04/10 Guía 5 Clase 1
Métrica de Campo Débil y Geodésicas Nulas (Rayos de Luz)

Ecuaciones de Einstein

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \overset{\text{const}}{\kappa} T_{\mu\nu}}$$

$$\nabla^\nu G_{\mu\nu} \equiv 0 \Rightarrow \nabla^\nu T_{\mu\nu} = 0$$

Métrica de campo débil

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| << 1 \qquad h_{\mu\nu} = 2\phi\,\delta_{\mu\nu}$$

$$\boxed{ds^2 = +(1+2\phi)\,dt^2 - (1-2\phi)(dx^2+dy^2+dz^2)} = \eta_{\mu\nu} dx^\mu dx^\nu + h_{\mu\nu} dx^\mu dx^\nu$$

$$c = 1 \,, \quad \phi = -\frac{GM}{r} \,, \quad r^2 = x^2+y^2+z^2 \,, \quad \partial_t \phi$$

$$|\phi| << 1 \,, \quad v^2 << 1$$

$$S_{PP} = -m \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \underset{v^\mu = \frac{dx^\mu}{dt}}{=} -m \int \sqrt{g_{\mu\nu} v^\mu v^\nu}\, dt \simeq \int \left(\frac{1}{2} m v^2 - m\phi\right) dt$$

Ecuaciones de Einstein en la aproximación de campo débil (Problema 1)

$$\boxed{g^{\mu\nu} = \frac{1}{1+2\phi} \delta^\mu_0 \delta^\nu_0 - \frac{1}{1-2\phi} \delta^\mu_i \delta^\nu_j}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \quad |h_{\mu\nu}| << 1 \,, \quad h_{\mu\nu} = +2\phi\,\delta_{\mu\nu}$$

- Podemos utilizar el principio variacional de las geodésicas para obtener los símbolos de Christoffel:

$$\tilde{S}_{PP} = \frac{1}{2} \int g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\lambda \,, \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda} \qquad \lambda : \text{parámetro afín}$$

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left[(1+2\phi)\,\dot{t}^2 - (1-2\phi)\,\delta_{kj}\,\dot{x}^k \dot{x}^j\right]$$

$$\text{E-L:} \quad \frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = 0$$

- $\mu = t$ $\quad \frac{\partial L}{\partial t} = 0 \,, \quad \frac{\partial L}{\partial \dot{t}} = (1+2\phi)\,\dot{t} \rightsquigarrow \frac{d}{d\lambda}\left((1+2\phi)\,\dot{t}\right) = 0$

$$\Rightarrow (1+2\phi)\,\ddot{t} + 2\,\partial_i\phi\,\dot{t}\,\dot{x}^i = 0 \Leftrightarrow \ddot{t} + 2\,\frac{\partial_i \phi}{1+2\phi}\,\dot{t}\,\dot{x}^i = 0$$

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0$$

$$\boxed{\Gamma^t_{ti} = \frac{\partial_i \phi}{1+2\phi} \simeq \partial_i \phi}$$

- $\mu = i$

$$\frac{\partial L}{\partial x^i} = \partial_i \phi \, \dot{t}^2 + \partial_i \phi \, \delta_{jk} \dot{x}^j \dot{x}^k$$

$$\frac{\partial L}{\partial \dot{x}^i} = -(1-2\phi)\,\delta_{ki}\,\dot{x}^k$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = -(1-2\phi)\,\delta_{ki}\,\ddot{x}^k + 2\,\partial_j \phi\,\delta_{ki}\,\dot{x}^k \dot{x}^j$$

$$\Rightarrow \quad 0 = -(1-2\phi)\,\delta_{ki}\,\ddot{x}^k + 2\,\partial_j\phi\,\delta_{ki}\,\dot{x}^k\dot{x}^j - \partial_i\phi\,(\dot{t}^2 + \delta_{jk}\dot{x}^j\dot{x}^k)$$

$$0 = \delta_{ki}\,\ddot{x}^k - 2\,\frac{\partial_j\phi}{1-2\phi}\,\delta_{ki}\,\dot{x}^k\dot{x}^j + \frac{\partial_i\phi}{1-2\phi}(\dot{t}^2 + \delta_{jk}\dot{x}^j\dot{x}^k)$$

