0.1 Ejercicio 5

La conexión en las variables y^i, $\bar{\Gamma}^n_{lm}$ está relacionada con la de las variables viejas Γ^k_{ij} de la siguiente manera

$$\Gamma^k_{ij} = \frac{\partial x^k}{\partial y^n} \left(\frac{\partial^2 y^n}{\partial x^i \partial x^j} + \frac{\partial y^l}{\partial x^i} \frac{\partial y^m}{\partial x^j} \bar{\Gamma}^n_{lm} \right). \tag{1}$$

Teniendo en cuenta que

$$\frac{\partial y^l}{\partial x^i} = \delta^l_i + \Gamma^l_{ij}(x_p) \left(x^j - x^j_p\right) \tag{2}$$

y que

$$\frac{\partial^2 y^l}{\partial x^i \partial x^j} = \Gamma^l_{ij}(x_p) \tag{3}$$

se tiene entonces

$$\Gamma^k_{ij}(x_p) = \frac{\partial x^k}{\partial y^n}(x_p) \left(\Gamma^n_{ij}(x_p) + \delta^l_i \delta^m_j \bar{\Gamma}^n_{lm}(x_p)\right) = \frac{\partial x^k}{\partial y^n}(x_p) \left(\Gamma^n_{ij}(x_p) + \bar{\Gamma}^n_{ij}(x_p)\right). \tag{4}$$

Además, como

$$\frac{\partial y^l}{\partial x^i}(x_p) = \delta^l_i, \tag{5}$$

entonces

$$\frac{\partial x^i}{\partial y^l}(x_p) = \delta^i_l. \tag{6}$$

Por lo tanto que finalmente

$$\Gamma^k_{ij}(x_p) = \delta^k_n \left(\Gamma^n_{ij}(x_p) + \bar{\Gamma}^n_{ij}(x_p)\right) = \Gamma^k_{ij}(x_p) + \bar{\Gamma}^k_{ij}(x_p), \tag{7}$$

con lo cual se deduce que

$$\bar{\Gamma}^k_{ij}(x_p) = 0. \tag{8}$$

0.2 Ejercicio 6

Si una conexión es métrica, entonces

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(g_{li,j} + g_{lj,i} - g_{ij,l}\right). \tag{9}$$

En el ítem a se tiene que

$$\begin{aligned} g_{ij;k} &= g_{ij,k} - \Gamma^l_{ik} g_{lj} - \Gamma^l_{jk} g_{il} = g_{ij,k} - \tfrac{1}{2} g^{lm} \left(g_{mi,k} + g_{mk,i} - g_{ik,m}\right) g_{lj} - \tfrac{1}{2} g^{lm} \left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right) g_{il} \\ &= g_{ij,k} - \tfrac{1}{2} \delta^m_j \left(g_{mi,k} + g_{mk,i} - g_{ik,m}\right) - \tfrac{1}{2} \delta^m_i \left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right) \\ &= g_{ij,k} - \tfrac{1}{2} \left(g_{ji,k} + g_{jk,i} - g_{ik,j}\right) - \tfrac{1}{2} \left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right) = 0. \end{aligned} \tag{10}$$

En cuanto al b,

$$\begin{aligned} \nabla_{\overline{W}} \left(\overline{U} \cdot \overline{V}\right) &= W^k \left(\overline{U} \cdot \overline{V}\right)_{,k} = W^k \left(\overline{U} \cdot \overline{V}\right)_{;k} = W^k \left(g_{ij} U^i V^j\right)_{;k} \\ &= W^k \left(g_{ij;k} U^i V^j + g_{ij} U^i_{;k} V^j + g_{ij} U^i V^j_{;k}\right). \end{aligned} \tag{11}$$

Como

$$W^k U^i_{;k} = W^k V^i_{;k} = 0, \tag{12}$$

y por lo probado en el ítem anterior

$$g_{ij;k} = 0 \tag{13}$$

se tiene entonces que

$$\nabla_{\overline{W}} \left(\overline{U} \cdot \overline{V}\right) = 0. \tag{14}$$

En cuanto al c,

$$\Gamma^k_{kj} = \frac{1}{2} g^{kl} \left(g_{lk,j} + g_{lj,k} - g_{kj,l}\right) = \frac{1}{2} g^{kl} g_{lk,j} = \frac{1}{2} \left(\ln |g|\right)_{,j} = \left(\ln \sqrt{|g|}\right)_{,j}, \tag{15}$$

ya que g^{kl} es simétrico y $g_{lj,k} - g_{kj,l}$ en antisimétrico en k, l.

