

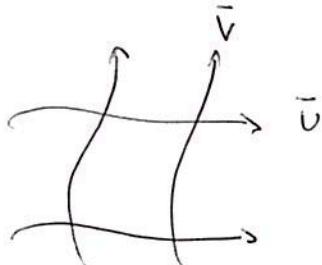
## Definiciones del tensor de Riemann

- Schutz  $R(\vec{c}; \vec{A}, \vec{B}) = [\nabla_{\vec{A}}, \nabla_{\vec{B}}] \vec{c} - \nabla_{[\vec{A}, \vec{B}]} \vec{c}$

- Transporte paralelo en curva infinitesimal

Schutz Gem methods Sec 6.9 p 212

$$\vec{U} = \frac{d}{d\lambda}, \quad \vec{V} = \frac{d}{d\mu}, \quad [\vec{U}, \vec{V}] = 0$$

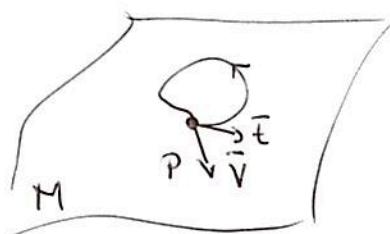


$$\delta \vec{A} = \delta \lambda \delta \mu [\nabla_{\vec{U}}, \nabla_{\vec{V}}] \vec{A}$$

$$\delta A^\alpha = \delta \lambda \delta \mu R^\alpha{}_{\beta \gamma \delta} A^\beta U^\gamma V^\delta$$

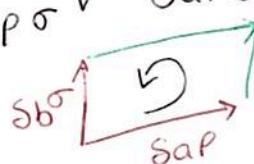
Weinberg Cap 6 Sec 3 p 135

en componentes OJO Convención distinta p/ el Riemann



$$\text{AVP. } \delta V^\mu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma} V^\nu A^\rho A^\sigma$$

$$= R^\mu{}_{\nu\rho\sigma} V^\nu \delta a^\rho \delta b^\sigma$$



- Comutador de derivadas (Emilio Lemos)

Carroll Sec 3.6 p 121

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$$

En componentes

$$R^{\mu}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} + \Gamma^{\kappa}_{\nu\sigma} \Gamma^{\mu}_{\kappa\rho} - \Gamma^{\kappa}_{\nu\rho} \Gamma^{\mu}_{\kappa\sigma}$$

antisym

Otras convenciones : Wald  $R_{\nu\rho\sigma}^{\mu}$   
 Weinberg  $\overset{(w)}{R^{\mu}}_{\nu\rho\sigma} = -\overset{(Schout)}{R^{\mu}}_{\nu\rho\sigma}$

Ver contratajo Misner, Thorne and Wheeler

Tensor de Ricci

$$R_{\mu\nu} \equiv R^{\lambda}_{\mu\lambda\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu}$$

Escalar de Ricci

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}_{\mu}$$

Propiedades:

$$R_{\lambda\mu\nu\rho} = -R_{\mu\lambda\nu\rho} = -R_{\lambda\mu\rho\nu}.$$

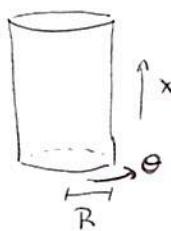
$$R_{\lambda\mu\nu\rho} = R_{\nu\rho\lambda\mu}$$

$$R_{\lambda[\mu\nu\rho]} = 0$$

$$\#_{\text{comp indep}} = \frac{1}{12} n^2(n^2 - 1)$$

Problema (13)

a) cilindro



$$ds^2 = dx^2 + R d\theta^2$$

es el plano identificado  $\Rightarrow R \lambda_{\mu\nu} \rho = 0$

b)  $S^2$        $ds^2 = a^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$   
 $\theta \in [0, \pi], \varphi \in [0, 2\pi]$

$$g_{\mu\nu} = a^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}, \quad g^{\mu\nu} = a^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2 \theta \end{pmatrix}$$

$$\partial_\theta S_{\varphi\varphi} = 2 \sin \theta \cos \theta$$

$$\Gamma^\theta_{\varphi\varphi} = -\sin \theta \cos \theta, \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{\cos \theta}{\sin \theta}$$

