31/08 **Relatividad General: Práctica**

Clase 1: Guía 1 Relatividad Especial

- Convenciones de la práctica:
  - Indices griegos indican coordenadas espacio-temporales

$$\mu, \nu, \rho, \sigma = \underset{\text{tiempo}}{0}, \underbrace{1, 2, 3}_{\text{espacio}}$$

ejemplo: $x^{\mu=0} = x^0 = ct$, $x^2 = y$

  - Indices latinos indican coordenadas espaciales

$$i, j, k = 1, 2, 3$$

ejemplo: $x^{j=3} = x^3 = z$

  - signatura de la métrica: mostly plus

Minkowski $\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}$ temporal

  - unidades $\boxed{c = 1}$ $\Rightarrow$ tiempo y longitudes tienen las mismas unidades

Cuadrivectores

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}$$

V : cuadrivector $\qquad V \cdot V \equiv \underbrace{V_\mu V^\mu}_{} = \eta_{\mu\nu} V^\mu V^\nu$ norma de V

→ convención suma de Einstein $V_\mu V^\mu = \sum_{\mu=0}^{3} V_\mu V^\mu$

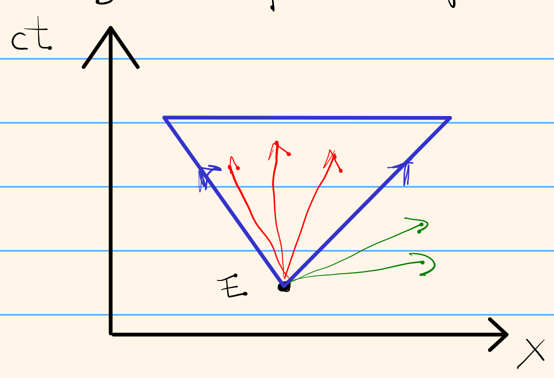
$\eta_{\mu\nu}$ provee un producto escalar para vectores q' NO es definido positivo

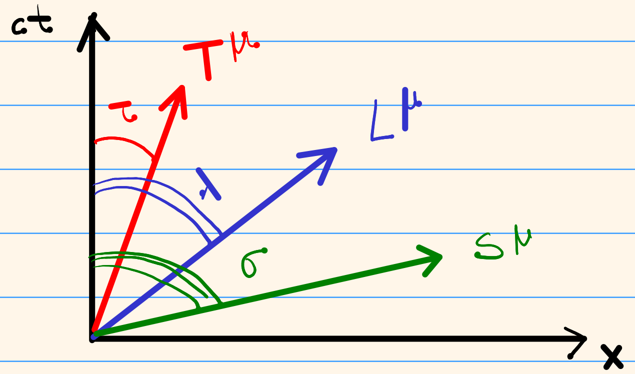
$V \cdot V > 0 \qquad V$ es tipo espacio (spacelike)

$V \cdot V = 0 \qquad V$ es tipo luz (lightlike or null)

$V \cdot V < 0 \qquad V$ es tipo tiempo (time like)

