Relatividad General : Segunda Entrega de Problemas

Problema (1)

Modelo de Poincaré para el plano hiperbólico:

$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, semi-plano superior de $\mathbb{R}^2$

Métrica $\quad ds^2 = \dfrac{dx^2 + dy^2}{y^2}$, $g_{ab} = 1/y^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $g^{ab} = y^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

a) Símbolos de Christoffel en la base coordenada $\{\partial_x, \partial_y\}$

Utilizando el principio variacional $\quad I = \frac{1}{2} \int d\lambda \, g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}$

$$I = \frac{1}{2} \int d\lambda \, \frac{1}{y^2} \left\{ \left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 \right\} \equiv \int d\lambda \, L(x, y, \dot{x}, \dot{y})$$

Notación $\frac{dx}{d\lambda} \equiv \dot{x}$, $\frac{dy}{d\lambda} \equiv \dot{y}$

Ecuaciones de Euler-Lagrange

$$0 = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^a} \right) - \frac{\partial L}{\partial x^a}$$

- $a = x \rightsquigarrow \dfrac{\partial L}{\partial x} = 0$, $\quad \dfrac{\partial L}{\partial \dot{x}} = \dfrac{1}{y^2}\left(\dfrac{dx}{d\lambda}\right) \Rightarrow \dfrac{d}{d\lambda}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = 0$

$\dfrac{\partial L}{\partial \dot{x}}$ es una cantidad conservada

Ecuación de la geodésica

$$0 = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{1}{y^2} \ddot{x} - \frac{2}{y^3} \dot{x}\dot{y} \Rightarrow \boxed{\ddot{x} - \frac{2}{y} \dot{x}\dot{y} = 0} \quad (1)$$

comparando con $\quad \ddot{x} + \Gamma^x{}_{ab} \dot{x}^a \dot{x}^b = 0 \quad$ tenemos q'

$$\boxed{\Gamma^x{}_{xy} = -\frac{1}{y}}$$

- $a = y \rightsquigarrow \frac{\partial L}{\partial y} = -\frac{1}{y^3}(\dot{x}^2+\dot{y}^2)$, $\frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{y^2}$, $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\ddot{y}}{y^2} - \frac{2}{y^3}\dot{y}^2$

$$0 = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \frac{\ddot{y}}{y^2} - \frac{2}{y^3}\dot{y}^2 + \frac{1}{y^3}(\dot{x}^2+\dot{y}^2) = \frac{\ddot{y}}{y^2} + \frac{1}{y^3}(\dot{x}^2-\dot{y}^2)$$

$$\Rightarrow \boxed{\ddot{y} + \frac{1}{y}(\dot{x}^2-\dot{y}^2) = 0} \quad (2) \quad \Rightarrow \boxed{\Gamma^y{}_{xx} = \frac{1}{y}, \quad \Gamma^y{}_{yy} = -\frac{1}{y}}$$

- Cálculo por definición

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{am}(\partial_b g_{mc} + \partial_c g_{bm} - \partial_m g_{ab})$$

- $a = x \rightsquigarrow \Gamma^x{}_{bc} = \frac{1}{2} g^{xm}(\partial_b g_{mc} + \partial_c g_{bm} - \partial_m g_{ab})$

g_{ab} y g^{ab} son diagonales $\Rightarrow \Gamma^x{}_{bc} = \frac{1}{2} g^{xx}(\partial_b g_{xc} + \partial_c g_{bx} - \underbrace{\partial_x g_{ab}}_{=0})$

los componentes de g_{ab} solo dependen de $y \Rightarrow b = x$, $c = y$ ó $b = y$ y $c = x$

$$\Gamma^x{}_{xy} = \frac{1}{2} g^{xx} \partial_y g_{xx} = \frac{1}{2} y^2 \partial_y \frac{1}{y^2} = -\frac{1}{y} \quad ✓$$

