.}$$

z -pinch



$$\left\{ \begin{array}{l} B(r) = \frac{\mu_0 I(r)}{2\pi r}, \quad j = \frac{1}{2\pi r} \frac{dI}{dr} \\ \frac{dp}{dr} = -j B \end{array} \right.$$

Bennett:

$$I_0^2 = \frac{8\pi}{\mu_0} (T_i + 2T_e) \frac{N_i}{L}$$

Plasma stability

For ideal MHD :

$$\int \frac{1}{2} \rho u^2 dv + \int \frac{P}{\gamma-1} dv + \int \frac{B^2}{2\mu_0} dV = \text{const.}$$

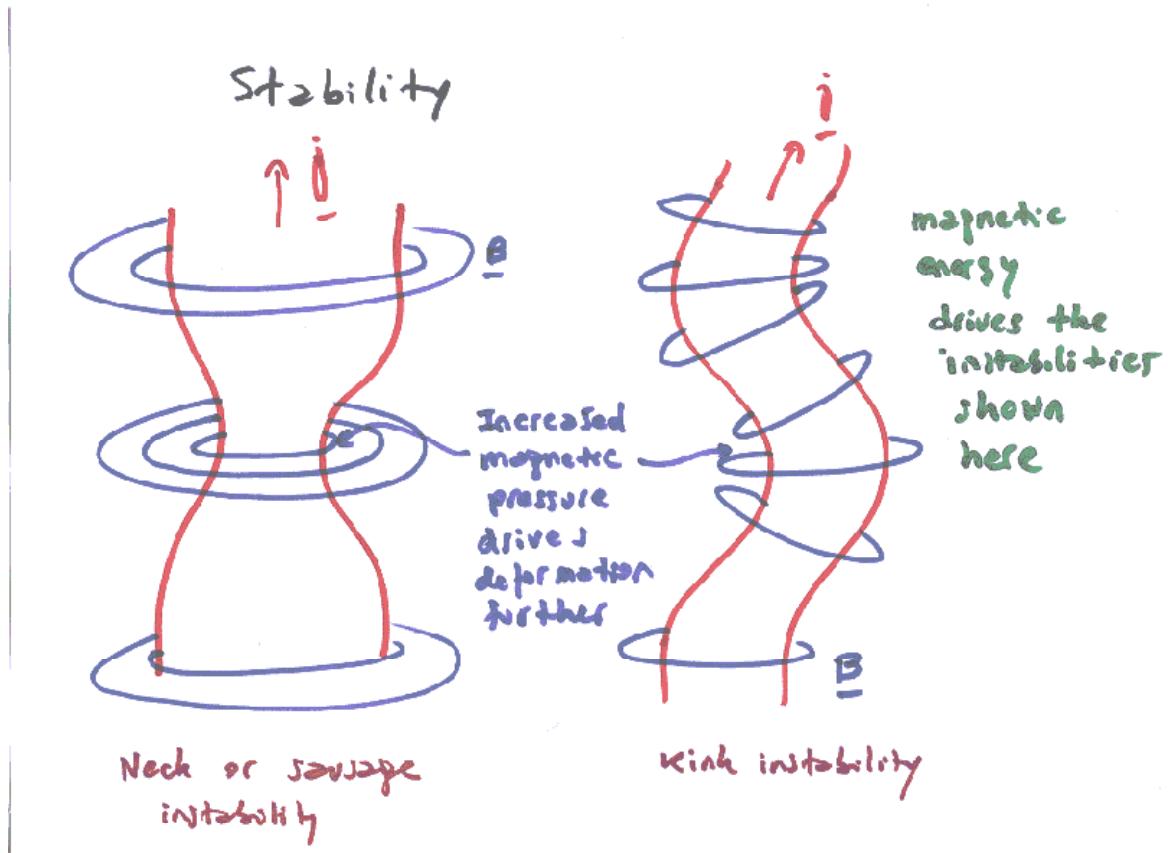
kinetic energy internal energy magnetic energy

All terms are positive definite.

Motion can develop in a static equilibrium ($\underline{v} = 0$)
driven by magnetic and/or internal energy.