$$\begin{aligned}
 R_{\theta\varphi\theta\varphi} &= \frac{1}{2} (g_{\theta\varphi,\varphi\theta} - g_{\theta\theta,\varphi\varphi} + g_{\varphi\varphi,\theta\theta}) \\
 &\quad + g_{\eta\eta} (\Gamma^\eta_{\varphi\theta} \Gamma^\eta_{\theta\varphi} - \Gamma^\eta_{\varphi\varphi} \Gamma^\eta_{\theta\theta}) \\
 &= -\frac{1}{2} \frac{\partial^2}{\partial \theta^2} (a^2 \sin^2 \theta) + g_{\theta\theta} (\Gamma^\theta_{\varphi\theta} \Gamma^\theta_{\theta\varphi} - \Gamma^\theta_{\varphi\varphi} \Gamma^\theta_{\theta\theta}) \\
 &\quad + g_{\varphi\varphi} (\Gamma^\varphi_{\varphi\theta} \Gamma^\varphi_{\theta\varphi} - \Gamma^\varphi_{\varphi\varphi} \Gamma^\varphi_{\theta\theta}) \\
 &= -\frac{a^2}{2} \frac{\partial}{\partial \theta} (2 \sin \theta \cos \theta) + a^2 \sin^2 \theta \frac{\cos^2 \theta}{\sin^2 \theta} \\
 &= -a^2 (\cos^2 \theta - \sin^2 \theta) + a^2 \cos^2 \theta = a^2 \sin^2 \theta = R_{\theta\varphi\theta\varphi}
 \end{aligned}$$

Tensor de Ricci

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = g^{\lambda\sigma} R_{\lambda\mu\nu}$$

$$R_{\theta\theta} = g^{\lambda\sigma} R_{\lambda\theta\sigma\theta} = g^{\theta\theta} R_{\theta\theta\theta\theta} + g^{\varphi\varphi} R_{\varphi\theta\varphi\theta} = \\ = \frac{1}{a^2 \sin^2 \theta} a^2 \sin^2 \theta = 1$$

$$R_{\theta\varphi} = g^{\lambda\sigma} R_{\lambda\theta\sigma\varphi} = 0$$

$$R_{\varphi\varphi} = \cancel{g^{\lambda\sigma} R_{\lambda\varphi\sigma\varphi}} \cancel{+ g^{\varphi\varphi}}$$

$$= g^{\lambda\sigma} R_{\lambda\varphi\sigma\varphi} = g^{\theta\theta} R_{\theta\varphi\theta\varphi} = \frac{1}{a^2} a^2 \sin^2 \theta \cdot \sin^2 \theta$$

$$R_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix} = \frac{1}{a^2} \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2 \theta \end{pmatrix} = \frac{1}{a^2} g_{\mu\nu}$$

$$R = g^{\mu\nu} R_{\mu\nu} = \frac{1}{a^2} g^{\mu\nu} g_{\mu\nu} = \frac{1}{a^2} \delta^\mu_\mu = \frac{2}{a^2} > 0$$

b') espacio hiperbólico del problema 4b

$$ds^2 = dz^2 + l^2 \sinh^2(z/\rho) d\varphi^2 \quad z \in \mathbb{R}_{\geq 0}, \varphi \in [0, 2\pi)$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & l^2 \sinh^2(z/\rho) \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1/l^2 \sinh^2(z/\rho) \end{pmatrix}$$

$$\partial_z g_{\varphi\varphi} = 2l \sinh(z/\rho) \cosh(z/\rho)$$

$$\Gamma_{\varphi\varphi}^z = -l \sinh(z/\rho) \cosh(z/\rho)$$

$$\Gamma_{z\varphi}^\varphi = \frac{1}{l} \frac{\cosh(z/\rho)}{\sinh(z/\rho)}$$

$$R_{\varphi\varphi\varphi\varphi} = -\ell^2 \sinh^2 \varphi/\ell$$

$$R_{\mu\nu} = -\frac{1}{\ell^2} g_{\mu\nu} \Rightarrow R = -\frac{2}{\ell^2} < 0$$

c)  $d\ell^2 = a^2 \left( dx^2 + \sin^2 x (d\theta^2 + \sin^2 \theta d\varphi^2) \right)$

$$x \in [0, \pi), \quad \theta \in [0, 2\pi), \quad \varphi \in [0, 2\pi)$$

$$\Gamma_{\theta\theta}^x = -\sin x \cos x, \quad \Gamma_{\varphi\varphi}^x = -\sin^2 \theta \sin x \cos x$$

$$\Gamma_{x\theta}^\theta = \frac{\cos x}{\sin x}, \quad \Gamma_{\varphi\theta}^\theta = -\sin \theta \cos \theta, \quad \Gamma_{x\varphi}^\theta = \frac{\cos x}{\sin x}, \quad \Gamma_{\varphi\varphi}^\theta$$

$$\Gamma_{\theta\varphi}^\varphi = \frac{\cos \theta}{\sin \theta}$$

$$R_{x\theta x\theta} = a^2 \sin^2 x, \quad R_{x\varphi x\varphi} = a^2 \sin^2 \theta \sin^2 x$$