- $a = y \rightsquigarrow \Gamma^y{}_{bc} = \frac{1}{2} g^{ym}(\partial_b g_{mc} + \partial_c g_{bm} - \partial_m g_{bc}) = \frac{1}{2} g^{yy}(\partial_b g_{yc} + \partial_c g_{by} - \partial_y g_{bc})$

las opciones que no dan cero son $b = c = y$
$b = c = x$

$$\Gamma^y{}_{xx} = \frac{1}{2} g^{yy}(-\partial_y g_{xx}) = -\frac{1}{2} y^2 \partial_y\left(\frac{1}{y^2}\right) = +\frac{1}{y} \quad ✓$$

$$\Gamma^y{}_{yy} = \frac{1}{2} g^{yy} \partial_y g_{yy} = -\frac{1}{y} \quad ✓$$

- Tensor de Riemann

$$R^a{}_{cbd} = \partial_b \Gamma^a_{cd} - \partial_d \Gamma^a_{cb} + \Gamma^a_{bm}\Gamma^m_{cd} - \Gamma^a_{dm}\Gamma^m_{cb}$$

En dos dimensiones solo hay una componente independiente

$$\#\text{ comp indptes Riemann} = \frac{1}{12} n^2(n^2-1) \underset{n=2}{=} 1$$

Por las simetrías del tensor de Riemann la componente no-nula es R_{xyxy}, como la métrica es diagonal calculamos $R^x{}_{yxy}$

$$R^x{}_{yxy} = \partial_x \underbrace{\Gamma^x_{yy}}_{=0} - \partial_y \Gamma^x_{yx} + \underbrace{\Gamma^x_{xm}}_{\neq 0 \text{ p/ } m=y} \Gamma^m_{yy} - \underbrace{\Gamma^x_{ym}}_{\neq 0 \text{ p/ } m=x} \Gamma^m_{yx}$$

$$R^x{}_{yxy} = -\partial_y \Gamma^x_{yx} + \Gamma^x_{xy}\Gamma^y_{yy} - \Gamma^x_{yx}\Gamma^x_{yx}$$

$$= +\partial_y \frac{1}{y} + \frac{1}{y}\cdot\frac{1}{y} - \frac{1}{y}\frac{1}{y} = \boxed{-\frac{1}{y^2} = R^x{}_{yxy}} \Rightarrow \boxed{R_{xyxy} = -\frac{1}{y^4}}$$

El resto de la componentes se relacionan algebráicamente mediante

- $R_{abcd} = -R_{abdc} = -R_{bacd}$
- $R_{abcd} = R_{cdab}$
- $R_{a[bcd]} = 0$

- Tensor de Ricci

$$R_{cd} = g^{mn} R_{mcnd}$$

$$R_{xx} = g^{mn} R_{mxnx} = g^{yy} R_{yxyx} = g^{yy} R_{xyxy} = -\frac{1}{y^2}$$

$$R_{xy} = g^{mn} R_{mxny} = 0 \quad \text{(cualquier combinación tiene tres índices iguales)}$$

$$R_{yy} = g^{mn} R_{myny} = g^{xx} R_{xyxy} = -\frac{1}{y^2}$$

$$\boxed{R_{cd} = -\frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -g_{cd}}$$

- Escalar de Ricci

$$R = g^{cd} R_{cd} = g^{xx} R_{xx} + g^{yy} R_{yy} = \boxed{-2 = R}$$

Observación: los espacios maximalmente simétricos (aquellos q' poseen el mayor número posible de vectores de Killing) tienen las siguientes expresiones p/ sus tensores de Riemann y Ricci en n-dim

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)} \left(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu} \right) \quad , \quad R_{\sigma\nu} = \frac{R}{n} g_{\sigma\nu}$$

El plano hiperbólico es un espacio maximalmente simétrico. En general los espacios hiperbólicos de n-dimensiones son maximalmente simétricos con $R<0$. Las n-esferas tambien lo son con $R>0$ (según la convención de MTW p/ el Riemann). El espacio plano en n-dimensiones tmb lo es con $R=0$. Los análogos de signatura lorentziana son los espacio-tiempos de Anti-de Sitter ($R<0$), de-Sitter ($R>0$) y Minkowski.

de Sitter será estudiado en la parte de Cosmología
Para más info ver Carroll sec 8.1 pag 323.

