

30/09

• Comentemos sobre notación de derivados covariantes

$$\begin{aligned}\nabla_{\bar{v}} \bar{v} &= v^\alpha \nabla_{\partial_\alpha} (v^\sigma \partial_\nu) \\ &= v^\alpha \left[ \underbrace{(\nabla_{\partial_\alpha} v^\sigma)}_{= v^\sigma_{,\alpha}} \partial_\nu + v^\sigma \underbrace{\nabla_{\partial_\alpha} \partial_\nu}_{\Gamma_{\nu\alpha}^\lambda \partial_\lambda} \right] \\ &= v^\alpha [ v^\sigma_{,\alpha} + \Gamma_{\nu\alpha}^\sigma v^\nu ] \partial_\nu \quad : \text{vector}\end{aligned}$$

$v^\sigma_{,\alpha}$  : componente del tensor  $\left( \begin{smallmatrix} 1 & \\ 1 & \end{smallmatrix} \right) \nabla v$

tmb se utiliza la notación  $\nabla_\alpha v^\nu$

$$\nabla v = v^\sigma_{,\alpha} \partial_\nu \otimes \tilde{dx}^\alpha$$

~~Obs~~ No confundir  $\nabla_\alpha v^\nu$  con  $\nabla_{\partial_\alpha} v^\nu = v^\nu_{,\alpha}$

Problema (7)

b)  $S^2$ , base  $= \{\partial_\theta, \partial_\varphi\}$

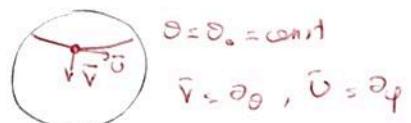
$$\Gamma_{\theta\varphi}^\varphi = \cot \theta, \quad \Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta$$

transporte paralelo  $\nabla_{\bar{v}} \bar{v} = 0$

vector inicial  $\bar{v}_0 = v^\theta \partial_\theta + v^\varphi \partial_\varphi \in \mathcal{D}_\theta \Rightarrow v^\theta = 1, v^\varphi = 0$   
 $\hookrightarrow (\theta = \theta_0, \varphi = 0)$

Componentes

$$(\nabla_{\bar{v}} \bar{v})^\theta = v^\alpha v^\sigma_{,\alpha} = v^\alpha (v^\sigma_{,\alpha} + \Gamma_{\sigma\alpha}^\nu v^\nu) = 0$$



$$1) \quad \text{Dado } \theta : \quad \nabla^\alpha (\nabla^\theta, \alpha + \Gamma_{\sigma\alpha}^\theta \nabla^\sigma) = 0$$

$$2) \quad \text{Dado } \varphi : \quad \nabla^\alpha (\nabla^\varphi, \alpha + \Gamma_{\sigma\alpha}^\varphi \nabla^\sigma) = 0$$

Dado q'  $\vec{U} = \partial \varphi$

$$1) \quad \nabla^\theta, \varphi + \Gamma_{\varphi\varphi}^\theta \nabla^\varphi = 0 \quad 2) \quad \nabla^\varphi, \varphi + \Gamma_{\theta\varphi}^\varphi \nabla^\theta = 0$$

P/ desacoplar el sistema derivamos (1)

$$\frac{\partial (1)}{\partial \varphi} \Rightarrow \nabla^\theta, \varphi \varphi + \underbrace{(-\cot \theta)(-\sin \theta \cos \theta)}_{\cos^2 \theta} \nabla^\theta = 0$$

$$\Rightarrow \quad \nabla^\theta(\varphi = 0) = A \cos(\omega \varphi) + B \sin(\omega \varphi) \quad \omega^2 = \omega^2 \theta_0$$

$$\nabla^\theta(\varphi = 0) = 1 \Rightarrow A = 1$$

nuevamente de (1) podemos obtener  $\nabla^\varphi = -\frac{1}{\Gamma_{\varphi\varphi}^\theta} \nabla^\theta, \varphi$

$$\Rightarrow \quad \nabla^\varphi = -\frac{1}{\sin \theta \cos \theta} \omega (-\sin(\omega \varphi) + B \cos(\omega \varphi))$$

y la condición inicial es  $\nabla^\varphi(\varphi = 0) = 0 \Rightarrow B = 0$

$$\boxed{\nabla^\varphi = -\frac{1}{\sin \theta} \sin(\omega \varphi), \quad \nabla^\theta = \cos(\omega \varphi)}$$

$$T = \frac{2\pi}{|\omega|} = \frac{2\pi}{\omega \theta_0} > 2\pi \quad \text{P/ } \theta_0 \neq \pi/2 \Rightarrow \text{no llegamos a completar una vuelta}$$