$$R_{\theta\varphi\theta\varphi} = a^2 \sin^4 x \sin^2 \theta$$

$$R_{\mu\nu} = \frac{2}{a^2} g_{\mu\nu}, \quad R = \frac{6}{a^2}$$

## Tensor de Weyl

$$C_{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho} - \frac{1}{n-2} (g_{\lambda\nu}R_{\mu\rho} + g_{\mu\rho}R_{\lambda\nu} - g_{\mu\nu}R_{\lambda\rho} - g_{\lambda\rho}R_{\mu\nu}) \\ - \frac{R}{(n-1)(n-2)} (g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu})$$

Tiene las mismas simetrías algebraicas q' el tensor de Riemann

$$C_{(\lambda\mu)\nu\rho} = C_{\lambda\mu(\nu\rho)} = 0 \quad + \quad C^{\lambda}_{\mu\nu\rho} = 0 \\ C_{\lambda[\mu\nu\rho]} = 0$$

$$\nabla_{\lambda} C^{\lambda}_{\mu\nu\rho} = 2(n-3) \nabla_{[\nu} S_{\rho]\mu} \quad S_{\mu\nu} = \frac{1}{n-2} \left( R_{\mu\nu} - \frac{R}{2(n-1)} g_{\mu\nu} \right)$$

Clasificación de Petrov → ~~separar/polarizar~~

describe las posibles simetrías algebraicas del tensor de Weyl en un evento de una variedad pseudo-Riemanniana

Autovectores del tensor ~~de Petrov~~ <sup>bivector</sup> tensor de Weyl actúan si & un bivector

$$X^{\mu} \quad \frac{1}{2} C^{\rho\sigma}_{\mu\nu} X^{\mu} = \lambda X^{\mu}$$

$$X^{\mu} = -X^{\nu} \epsilon^{\nu\mu}$$

se pueden escribir como comb lineal de productos wedge de vectores

$$\vec{B} = \vec{e}_1 \wedge \vec{e}_2 + \vec{e}_3 \wedge \vec{e}_4$$

Tiene la propiedad de  $g'$

$$c^\lambda_{\mu\nu\rho} (\tilde{g}_{\alpha\beta} = \Omega(x)^2 g_{\alpha\beta}) = c^\lambda_{\mu\nu\rho} (g_{\alpha\beta})$$

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{\Omega} \left( \delta^\lambda_\mu \nabla_\nu \Omega + \delta^\lambda_\nu \nabla_\mu \Omega - g_{\mu\nu} \nabla^\lambda \Omega \right)$$

$$\tilde{R}^{\lambda\mu}_{\nu\rho} = \frac{1}{\Omega^2} R^{\lambda\mu}_{\nu\rho} + \delta^{[\lambda}_{[\nu} \Omega^{\mu]}_{\rho]}$$

$$\Omega^\mu_{;\rho} \equiv \frac{1}{\Omega} (\Omega^{-1})_{;\rho\eta} g^{\mu\eta} - 2 g^{\eta\lambda} (\Omega^{-1})_{;\eta} (\Omega^{-1})_{;\lambda} \delta^\mu_\rho$$

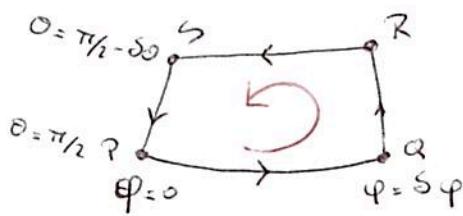
$$(\Omega^{-1})_{;\rho} = (\Omega^{-1})_{,\rho} = \frac{\partial}{\partial x^\rho} \Omega^{-1} = - \frac{1}{\Omega^2} \partial_\rho \Omega$$

$$(\Omega^{-1})_{;\rho\eta} = \left( - \frac{1}{\Omega^2} \partial_\rho \Omega \right)_{;\eta} = + \frac{2}{\Omega^3} \partial_\eta \Omega \partial_\rho \Omega - \frac{1}{\Omega^2} (\partial_\rho \Omega)_{;\eta}$$

$$(\partial_\rho \Omega)_{;\eta} = \partial_\eta \partial_\rho \Omega - \Gamma^\lambda_{\rho\eta} \partial_\lambda \Omega$$

Problema (15)