En el ítem d, usando lo probado en el ítem anterior se llega a

$$\begin{aligned} V^i_{;i} &= V^i_{,i} + \Gamma^i_{ki} V^k = V^i_{,i} + \Gamma^i_{ik} V^k = V^i_{,i} + \left(\ln \sqrt{|g|}\right)_{,k} V^k = V^i_{,i} + \tfrac{1}{\sqrt{|g|}} \sqrt{|g|}_{,k} V^k \\ &= \tfrac{1}{\sqrt{|g|}} \left(\sqrt{|g|} V^i_{,i} + \sqrt{|g|}_{,i} V^i\right) = \tfrac{1}{\sqrt{|g|}} \left(\sqrt{|g|} V^i\right)_{,i}. \end{aligned} \tag{16}$$

En el ítem e, sabiendo que Γ^k_{ij} es simétrico en i, j mientras que F^{ij} es antisimétrico se tiene que

$$\begin{aligned} F^{ij}{}_{;i} = F^{ij}{}_{,i} + \Gamma^i_{li}\ F^{lj} + \Gamma^j_{li}\ F^{il} = F^{ij}{}_{,i} + \Gamma^i_{il}\ F^{lj} = F^{ij}{}_{,i} + \left(\ln\sqrt{|g|}\right)_{,l}\ F^{lj} = F^{ij}{}_{,i} + \tfrac{1}{\sqrt{|g|}}\ \sqrt{|g|}_{,l}\ F^{lj} \\ = \tfrac{1}{\sqrt{|g|}}\ \left(\sqrt{|g|}\ F^{ij}{}_{,i} + \sqrt{|g|}_{,i}\ F^{ij}\right) = \tfrac{1}{\sqrt{|g|}}\ \left(\sqrt{|g|}\ F^{ij}\right)_{,i}. \end{aligned} \tag{17}$$

En cuanto al f,

$$\begin{aligned} g^{ij}\ f_{;ij} = g^{ij}\ (f_{;i})_{;j} = g^{ij}\ (f_{,i})_{;j} = g^{ij}\ \left(f_{,ij} - \Gamma^k_{ij}\ f_{,k}\right) = g^{ij}\ f_{,ij} - \tfrac{1}{2}\ g^{kl}\ g^{ij}\ (g_{li,j} + g_{lj,i} - g_{ij,l})\ f_{,k} \\ = g^{ij}\ f_{,ij} - \tfrac{1}{2}\ g^{kl}\ \left[g^{ij}\ (g_{li,j} + g_{lj,i}) - g^{ij}\ g_{ij,l}\right]\ f_{,k} = g^{ij}\ f_{,ij} - \tfrac{1}{2}\ g^{kl}\ \left[2\ g^{ij}\ g_{li,j} - \ln|g|_{,l}\right]\ f_{,k} \\ = g^{ij}\ f_{,ij} - g^{kl}\ \left[g^{ij}\ g_{li,j} - \ln\sqrt{|g|}_{,l}\right]\ f_{,k} = g^{ij}\ f_{,ij} + \left[g^{kl}\ \ln\sqrt{|g|}_{,l} - g^{kl}\ g^{ij}\ g_{li,j}\right]\ f_{,k}. \end{aligned} \tag{18}$$

Teniendo en cuenta que

$$g^{kl}\ g^{ij}\ g_{li,j} = g^{kl}\ \left[\left(g^{ij}\ g_{li}\right)_{,j} - g^{ij}{}_{,j}\ g_{li}\right] = g^{kl}\ \left[\left(\delta^j_l\right)_{,j} - g^{ij}{}_{,j}\ g_{li}\right] = -g^{kl}\ g^{ij}{}_{,j}\ g_{li} = -\delta^k_i\ g^{ij}{}_{,j} = -g^{kj}{}_{,j} \tag{19}$$

se tiene entonces que

$$\begin{aligned} g^{ij}\ f_{;ij} = g^{ij}\ f_{,ij} + \left(g^{kj}{}_{,j} + g^{kl}\ \ln\sqrt{|g|}_{,l}\right)\ f_{,k} = g^{ij}\ f_{,ij} + \left(g^{ij}{}_{,j} + g^{ij}\ \ln\sqrt{|g|}_{,j}\right)\ f_{,i} \\ = g^{ij}\ f_{,ij} + \tfrac{1}{\sqrt{|g|}}\ \left(\sqrt{|g|}\ g^{ij}{}_{,j} + g^{ij}\ \sqrt{|g|}_{,j}\right)\ f_{,i} = g^{ij}\ f_{,ij} + \tfrac{1}{\sqrt{|g|}}\ \left(\sqrt{|g|}\ g^{ij}\right)_{,j}\ f_{,i} \\ = \tfrac{1}{\sqrt{|g|}}\ \left[\sqrt{|g|}\ g^{ij}\ (f_{,i})_{,j} + \left(\sqrt{|g|}\ g^{ij}\right)_{,j}\ f_{,i}\right] = \tfrac{1}{\sqrt{|g|}}\ \left(\sqrt{|g|}\ g^{ij}\ f_{,i}\right)_{,j}. \end{aligned} \tag{20}$$

Finalmente, en el ítem g se tiene que

$$\begin{aligned} \mathrm{div}_{\tilde{\Omega}}\overline{V} = \tfrac{1}{\sqrt{|g|}}\ \left(\sqrt{|g|}\ V^i\right)_{,i} = \tfrac{1}{\sqrt{|g|}}\ \left(\sqrt{|g|}_{,i}\ V^i + \sqrt{|g|}\ V^i{}_{,i}\right) = \frac{\sqrt{|g|}_{,i}}{\sqrt{|g|}}\ V^i + V^i{}_{,i} \\ = \left(\ln\sqrt{|g|}\right)_{,i}\ V^i + V^i{}_{,i} = \Gamma^k_{ki}\ V^i + V^i{}_{,i} = V^i{}_{,i} + \Gamma^k_{ik}\ V^i = V^i_{;i}. \end{aligned} \tag{21}$$