$$\text{P/ } \theta_0 = \pi/3$$



Problema 4)

sist coords  $g_{0\mu} = 0 \quad \forall \mu \neq 0, \quad g_{00} = \text{const.}$

queremos ver q' las líneas coordenadas asociadas a  $x^0$  son geodésicas

$$\bar{U} = \partial_{x^0} \quad (x^i = \text{const}, i \neq 0)$$

a)  $\nabla_{\bar{U}} \bar{U} = 0$  resolver como el ejercicio q' recién vimos

b) Coordenadas normales de Gauss

$$ds^2 = -c^2 dt^2 + g_{ik} dx^i dx^k, \quad i, k = 1, 2, 3$$

q' rqq líneas  $x^i = \text{const}$  ( $\bar{U} = \partial_i$ ) son:

i) geodésicas  $\checkmark$  (parte a)

ii) temporales  $\bar{U} \cdot \bar{U} < 0$

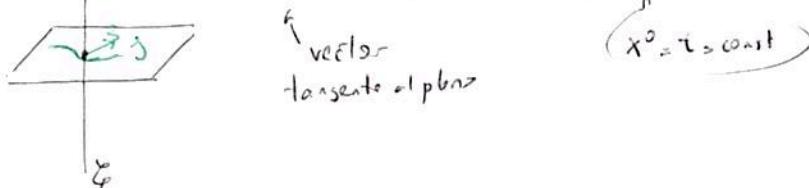
iii) ortogonales a hipersuperficies de  $t = \text{const}$

iv)  $t$  mide el tiempo propio

i)  $\checkmark$

$$ii) \bar{U} \cdot \bar{U} = g_{11}(\bar{U}, \bar{U}) = g_{00} = -c^2 < 1$$

$$iii) \bar{U} = \partial_t \quad \bar{s} = \frac{d}{d\sigma} = \frac{dx^\mu}{d\sigma} \partial_\mu = \frac{dx^i}{d\sigma} \partial_i \Rightarrow \bar{U} \cdot \bar{s} = 0$$



$$iv) \text{tiempo propio} \quad d\tau^2 = -\frac{ds^2}{c^2} \Big|_{\xi} = -\frac{1}{c^2} (d\bar{x} \cdot d\bar{x}) \Big|_{\xi}$$

$$\bar{U} = \frac{dx^\mu(\lambda)}{d\lambda} \partial_\mu = \frac{d\bar{x}}{d\lambda} \Rightarrow d\lambda \bar{U} \approx d\bar{x}$$

pequeños desplazamientos (Nº 1-forma)

$$U^{\circ} = \frac{dt}{d\lambda} = 1 \Rightarrow d\bar{x} \approx dt \bar{U} \Rightarrow d\tau^2 = -\frac{1}{c^2} dt^2 \underbrace{\bar{U} \cdot \bar{U}}_{=1} = \cancel{-dt^2} = d\tau^2$$

### Problema 9

Tensión tensor (1)

$$T(\ ; \bar{U}, \bar{V}) = \underbrace{\nabla_{\bar{U}} \bar{V}}_{\Gamma_{\bar{U}\bar{V}}^{\bar{W}}} - \nabla_{\bar{V}} \bar{U} - [\bar{U}, \bar{V}]$$

$$\text{linealidad: } T(\ ; f\bar{U}, g\bar{V}) = fg T(\ ; \bar{U}, \bar{V})$$

a)  $\{e_\theta, e_\varphi\}$  bon de  $S^2$

componentes  $T^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}}$  en bon ( $T^\mu{}_{\alpha\beta}$  comp en base coord  $\{\partial_\theta, \partial_\varphi\}$ )

$$\text{notación } \bar{e}_\theta = \hat{e}_\theta = e_\theta$$

$$T(\ ; e_{\hat{\mu}}, e_{\hat{\nu}}) = \underbrace{\nabla_{e_{\hat{\mu}}} e_{\hat{\nu}}}_{\Gamma_{\hat{\mu}\hat{\nu}}^{\hat{\lambda}} e_{\hat{\lambda}}} - \nabla_{e_{\hat{\nu}}} e_{\hat{\mu}} - [e_{\hat{\mu}}, e_{\hat{\nu}}]$$