b) $ds^2 = g_{ab}\, dx^a dx^b = \frac{1}{y^2}(dx^2 + dy^2)$

$x \to x + a$ deja el intervalo invariante

Dilatación $(x,y) \to (e^\lambda x, e^\lambda y)$ con $\lambda = \text{const}$

$dx \to e^\lambda dx, \quad dy \to e^\lambda dy$

$$ds^2 \longrightarrow \frac{1}{e^{2\lambda}} \left(e^{2\lambda} dx^2 + e^{2\lambda} dy^2 \right) = \frac{1}{y^2}(dx^2 + dy^2)$$

La dilatación tambien es una simetría del plano hiperbólico

c) comprobar g'

$$\{\bar{P} = \partial_x, \quad \bar{D} = x\partial_x + y\partial_y, \quad \bar{K} = (x^2 - y^2)\partial_x + 2xy\partial_y\}$$

son vectores de Killing, es decir, $\pounds_\xi g = 0$ p/ $\xi = \{\bar{P}, \bar{D}, \bar{K}\}$

$$(\pounds_\xi g)_{ab} = \xi^m \partial_m g_{ab} + g_{am}\partial_b \xi^m + g_{mb}\partial_a \xi^m$$

- $\bar{P}$: $\quad (\pounds_{\bar{P}} g)_{ab} = \partial_x g_{ab} + g_{ax}\partial_b P^x + g_{xb}\partial_a P^x = 0$ ✓

- $\bar{D}$: $\quad (\bar{D})^x = x, \ (\bar{D})^y = y$, recordar g' g_{ab} sólo depende de y

$$(\pounds_{\bar{D}} g)_{ab} = D^y \partial_y g_{ab} + g_{am}\partial_b D^m + g_{mb}\partial_a D^m$$

$$(\pounds_{\bar{D}} g)_{xx} = y\partial_y g_{yy} + g_{xm}\partial_x D^m + g_{mx}\partial_x D^m$$

$$= y\partial_y(1/y^2) + 2g_{xx}\partial_x x = -2/y^2 + 2/y^2 = 0 \ ✓$$

$$(\pounds_{\bar{D}} g)_{xy} = y\partial_y g_{xy} + g_{xm}\partial_y{}^m + g_{my}\partial_x{}^m = 0 \ ✓ \ (g_{ab} \text{ es diagonal})$$

$$(\pounds_{\bar{D}} g)_{yy} = y\partial_y \partial_{yy} + g_{ym}\partial_y \xi^m + g_{my}\partial_y \xi^m$$

$$= y\partial_y(1/y^2) + 2g_{yy}\partial_y y = -2/y^2 + 2/y^2 = 0 \ ✓$$

- $\bar{K}$: $\quad (\bar{K})^x = x^2 - y^2, \ (\bar{K})^y = 2xy$

$$(\pounds_{\bar{K}} g)_{ab} = K^y \partial_y g_{ab} + g_{am}\partial_b K^m + g_{mb}\partial_a K^m$$

$$(\pounds_{\bar{K}} g)_{xx} = 2xy\partial_y(1/y^2) + g_{xm}\partial_x K^m + g_{mx}\partial_x K^m$$

$$= -\frac{4x}{y^2} + 2\underbrace{g_{xx}\partial_x K^x}_{= 1/y^2 \, \partial_x(x^2 - y^2) = 2x/y^2}$$

$$= -\frac{4x}{y^2} + \frac{4x}{y^2} = 0 \ ✓$$

$$(\mathcal{L}_K g)_{xy} = K^Y \underbrace{\partial_y g_{xy}}_{=0} + g_{xm}\partial_y K^m + g_{my}\partial_x K^m$$

$$= g_{xx}\partial_y K^x + g_{yy}\partial_x K^Y$$

$$= 1/y^2\, \partial_y (x^2-y^2) + 1/y^2\, \partial_x (2xy) = -2/y + 2/y = 0 \checkmark$$

$$(\mathcal{L}_K g)_{yy} = K^Y \partial_y g_{yy} + g_{ym}\partial_y K^m + g_{my}\partial_y K^m$$

$$= 2xy\,\partial_y(1/y^2) + 2 g_{yy} \underbrace{\partial_y K^Y}_{= \partial_y(2xy) = 2x} = -4x/y^2 + 4x/y^2 = 0 \checkmark$$