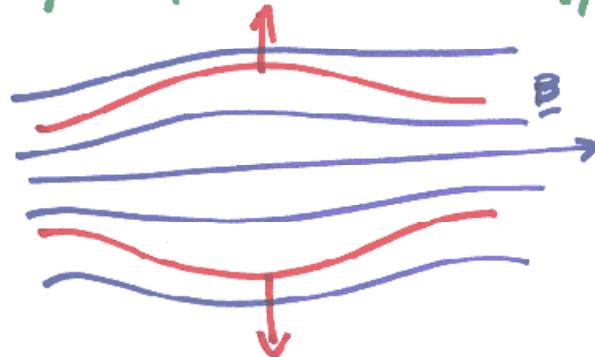
Equilibrium is (linearly) stable if internal + magnetic
energy increases for every possible small perturbation
of the equilibrium configuration.

Electromagnetic instabilities



Flute modes

Instabilities can be driven by the plasma internal energy

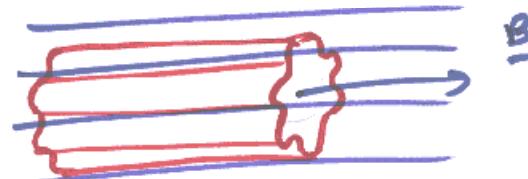


B frozen in the plasma.
Magnetic tension resists line bending.

For $\beta \ll 1$ plasma internal energy is small compared to magnetic energy.

More unstable modes do not bend magnetic lines:

"Flute" modes:



Interchange instability

Plasma tube

$$U_M = \frac{1}{2\mu_0} \int B^2 dV$$

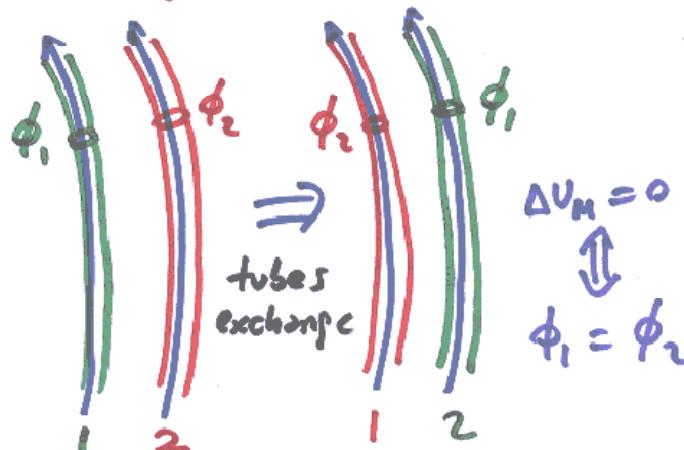
$$= \frac{1}{2\mu_0} \int B^2 ds dr$$

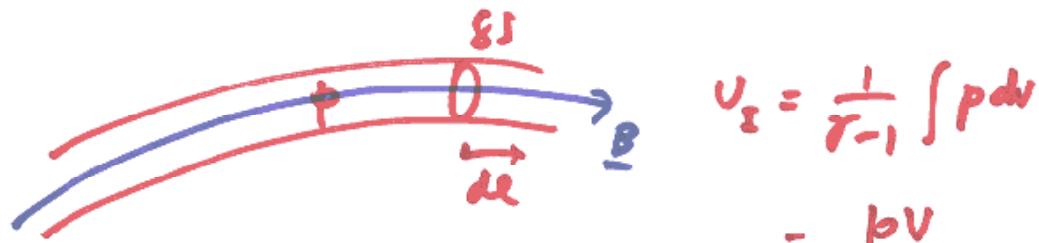
$$\phi = B \delta S = \text{const} \Rightarrow B = \frac{\phi}{\delta S}$$

$$U_M = \frac{\phi^2}{2\mu_0} \int \frac{dr}{\delta S}$$

$$2\mu_0 \Delta U_M = \phi_1^2 \int_2 \frac{dr}{\delta S} + \phi_2^2 \int_1 \frac{dr}{\delta S}$$