$$\times\ \delta^{il}$$

$$0 = \ddot{x}^l - 2\,\frac{\partial_j\phi}{1-2\phi}\,\dot{x}^l\dot{x}^j + \frac{\delta^{li}\partial_i\phi}{1-2\phi}(\dot{t}^2 + \delta_{jk}\dot{x}^j\dot{x}^k)$$

$$\boxed{\Gamma^l_{tt} = \frac{\partial_l \phi}{1-2\phi} \simeq \partial_l \phi}$$

$$\boxed{\Gamma^l_{jj} = \frac{\partial_l \phi}{1-2\phi} \simeq \partial_l \phi}$$

$$\boxed{\Gamma^l_{lj} = -\frac{\partial_j \phi}{1-2\phi} \simeq -\partial_j \phi}$$

$$\boxed{\Gamma^l_{ll} = -\frac{\partial_l \phi}{1-2\phi} \simeq -\partial_l \phi}$$

No hay suma implícita

- Tensores de curvatura (Ricci y Einstein)

$$R^\lambda{}_{\mu\rho\nu} = \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \underbrace{\Gamma^\lambda_{\alpha\rho}\Gamma^\alpha_{\mu\nu} - \Gamma^\lambda_{\alpha\nu}\Gamma^\alpha_{\mu\rho}}_{\mathcal{O}(\phi^2)}$$

$$\Rightarrow \quad R^\lambda{}_{\mu\rho\nu} \simeq \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\rho}$$

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \simeq \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda}$$

$\partial_t \phi = 0$

- $R_{00} = \partial_\lambda \Gamma^\lambda_{00} - \underbrace{\partial_0 \Gamma^\lambda_{0\lambda}}_{=0} = \partial_\ell \Gamma^\ell_{00} = \partial_\ell \partial_\ell \phi = \Delta \phi$

- $R_{i0} = \underbrace{\partial_\ell \Gamma^\ell_{i0}}_{=0} - \underbrace{\partial_0 \Gamma^\lambda_{i\lambda}}_{=0} = 0$

- $R_{ii} = \partial_\ell \Gamma^\ell_{ii} - \partial_i \Gamma^\lambda_{i\lambda}$

$$R_{xx} = \partial_x \Gamma^x_{xx} + \partial_y \Gamma^y_{xx} + \partial_z \Gamma^z_{xx} - \partial_x \Gamma^0_{x0} - \partial_x \Gamma^x_{xx} - \partial_x \Gamma^y_{xy} - \partial_x \Gamma^z_{xz}$$
$$= -\cancel{\partial_x^2 \phi} + \partial_y^2 \phi + \partial_z^2 \phi - \cancel{\partial_x^2 \phi} + \cancel{\partial_x^2 \phi} + \cancel{\partial_x^2} \phi + \partial_x^2 \phi$$
$$= \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi = \Delta \phi$$

- $R_{ij\neq i} = \partial_\ell \Gamma^\ell_{ij} - \partial_i \Gamma^\lambda_{j\lambda}$

$R_{xy} = 0$

- $R_{\mu\nu} = \Delta\phi\, \delta_{\mu\nu} = \Delta\phi \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$$R = g^{\mu\nu} R_{\mu\nu} \simeq \eta^{\mu\nu} R_{\mu\nu} = \Delta\phi\, \eta^{\mu\nu}\delta_{\mu\nu} = -2\Delta\phi$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \simeq R_{\mu\nu} - \frac{1}{2} R \eta_{\mu\nu}$$
$$= \Delta\phi\, \delta_{\mu\nu} + \Delta\phi\, \eta_{\mu\nu} = \begin{pmatrix} 2\Delta\phi & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

- Ecuaciones de Einstein

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}}$$

Tensor de energía-momentos

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu - P g_{\mu\nu}, \quad u^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right) = \gamma(1, \vec{v})$$

Aproximación no relativista $|\vec{v}| << 1 \rightsquigarrow P \simeq 0 \Rightarrow \boxed{T_{\mu\nu} \simeq \rho u_\mu u_\nu}$

$$\Rightarrow \quad 2\Delta\phi \simeq \kappa\rho$$

Ecuación de Poisson para el potencial gravitatorio

$$\Delta\phi = 4\pi G\rho_m \quad \Rightarrow \quad \boxed{\kappa = 8\pi G} \qquad \boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