- Otro modo $(\mathcal{L}_\xi g)_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} = 2\,\xi_{(\mu;\nu)} = 0$

$$\xi_{a;b} = \partial_b \xi_a - \Gamma^m_{ab}\xi_m$$

Notemos q'

$$\xi_{x;x} = \partial_x \xi_x - \Gamma^m_{xx}\xi_m = \partial_x \xi_x - \Gamma^Y_{xx}\xi_y$$

$$\xi_{y;y} = \partial_y \xi_y - \Gamma^m_{yy}\xi_m = \partial_y \xi_y - \Gamma^Y_{yy}\xi_y$$

$$\xi_{x;y} = \partial_y \xi_x - \Gamma^m_{xy}\xi_m = \partial_y \xi_x - \Gamma^x_{xy}\xi_x$$

$$\xi_{y;x} = \partial_x \xi_y - \Gamma^m_{yx}\xi_m = \partial_x \xi_y - \Gamma^x_{yx}\xi_x$$

Sabiendo del item (a) cuales son los Γ^a_{bc} no nulos.

Vamos a necesitar las componentes covariantes de los vectores $\bar{P}$, $\bar{D}$ y $\bar{K}$, recordemos q' $\xi_\mu = g_{\mu\nu}\xi^\nu$

$$P_x = g_{xx} = 1/y^2$$

$$D_x = x/y^2 \quad , \quad D_y = 1/y$$

$$K_x = x^2/y^2 - 1 \, , \quad K_y = 2x/y$$

$\bar{P}$: $P_{x;x} = \partial_x P_x - \Gamma^{y}_{xx} P_y = 0$ ✓

$P_{y;y} = \partial_y P_y - \Gamma^{y}_{yy} P_y = 0$ ✓

$$\left.\begin{aligned} P_{x;y} &= \partial_y P_x - \Gamma^{x}_{xy} P_x = -1/y^3 \\ P_{y;x} &= \partial_x P_y - \Gamma^{x}_{yx} P_x = +1/y^3 \end{aligned}\right\} ✓$$

$\bar{D}$: $D_{x;x} = \partial_x D_x - \Gamma^{y}_{xx} D_y = 1/y^2 - 1/y \; 1/y = 0$ ✓

$D_{y;y} = \partial_y D_y - \Gamma^{y}_{yy} D_y = -1/y^2 - (-1/y) 1/y = 0$ ✓

$$\left.\begin{aligned} D_{x;y} &= \partial_y D_x - \Gamma^{x}_{xy} D_x = -x/y^3 \\ D_{y;x} &= \partial_x D_y - \Gamma^{x}_{yx} D_x = +x/y^3 \end{aligned}\right\} ✓$$

$\bar{K}$: $K_{x;x} = \partial_x K_x - \Gamma^{y}_{xx} K_x = 2x/y^2 - 1/y \; 2x/y = 0$ ✓

$K_{y;y} = \partial_y K_y - \Gamma^{y}_{yy} K_y = -2x/y^2 - (-1/y) 2x/y = 0$ ✓

$$\left.\begin{aligned} K_{x;y} &= \partial_y K_x - \Gamma^{x}_{xy} K_x = -2x^2/y^3 - (-1/y)(x^2/y^2 - 1) = -x^2/y^3 - 1/y \\ K_{y;x} &= \partial_x K_y - \Gamma^{x}_{yx} K_x = 2/y - (-1/y)(x^2/y^2 - 1) = 1/y + x^2/y^3 \end{aligned}\right\} ✓$$

- Relaciones de conmutación

Recordemos q el conmutador de dos campos vectoriales es

$[\bar{U}, \bar{V}](f) = \bar{U}(\bar{V}(f)) - \bar{U}(\bar{V}(f))$

en componentes (base coordenada)