Transporte paralelo



Conveniencia

$$V^{\mu}(\theta) - V^{\mu}(\text{initial})$$



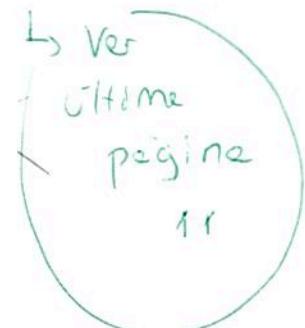
$$\delta V^{\mu} = R^{\mu}_{\nu} \nu^{\nu}_{\delta \varphi} S_{\theta}$$

↳ dirección en la

g' primero me muevo

Importa el sentido

en g' se recorre la curva



$\bullet P \rightarrow Q$  parametrizado con  $\varphi$   $x^{\mu} = (\theta_0 = \pi/2, \varphi)$

$$V_{\text{initial}} = \partial_{\theta}$$

$$\varphi: \theta \rightarrow \delta \varphi$$

$$V^{\theta}_0 = 1, \quad V^{\varphi}_0 = 0$$

initial

expresión g' due Emilio

$$\nabla_{\partial \varphi} \bar{V} = 0$$

$$V''(\varphi) = \cos(\omega \varphi) \partial_{\theta} - \frac{1}{\sin \theta_0} \sin(\omega \varphi) \partial_{\varphi}$$

$$\omega = \omega \sin \theta_0 = \omega \sin \pi/2 = 0, \quad \sin \theta_0 = 1$$

$$\Rightarrow V''(\varphi) = \partial_{\theta} \quad \forall \varphi \Rightarrow \boxed{V''_{P \rightarrow Q} = \partial_{\theta}}$$

Recordar  $\nabla_{\partial \varphi} \partial_{\theta} = T^{\varphi}_{\theta \varphi} = 0 \quad \theta / \varphi = \pi/2$   
(Problema 7)

②  $Q \rightarrow R$

parámetros  $\theta$        $x^\mu = (\theta, \phi_0 = \delta\varphi)$   
 $\nabla_{\partial_\theta} \bar{V} = 0$        $\theta : \pi/2 \rightarrow \pi/2 - \delta\theta$

$$\Rightarrow \frac{\partial V^\mu}{\partial \theta} + \Gamma_{\lambda\theta}^\mu V^\lambda = 0$$

$$\mu = \theta \quad \frac{\partial V^\theta}{\partial \theta} + \Gamma_{\lambda\theta}^\theta V^\lambda = 0 \quad \Rightarrow \frac{\partial V^\theta}{\partial \theta} = 0 \quad \Rightarrow V^\theta = \text{const}$$

$$\mu = \varphi \quad \frac{\partial V^\varphi}{\partial \theta} + \Gamma_{\lambda\theta}^\varphi V^\lambda = 0 \quad \text{initial}$$

$$\Rightarrow \frac{\partial V^\varphi}{\partial \theta} + \frac{\cos \theta}{\sin \theta} V^\varphi = 0 \quad \Rightarrow V^\varphi(\theta) = V_0^\varphi \frac{\sin \theta_0}{\sin \theta} \delta\varphi$$

$$V''(\theta) = V_0^\theta \partial_\theta + V_0^\varphi \frac{\sin \theta_0}{\sin \theta} \partial_\varphi$$

$$V''_0 = \partial_\theta \quad \Rightarrow V_0^\theta = 1, \quad V_0^\varphi = 0$$

$$\Rightarrow \boxed{V''_{Q \rightarrow R} = \partial_\theta} \quad \text{Recuerde Problema 7} \quad \nabla_{\partial_\theta} \partial_\theta = 0$$

③  $R \rightarrow S$

$\theta_0 = \pi/2 - \delta\phi$

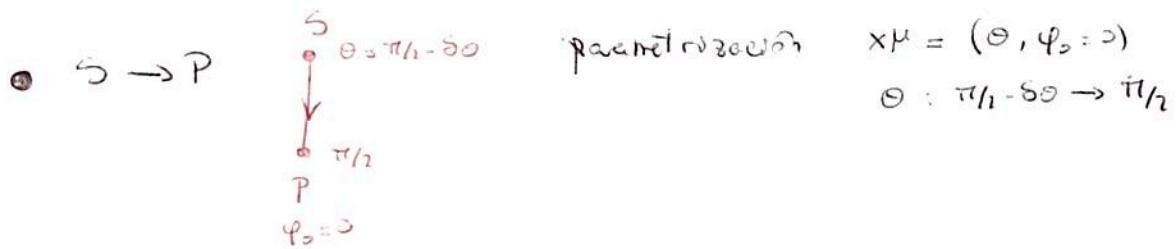
Ahora las condiciones iniciales están a  $\phi_0 = \delta\varphi$

$$V_{\text{initial}} = \partial_\theta \quad \text{hacer de nuevo la cuenta con } \nabla_{\partial_\phi} \bar{V} = 0$$

es como recorrer al revés el primer camino a  $\theta_0 = \pi/2 - \delta\phi$

$$\boxed{V''_{R \rightarrow S} = \cancel{\cos(\delta\phi \sin \theta_0)} \partial_\theta + \frac{\sin(\delta\phi \sin \theta_0)}{\cos \theta_0} \partial_\varphi}$$

$$\nabla_{\partial_\varphi} \partial_\theta \neq 0 \quad \text{p/ } \theta \neq \pi/2$$