- Diagrama espacio-temporal



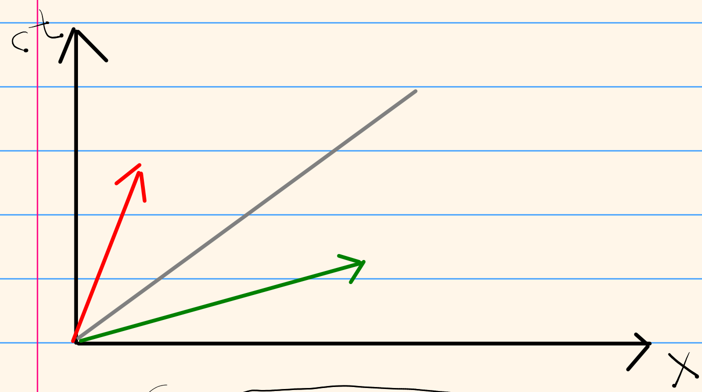


$\tau < \pi/4 \qquad T_\mu T^\mu < 0$

$\lambda = \pi/4 \qquad L_\mu L^\mu = 0$

$\sigma > \pi/4 \qquad S_\mu S^\mu > 0$

$$\tan \alpha = \frac{\text{" comp } x\text{"}}{\text{" comp } t\text{"}}$$



2D

$$\|T\|^2 = -(T^0)^2 + (T^x)^2 < 0$$

$$T^0 = |T| \, ch \, \Theta_T$$
$$T^x = |T| \, sh \, \Theta_T$$

$$\|S\|^2 = -(S^0)^2 + (S^x)^2 > 0$$

$$S^0 = |S| \, sh \, \Theta_S$$
$$S^x = |S| \, ch \, \Theta_S$$

(ver Ferraro sec 7.3)

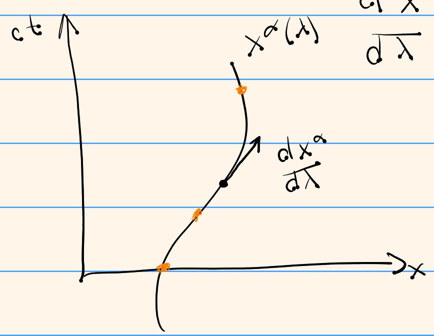
Resumen:

	signatura \ tipo de vector	tiempo	nulo	espacios
práctica	mostly plus	$T \cdot T < 0$	$L \cdot L = 0$	$S \cdot S > 0$
teórica	mostly minus	$T \cdot T > 0$	$L \cdot L = 0$	$S \cdot S < 0$

Problema ①

trayectoria de un detector $x^a(\lambda)$ — λ: parametro

vector tangente a la trayectoria



$$\frac{dx^\alpha(\lambda)}{d\lambda} = \underbrace{\gamma(v)}_{\gamma(v)\equiv(1-v^2/c^2)^{-1/2}}\left(\overset{t}{c\,\mathrm{ch}\left(\frac{a\lambda}{c}\right)}, \overset{x}{c\,\mathrm{sh}(a\lambda/c)}, \overset{y}{0}, \overset{z}{v}\right)$$

(a partir de ahora tomo $c = 1$)!

a) ver q! λ es el tiempo propio

tiempo propio $d\tau^2 \equiv -\underbrace{ds^2}_{\text{intervalo}} = -\eta_{\mu\nu}dx^\mu dx^\nu$

τ: tiempo propio parametriza curvas temporales.

escribimos el intervalo

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda^2 = \left\{\overset{\eta_{00}}{-}\left(\frac{dx^0}{d\lambda}\right)^2 \overset{\eta_{ii}=1}{+}\left(\frac{dx^1}{d\lambda}\right)^2+\left(\frac{dx^2}{d\lambda}\right)^2+\left(\frac{dx^3}{d\lambda}\right)^2\right\}d\lambda^2$$

$$= \gamma^2(v)\left(-\mathrm{ch}^2(a\lambda)+\mathrm{sh}^2(a\lambda)+v^2\right)d\lambda^2 = \gamma^2(v)\underbrace{(-1+v^2)}_{=-\gamma^{-2}(v)}d\lambda^2$$

$$ds^2 = -d\lambda^2 = -d\tau^2 \Rightarrow \boxed{\lambda = \tau}\,!$$

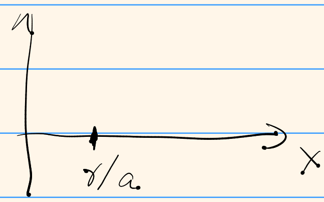
b) hallar $x^a(\tau)$ $\longrightarrow$ simplemente integramos ($v =$ const)

$$x^\alpha(\tau) = \int \frac{dx^a}{d\tau}\, d\tau$$

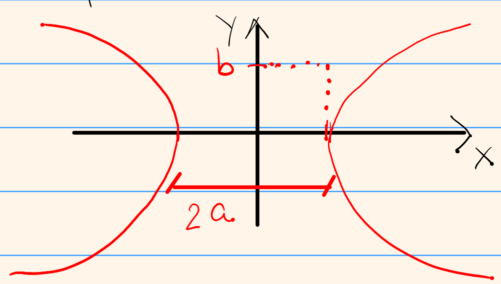
$$\begin{cases} x^0(\tau) = \dfrac{\gamma(v)}{a}\, sh(a\tau) + \cancel{const} \\ x^1(\tau) = \gamma/a\; ch(a\tau) + \cancel{const} \end{cases}$$