$[\bar{U}, \bar{V}]^\mu = U^\nu \partial_\nu V^\mu - V^\nu \partial_\nu U^\mu$

▸ $[\bar{D}, \bar{P}] = \bar{D}(\bar{P}) - \bar{P}(\bar{D}) = -\partial_x = -\bar{P}$ ✓

$\bar{D}(\bar{P}) = x\partial_x^2 + y\partial_x\partial_y$, $\bar{P}(\bar{D}) = \partial_x + x\partial_x^2 + y\partial_x\partial_y$

▸ $[\bar{P}, \bar{K}] = \bar{P}(\bar{K}) - \bar{K}(\bar{P}) = 2\bar{D}$ ✓

$$\left.\begin{aligned} \bar{P}(\bar{K}) &= 2x\partial_x + (x^2-y^2)\partial_x^2 + 2y\partial_y + 2xy\,\partial_x\partial_y \\ \bar{K}(\bar{P}) &= (x^2-y^2)\partial_x^2 + 2xy\,\partial_y\partial_x \end{aligned}\right\} \rightsquigarrow 2(x\partial_x + y\partial_y)$$

▸ $[\bar{D}, \bar{K}] = \bar{D}(\bar{K}) - \bar{K}(\bar{D}) = (x^2-y^2)\partial_x + 2xy\,\partial_y = \bar{K}$ ✓

$\bar{D}(\bar{K}) = 2x^2\partial_x - 2y^2\partial_x + x(x^2-y^2)\partial_x^2 + 4xy\partial_y + y(x^2-y^2)\partial_y\partial_x + 2xy^2\partial_y^2$

$\bar{K}(\bar{D}) = (x^2-y^2)\partial_x + x(x^2-y^2)\partial_x^2 + 2xy\partial_y + y^2(x^2-y^2)\partial_x\partial_y + 2xy^2\partial_y^2$

• Generadores de isometrías infinitesimales

• Traslaciones $x \to x' = x + a = \phi^x(x,a)$

según la definición (5)

$$\bar{P} = \left.\frac{\partial \phi^x}{\partial a}\right|_{a=0} \partial_x = \partial_x \checkmark$$

• dilataciones: $(x,y) \to (x',y') = (e^\lambda x, e^\lambda y)$

$$x' = e^\lambda x \simeq (1+\lambda)x = \phi^x(x,\lambda) \qquad p/\ |\lambda| \ll 1$$
$$y' = e^\lambda y \simeq (1+\lambda)y = \phi^y(y,\lambda)$$

$$\bar{D} = \left.\frac{\partial \phi^x}{\partial \lambda}\right|_{\lambda=0} \partial_x + \left.\frac{\partial \phi^y}{\partial \lambda}\right|_{\lambda=0} \partial_y = x\partial_x + y\partial_y \checkmark$$

• Las transformaciones conformes especiales generadas por $\bar{K} = (x^2-y^2)\partial_x + 2xy\partial_y$ no son tan fáciles de interpretar.

La versión finita es el resultado componer una inversión ($x^\mu \to x^\mu/\|x\|^2$) una traslación y nuevamente una inversión. Esto excede al curso pero aquellas/os interesadas/os pueden consultar la Wikipedia o el libro Conformal Field Theory de Di Francesco et al.

d) Geodésicas

si ξ es un vector de Killing $\Rightarrow$ $g(\xi, d/d\lambda)$ es constante a lo largo de la geodésica de $\bar{U} = d/d\lambda$.

$$\bar{P} = \partial_x \Rightarrow g(\bar{P}, d/d\lambda) = g(\partial_x, \dot{x}\partial_x + \dot{y}\partial_y) = \boxed{\dot{x}/y^2 \equiv 1/b = \text{const}} \quad (3)$$

$$\bar{D} = x\partial_x + y\partial_y \Rightarrow g(\bar{D}, d/d\lambda) = g(x\partial_x + y\partial_y, \dot{x}\partial_x + \dot{y}\partial_y)$$
$$= \boxed{(x\dot{x} + y\dot{y})/y^2 \equiv 1/c = \text{const}} \quad (4)$$

• Primero notemos q' la curva con $\dot{x}(\lambda) = 0$ resuelve (1)

$$(1): \quad \ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0 \quad \checkmark \ (\text{trivial})$$