$$- \phi_1^2 \int_1 \frac{dr}{\delta S} - \phi_2^2 \int_2 \frac{dr}{\delta S}$$

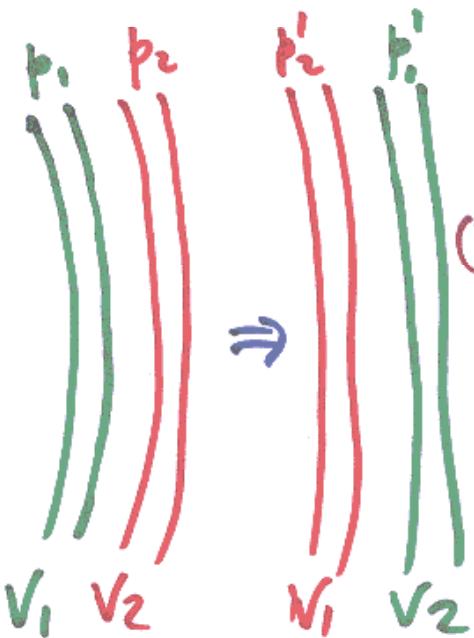




$$V_I = \frac{1}{r-1} \int p dV$$

$$= \frac{pV}{r-1}$$

linear of $\underline{\sigma}$
or $p = \text{const.}$
surface.



$$(r-1) \Delta V_I = p'_1 V_2 + p'_2 V_1 - p_1 V_1 - p_2 V_2$$

$$pV^r = \text{const} \Rightarrow \begin{cases} p_1 \equiv p & p_2 \equiv p + \delta p \\ V_1 \equiv V & V_2 \equiv V + \delta V \end{cases}$$

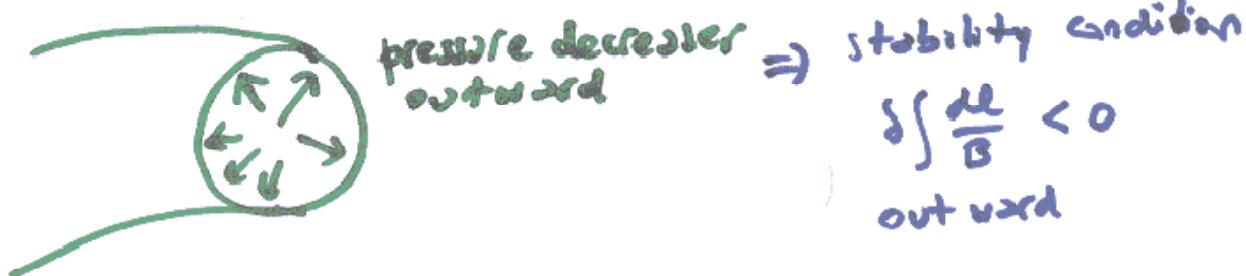
$$\Delta V_I = r p \frac{\delta V^r}{V} + \delta p \delta V$$