$$x^2(\tau) = \cancel{const}$$

$$x^3(\tau) = v\tau + \cancel{const}$$



Hipérbola

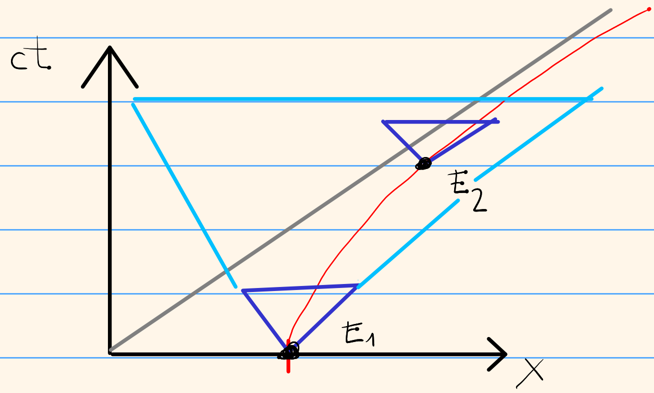


$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

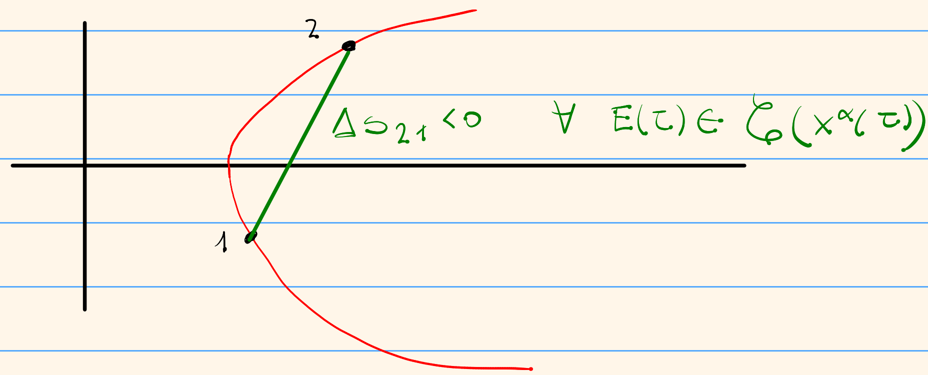
$$\underbrace{-(x^0)^2 + (x^1)^2} = -\frac{\gamma^2}{a^2} sh^2(a\tau) + \frac{\gamma^2}{a^2} ch^2(a\tau) = \frac{\gamma^2}{a^2} > 0$$

$$\boxed{-(x^0(\tau))^2 + (x^1(\tau))^2 = \gamma^2/a^2 > 0}$$ hipérbola en (t, x)

c) $x^a(\tau)$ describe una curva temporal?



Para corroborar q.' $x^\alpha(\tau)$ describe una curva tipo tiempo la separación entre dos eventos cualesquiera pertenecientes a trayectoria tiene q.' ser negativo



la separación entre eventos es temporal

$$\Delta s_{21}^2 = \| x^\mu(\tau_2) - x^\mu(\tau_1) \|^2_{Mink} = \eta_{\mu\nu} \left(x^\mu(\tau_2) - x^\mu(\tau_1) \right) \left(x^\nu(\tau_2) - x^\mu(\tau_1) \right) < 0$$

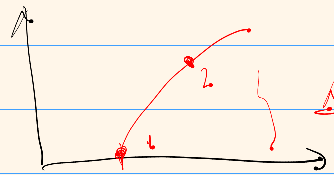
$$= -\left(x^0(\tau_2) - x^0(\tau_1) \right)^2 + \left(x^1(\tau_2) - x^1(\tau_1) \right)^2 + \left(x^3(\tau_2) - x^3(\tau_1) \right)^2$$

$$= -\gamma^2(v) \left(sh(a\tau_2) - sh(a\tau_1) \right)^2 + \gamma^2(v) \left(ch(a\tau_2) - ch(a\tau_1) \right)^2 + \gamma^2(v) v^2 \overbrace{(\tau_2 - \tau_1)}^{\Delta\tau}{}^2$$

$$= \gamma^2(v) \left\{ \begin{array}{l} -sh^2(a\tau_2) - sh^2(a\tau_1) + 2\, sh(a\tau_2) sh(a\tau_1) \\ + ch^2(a\tau_2) + ch^2(a\tau_1) - 2 ch(a\tau_2) ch(a\tau_1) \end{array} + v^2 \Delta\tau^2 \right\}$$