$\hookrightarrow$  problema 7

opción ①: cálculo a fuerza bruta

$$\text{opción ②: } T^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}} = \hat{N}^{\hat{\mu}}{}_{\hat{\nu}} \hat{N}^{\hat{\nu}}{}_{\hat{\rho}} T^\mu{}_{\alpha\beta}$$

$$\text{en base coord } [\partial_\mu, \partial_\nu] = 0$$

~~$$T^\mu{}_{\alpha\beta} = T^\mu{}_\nu$$~~

$$T^\lambda{}_{\mu\nu} = \underbrace{(\Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu})}_{2\Gamma^\lambda{}_{[\mu\nu]}} \quad \text{dado que es una conexión métrica}$$

$$\text{Obs/ii) } [\nabla(\bar{v}, \bar{v})]^{\bar{v}} = \bar{v}^{\mu} v^{\sigma}_{;\mu} - v^{\mu} v^{\sigma}_{;\mu} - [\bar{v}, \bar{v}]^{\bar{v}}$$

$$[\bar{v}, \bar{v}]^{\bar{v}} = \bar{v}^{\mu} v^{\sigma}_{;\mu} - v^{\mu} v^{\sigma}_{;\mu} - [\nabla(\bar{v}, \bar{v})]^{\bar{v}}$$

$$\nabla = \bar{v}^{\mu} v^{\sigma}_{;\mu} - v^{\mu} v^{\sigma}_{;\mu}$$

iii) Si la torsión es nula puedo reemplazar "j" por ";" en  $\mathcal{L} = \mathcal{E} + J$

$$[\bar{v}, \bar{v}]^{\bar{v}} = \bar{v}^{\mu} v^{\sigma}_{;\mu} - v^{\mu} v^{\sigma}_{;\mu} = \cancel{\bar{v}^{\mu} v^{\sigma}_{;\mu} - v^{\mu} v^{\sigma}_{;\mu}} \quad \text{de tensiones}$$

$$= \bar{v}^{\mu} v^{\sigma}_{;\mu} - v^{\mu} v^{\sigma}_{;\mu}$$

### Problema (1)

$$c) v^{\lambda}_{;\rho} - v^{\lambda}_{;\rho} = ? \quad (*)$$

$$(v^{\lambda}_{;\rho})_{;\rho} = (v^{\lambda}_{,\rho} + \Gamma^{\lambda}_{\sigma\rho} v^{\sigma})_{;\rho} = \underbrace{v^{\lambda}_{,\rho\rho}}_{(1)} + \underbrace{\Gamma^{\lambda}_{\sigma\rho} v^{\sigma}_{,\rho}}_{(2)} - \underbrace{\Gamma^{\sigma}_{\sigma\rho} v^{\lambda}_{,\sigma}}_{(3)}$$

$$+ \underbrace{\Gamma^{\lambda}_{\sigma\rho} v^{\sigma}}_{(4)} + \underbrace{\Gamma^{\lambda}_{\sigma\rho} v^{\sigma}_{,\rho}}_{(5)} + \underbrace{\Gamma^{\lambda}_{\delta\rho} \Gamma^{\delta}_{\sigma\rho} v^{\sigma}}_{(6)} - \underbrace{\Gamma^{\delta}_{\sigma\rho} \Gamma^{\lambda}_{\delta\sigma} v^{\sigma}}_{(7)}$$

los términos simétricos en  $\rho\rho$  se van

$$(*) = (\Gamma^{\lambda}_{\sigma\rho,\rho} - \Gamma^{\lambda}_{\sigma\rho,\rho} + \Gamma^{\lambda}_{\delta\rho} \Gamma^{\delta}_{\sigma\rho} - \Gamma^{\lambda}_{\delta\rho} \Gamma^{\delta}_{\sigma\rho}) v^{\sigma}$$

$$+ (-\Gamma^{\sigma}_{\sigma\rho} + \Gamma^{\sigma}_{\rho\rho}) (v^{\lambda}_{,\sigma} + \Gamma^{\lambda}_{\sigma\rho} v^{\sigma})$$

$$= R^{\lambda}_{\sigma\rho} v^{\sigma} + T^{\sigma}_{\rho\rho} v^{\lambda}_{;\sigma}$$

$\downarrow$   
base coord

$$\underbrace{v^{\lambda}_{;\rho} - v^{\lambda}_{;\rho}}_{\nabla_{\rho} v^{\lambda}} = R^{\lambda}_{\sigma\rho} v^{\sigma} - T^{\sigma}_{\rho\rho} v^{\lambda}_{;\sigma}$$

$$= [\nabla_{\rho}, \nabla_{\rho}] v^{\lambda}$$

$$= \nabla_{\rho} (\nabla_{\rho} v^{\lambda}) - \nabla_{\rho} (\nabla_{\rho} v^{\lambda})$$