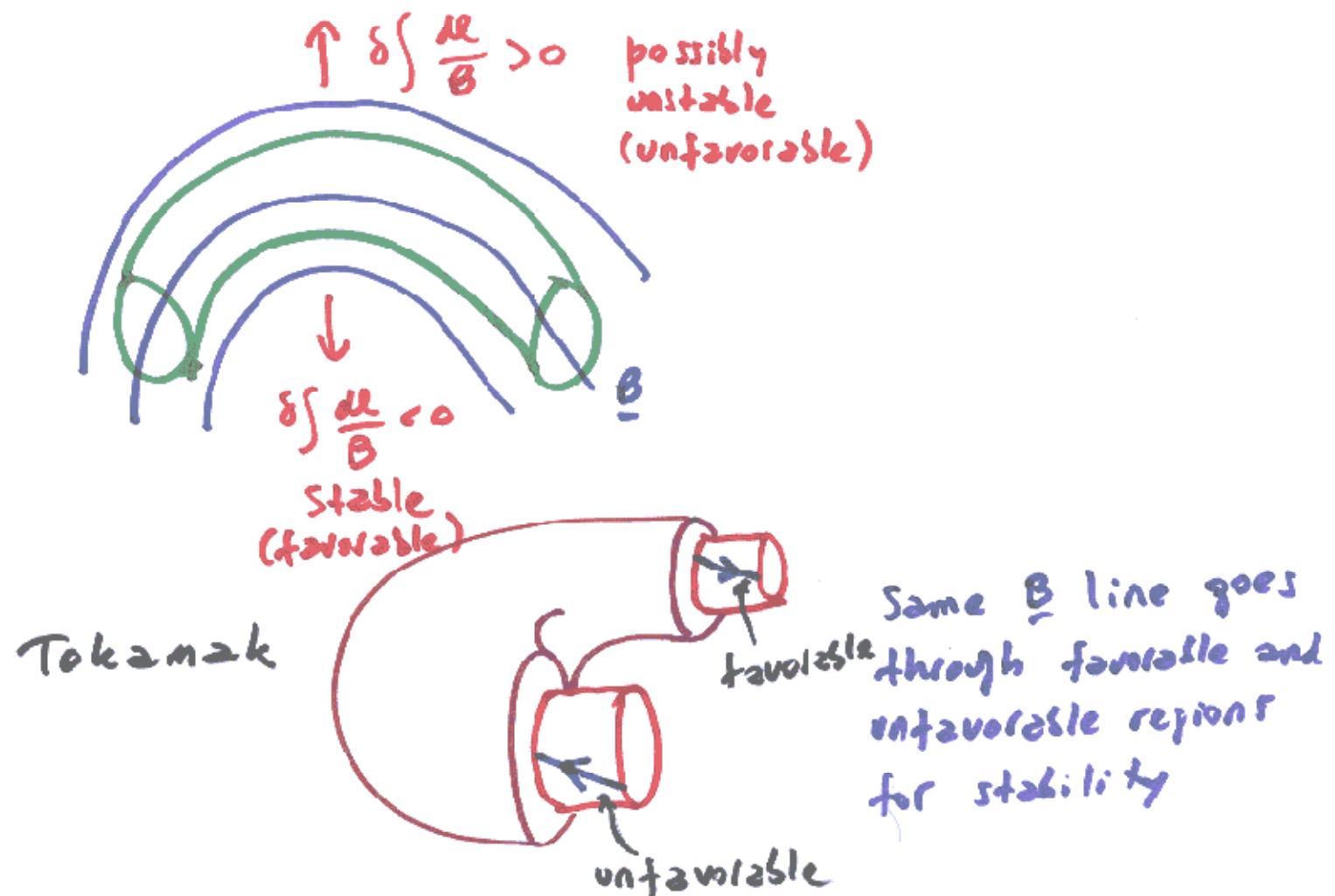
With $\phi_1 = \phi_2$ $\Delta U_H = 0 \Rightarrow$

stability if $\Delta U_1 > 0$

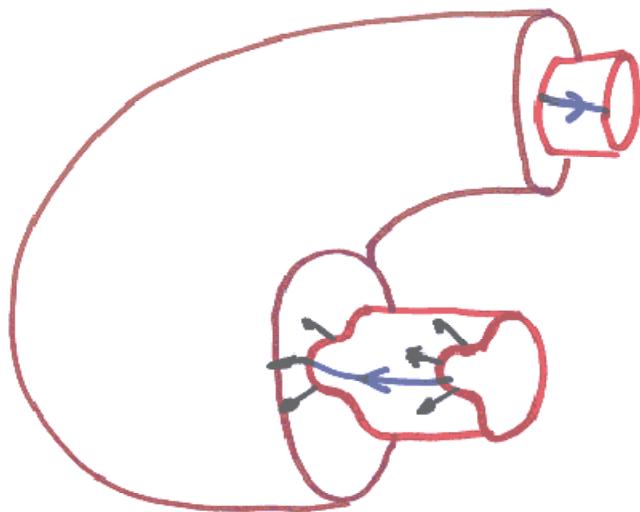
Sufficient condition: $\delta p \delta V > 0$

$$\left. \begin{aligned} v_1 &= \int_1 \delta s \, dl = \phi \int_1 \frac{dl}{B} \\ v_2 &= \int_2 \delta s \, dl = \phi \int_2 \frac{dl}{B} \end{aligned} \right\} \quad \delta V = \phi \int \frac{dl}{B}$$