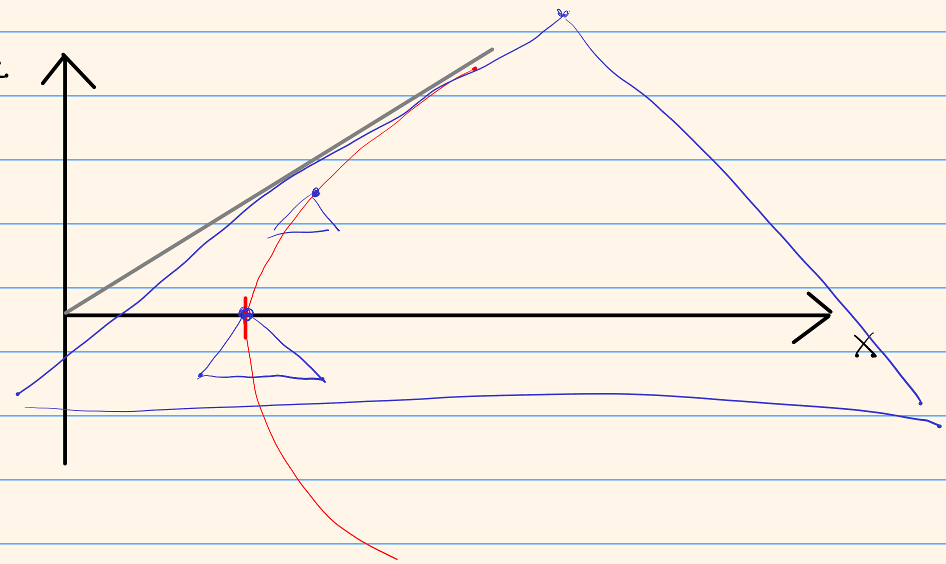
$$\Delta s_{21}^2 = \gamma^2(v) \left\{ \underbrace{2 - 2\, ch(a \Delta\tau)}_{<0} \right\} + \gamma^2(v) v^2 \Delta\tau^2 < 0$$

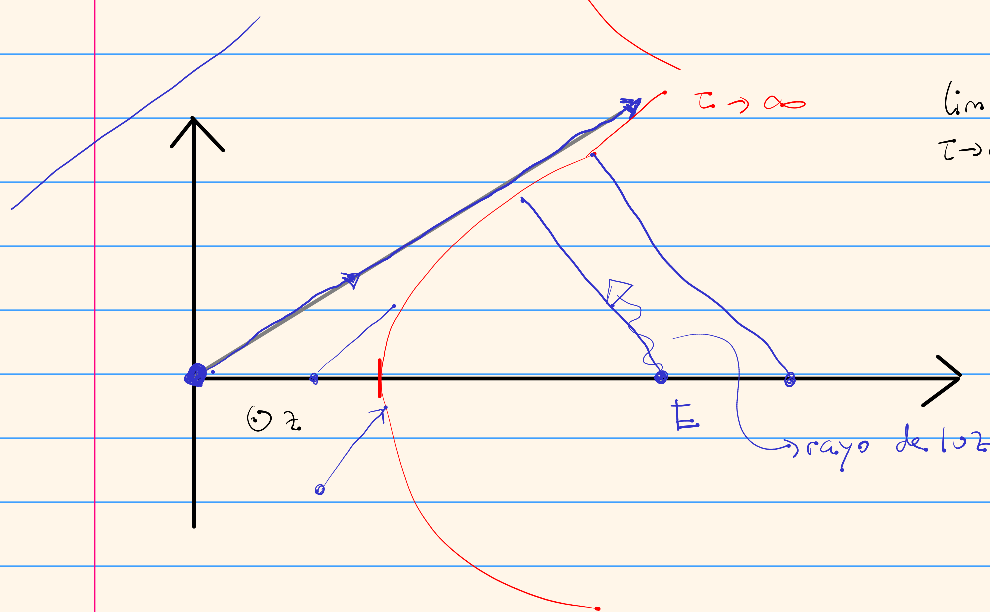
$$ch(\alpha x) \cong 1 + \frac{\alpha^2}{2} x^2$$