Reemplazando en (2)

$$\ddot{y} - \frac{1}{y}\dot{y} = 0 \Rightarrow \text{se integra a} \quad y(\lambda) = y_0 e^{\alpha\lambda} \qquad \alpha = \text{const}$$

tmb podemos notar q' $\frac{d}{d\lambda}\ln(\dot{y}/y) = 0$ es equivalente a (2)

y da el mismo resultado

⇒ la curva $(x(\lambda), y(\lambda)) = (x_0 = \text{const}, y_0 e^{\alpha\lambda})$

es una recta vertical q' corta el eje x en x_0. ✓

- Por otro lado, notemos $x\dot{x} = \frac{d}{d\lambda}\left(\frac{1}{2}x^2\right)$ e $y\dot{y} = \frac{d}{d\lambda}\left(\frac{1}{2}y^2\right)$

entonces podemos reescribir (4) del siguiente modo

$$\frac{y^2}{c} = x\dot{x} + y\dot{y} = \frac{1}{2}\frac{d}{d\lambda}(x^2+y^2) \Leftrightarrow \frac{d}{d\lambda}(x^2+y^2) = \frac{2y^2}{c} \qquad (5)$$

Notemos tmb q' de (3) $y^2 = b\dot{x}$, Reemplazando en (5)

$$\frac{d}{d\lambda}(x^2+y^2) = \frac{2}{c}b\dot{x} = \frac{d}{d\lambda}\left(\frac{2b}{c}x\right) \Leftrightarrow \frac{d}{d\lambda}\left(x^2 - \frac{2xb}{c} + y^2\right) = 0$$

$$\Rightarrow \quad x^2 - 2x\frac{b}{c} + y^2 = \text{const}.$$

esto es equivalente a

$$\boxed{(x - x_0)^2 + y^2 = r^2} \quad (6) \rightsquigarrow \text{círculos centrados en } x_0 \quad ✓$$

donde identificamos $x_0 = b/c$

Otra forma de obtener el mismo resultado es utilizar la conservación de $g(\partial_x, d/d\lambda)$ y la condición $g(d/d\lambda, d/d\lambda) = +1$ p/ resolver (1) y (2) explícitamente. El resultado es

$$x(\lambda) = x_0 \pm r\frac{1}{\operatorname{tgh}\lambda}, \qquad y(\lambda) = r\frac{\sqrt{\operatorname{tgh}^2(\lambda) - 1}}{\operatorname{tgh}(\lambda)} \qquad (7)$$

con x_0 y r constantes de integración.

Se puede chequear q' $x(\lambda)$ e $y(\lambda)$ de (7) satisfacen (6)

Por último, notemos que podemos obtener las ecuaciones de la geodésica derivando $g(\bar{P}, d/d\lambda)$ y $g(\bar{D}, d/d\lambda)$

$\Rightarrow g(\bar{P}, d/d\lambda) = \dot{x}/y^2 \Rightarrow \frac{d}{d\lambda}\left(\dot{x}/y^2\right) = \underbrace{\frac{\ddot{x}}{y^2} - \frac{2}{y^3}\dot{x}\dot{y}}_{= \frac{1}{y^2}\underbrace{\left(\ddot{x} - \frac{2}{y}\dot{x}\dot{y}\right)}_{=(1)\checkmark}} = 0$

$\Rightarrow g(\bar{K}, d/d\lambda) = \frac{x\dot{x} + y\dot{y}}{y^2}$

$\Rightarrow \frac{d}{d\lambda}\left[g(\bar{K}, d/d\lambda)\right] = \frac{x}{y}\underbrace{\left(\ddot{x} - \frac{2}{y}\dot{x}\dot{y}\right)}_{=(1)} + \underbrace{\frac{\ddot{y}}{y} + \frac{\dot{x}^2 - \dot{y}^2}{y^2}}_{= \frac{1}{y}\underbrace{\left(\ddot{y} + \frac{1}{y}(\dot{x}^2 - \dot{y}^2)\right)}_{=(2)\checkmark}} = 0$