Ballooning instability



If instability develops in unfavorable region it must bend the magnetic line (ballooning instability)



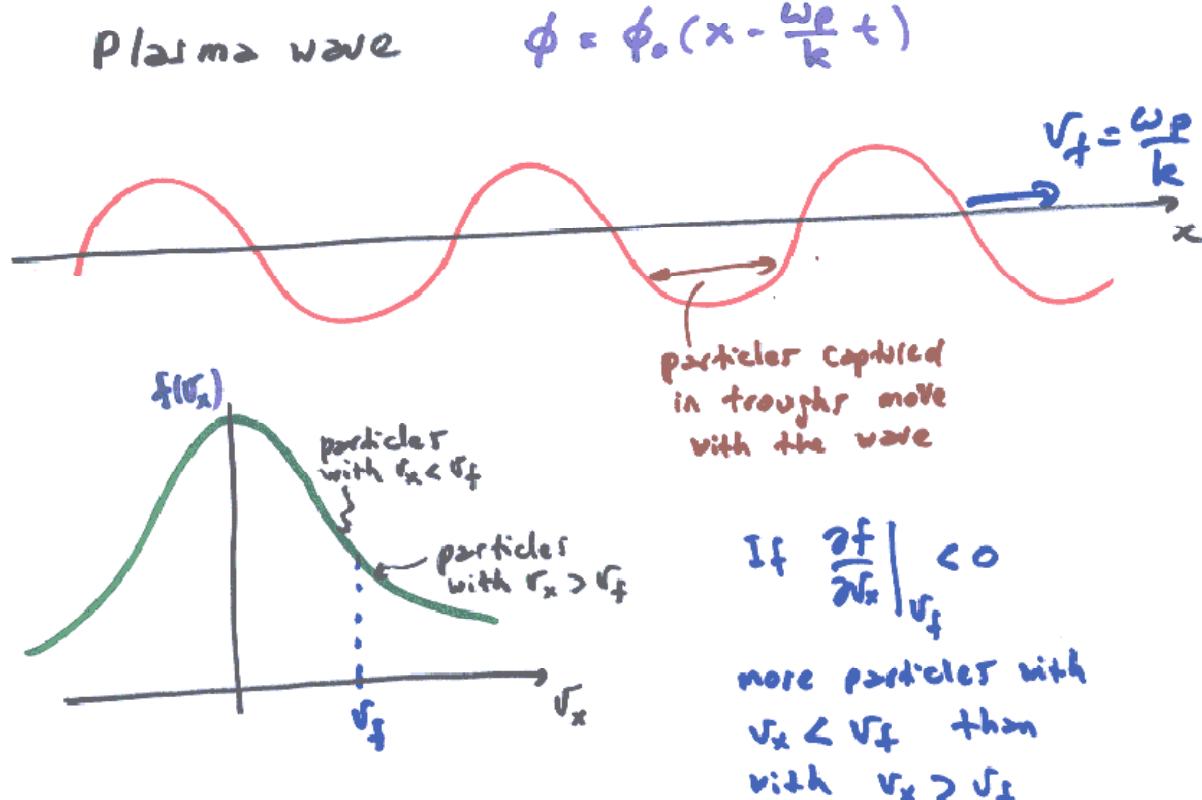
Internal energy must be sufficient



stable for low enough β

typically $\beta < 0.06$

Kinetic effects



Landau damping

According to Landau, wave amplitude decays as

$$\phi = \phi_0(x - \frac{\omega_p}{k} t) e^{-\gamma t}$$

$$\gamma = -\frac{\pi}{2} \frac{\omega_p^3}{k^2} \left. \frac{\partial \hat{f}}{\partial k_x} \right|_{k_x = \frac{\omega_p}{k}}$$

Fourier transform of f in (x, t)

Landau damping if $\left. \frac{\partial \hat{f}}{\partial k_x} \right|_{\omega_p/k} < 0$

WAVE ENERGY
TO
PARTICLES

Instability if $\left. \frac{\partial \hat{f}}{\partial k_x} \right|_{\omega_p/k} > 0$

PARTICLES KINETIC
ENERGY TO WAVE

Two-stream instability