Même expression qu'en  $Q \rightarrow R$

$$V''(\theta) = V_{\theta}^{\theta} \partial_{\theta} + V_{\theta}^{\varphi} \frac{\sin \theta}{\cos \theta} \partial_{\varphi}$$

where  $V_{\theta}^{\theta} = \cos(\delta\varphi \sin \delta\theta)$  et  $V_{\theta}^{\varphi} = \frac{\sin(\delta\varphi \sin \delta\theta)}{\cos \delta\theta}$

$$\theta_0 = \pi/2 - \delta\theta \Rightarrow \sin \theta_0 = \cos \delta\theta$$

$$V''(\theta) = \underbrace{\cos(\delta\varphi \sin \delta\theta)}_{V_{\theta}^{\theta}} \partial_{\theta} + \underbrace{\frac{\sin(\delta\varphi \sin \delta\theta)}{\cos \delta\theta}}_{V_{\theta}^{\varphi}} \frac{\cos \delta\theta}{\sin \theta} \partial_{\varphi}$$

$$V''_{S \rightarrow P} = \cos(\delta\varphi \sin \delta\theta) \partial_{\theta} + \sin(\delta\varphi \sin \delta\theta) \partial_{\varphi}$$

$$\Rightarrow V''_{\theta} = \cos(\delta\varphi \sin \delta\theta) \partial_{\theta} + \sin(\delta\varphi \sin \delta\theta) \partial_{\varphi}$$

Ahora aproximemos al orden más bajo  $\theta, \delta\theta, \delta\varphi \ll \epsilon$

expresión del Riemann es orden 2

$$\sin \delta\theta \approx \delta\theta + O(\delta\theta^3)$$

$$\cos(\delta\varphi \sin \delta\theta) \approx \cos(\delta\varphi \delta\theta) \approx 1 + O(\delta\theta^2 \delta\varphi^2)$$

possible término  $\frac{\partial^2 f}{\partial \theta \partial \varphi} \Big|_{\delta\theta = \delta\varphi = 0} = 0$   
orden  $\delta\theta \delta\varphi$

$$\sin(\delta\varphi \sin \delta\theta) \approx \sin(\delta\varphi \delta\theta) \approx \delta\theta \delta\varphi \quad \frac{\partial^2 f}{\partial \theta \partial \varphi} \Big|_{\delta\theta = \delta\varphi} \neq 0$$

$$\boxed{V''_\theta \approx \partial_\theta + \delta\theta \delta\varphi \partial_\varphi}$$

Viere printerd)

$$\partial V^\mu = R^\mu_{\nu\varphi\theta} V^\nu \delta\varphi \delta\theta$$

$$V^\nu_0 = \delta^\nu_\theta \Rightarrow \delta V^\theta = R^\theta_{\nu\varphi\theta} V^\nu \delta\varphi \delta\theta \\ = R^\theta_{\theta\varphi\theta} \delta\varphi \delta\theta$$

$$R^\theta_{\theta\varphi\theta} = g^{\theta\lambda} R_{\lambda\theta\varphi\theta} = g^{\theta\theta} \underbrace{R_{\theta\theta\varphi\theta}}_{=0} \Rightarrow \boxed{\delta V^\theta = 0}$$

$$\delta V^\varphi = R^\varphi_{\nu\varphi\theta} V^\nu \delta\varphi \delta\theta = R^\varphi_{\theta\varphi\theta} \delta\varphi \delta\theta$$

$$R^\varphi_{\theta\varphi\theta} = g^{\varphi\lambda} R_{\lambda\theta\varphi\theta} = g^{\varphi\varphi} R_{\varphi\theta\varphi\theta} = R_{\varphi\theta\varphi\theta} \cancel{R_{\varphi\theta\varphi\theta}} \frac{1}{\sin^2\theta} \\ = -\frac{1}{\sin^2\theta} R_{\theta\varphi\varphi\theta} = +\frac{R_{\theta\varphi\theta\varphi}}{\sin^2\theta} = 1$$

$$\Rightarrow \boxed{\delta V^\varphi = \delta\theta \delta\varphi}$$