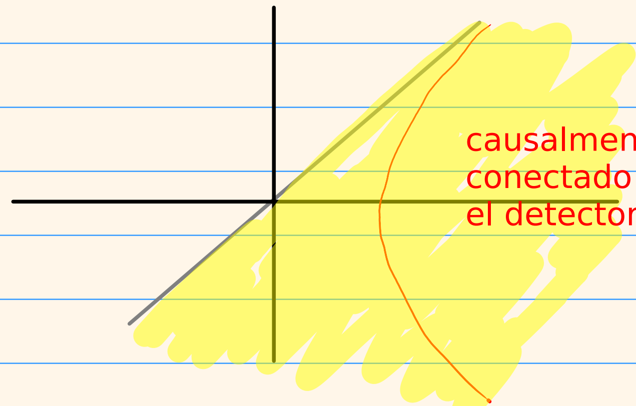
$\Delta s_{21}^2 < 0$ $\forall$ evento de la trayectoria del detector

d) Región del espacio tiempo q. puede influir s/ el detector





$$\lim_{\tau\to\infty} \frac{x^0(\tau)}{x^1(\tau)} = 1 \quad (\text{recta a } 45°)$$



e) Temperatura $T(t, x, y, z) = \dfrac{G' t}{x^2 + y^2}$ calcular $\dfrac{dT}{d\tau}$

$x^2 \to (x^1(\tau))^2$, $y^2 \to (x^2(\tau))^2$

4-velocidad $U^\mu \equiv \dfrac{dx^\mu}{d\tau}$

$$\frac{dT}{d\tau} = \frac{dx^\mu}{d\tau}\frac{\partial T}{\partial x^\mu}(t,x,y,z) = U^\mu \partial_\mu T$$

$\partial_\mu = \dfrac{\partial}{\partial x^\mu}$

$$= U^0 \partial_t T + U^x \partial_x T + U^y \partial_y T + U^z \partial_z T$$

$\dfrac{dx^\alpha}{d\tau}$

$\dfrac{\partial}{\partial x^0} = \dfrac{\partial}{\partial t}$

Comentarios s/ problemas 2, 3 y 4

Notación: transf de Lorentz $\underbrace{\overset{x^{0'}}{ct'} = \gamma(v)(\overset{x^0}{ct} - \beta x)}$, $\underbrace{x' = \gamma(v)(x - \beta ct)}$

se suele denotar $x^{0'} = \gamma(v)(x^0 - \beta x^1)$, $x^{1'} = \gamma(v)(x^1 - \beta x^0)$

$\Rightarrow \quad \overset{S'}{x^{\mu'}} = \Lambda^{\mu'}{}_{\nu} \overset{S}{x^{\nu}} \qquad x^{\mu'} \equiv (x')^{\mu'}$

Coords en S x^μ, Coords en S' $x^{\mu'}$

Ejemplo: $T_{\mu\nu} \Rightarrow T_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} T_{\mu\nu}$

$$U^\mu = \frac{dx^\mu}{d\tau} \qquad u^\mu = \frac{dx^\mu}{dt} = (u^0, \underset{\text{3-vector}}{\vec{u}})$$

$$a^\mu = \frac{dU^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} \qquad a^\mu = \frac{du^\mu}{dt} = \frac{d^2x^\mu}{dt^2}$$

α^μ

4-vectores

$$U^\mu = (U^0, \overset{U^x, U^y, U^z}{U^i}) \longrightarrow \vec{u}$$

$$U^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \overset{\gamma}{\frac{dt}{d\tau}} u^\mu = \gamma(u) u^\mu$$