Otra forma de calcular los símbolos de Christoffel y los tensores de curvatura para la métrica de campo débil

$$g^{\mu\nu} = \frac{1}{1+2\phi}\delta^\mu_0\delta^\nu_0 - \frac{1}{1-2\phi}\delta^\mu_i\delta^\nu_j$$

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}, \quad |h_{\mu\nu}| << 1, \quad h_{\mu\nu} = +2\phi\,\delta_{\mu\nu}$$

$$g^{\mu\nu} = \eta^{\mu\nu} + H^{\mu\nu} \rightsquigarrow \delta^\mu_\nu = g^{\mu\alpha}g_{\alpha\nu} = (\eta^{\mu\alpha} + H^{\mu\alpha})(\eta_{\alpha\nu} + h_{\alpha\nu})$$

$$\cancel{\delta^\mu_\nu} = \cancel{\delta^\mu_\nu} + H^{\mu\alpha}\eta_{\alpha\nu} + \eta^{\mu\alpha}h_{\alpha\nu} + O(h^2)$$

$$\Rightarrow \quad H^{\mu\alpha}\eta_{\alpha\nu} = -\eta^{\mu\alpha}h_{\alpha\nu}$$

$$\boxed{H^{\mu\rho} = -\eta^{\mu\alpha}h_{\alpha\nu}\eta^{\nu\rho}} \Rightarrow \boxed{g^{\mu\nu} \simeq \eta^{\mu\nu} - \underbrace{\eta^{\mu\alpha}h_{\alpha\nu}\eta^{\nu\rho}}_{h^{\mu\nu}}}$$

- Símbolos de Christoffel

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) = \frac{1}{2}g^{\lambda\alpha}(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu})$$

$$\simeq \frac{1}{2}\eta^{\lambda\alpha}(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu})$$

$$\boxed{\Gamma^\lambda_{\mu\nu} \simeq \partial_{(\mu}h^\lambda_{\nu)} - \frac{1}{2}\partial^\lambda h_{\mu\nu}}$$

$$(\mu\nu) = \frac{1}{2}(\mu\nu + \nu\mu)$$

$$h^\lambda_\nu = \eta^{\lambda\alpha}h_{\alpha\nu}$$

$$\partial^\lambda = \eta^{\lambda\alpha}\partial_\alpha$$

- Tensores de curvatura

$$R^\lambda{}_{\mu\rho\nu} \simeq \partial_\rho\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\rho}$$

$$\simeq \partial_\rho\partial_{(\mu}h^\lambda_{\nu)} - \frac{1}{2}\partial_\rho\partial^\lambda h_{\mu\nu} - \partial_\nu\partial_{(\mu}h^\lambda_{\rho)} + \frac{1}{2}\partial_\nu\partial^\lambda h_{\mu\rho}$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \simeq \partial_\lambda \partial_{(\mu} h^{\lambda}_{\nu)} - \frac{1}{2}\partial_\lambda \partial^\lambda h_{\mu\nu} - \underbrace{\partial_\nu \partial_{(\mu} h^{\lambda}_{\lambda)}}_{\frac{1}{2}\partial_\nu\partial_\mu h^\lambda_\lambda + \frac{1}{2}\partial_\nu \partial_\lambda h^\lambda_\mu} + \frac{1}{2}\partial_\nu \partial^\lambda h_{\mu\lambda}$$

$$\Rightarrow \boxed{R_{\mu\nu} \simeq -\frac{1}{2}\Box h_{\mu\nu} + \partial_\lambda \partial_{(\mu} h^{\lambda}_{\nu)} - \frac{1}{2}\partial_\mu \partial_\nu h}$$

$$\Box \equiv \partial_\lambda \partial^\lambda = \eta^{\lambda\rho}\partial_\lambda \partial_\rho \quad , \quad h \equiv \eta^{\mu\nu} h_{\mu\nu}$$

$$R = g^{\mu\nu} R_{\mu\nu} \simeq \eta^{\mu\nu} R_{\mu\nu} = \boxed{-\Box h + \partial_\lambda \partial_\rho h^{\lambda\rho} \simeq R}$$

▶ Geodésicas

La ecuación de la geodésica (o autoparalela) en una variedad dotada de una métrica y una conexión compatible con la misma es

$$\boxed{\nabla_U U = 0} \quad , \quad U = \frac{d}{d\lambda}$$

$$(\nabla_U U)^\mu = U^\nu U^\mu{}_{;\nu} = U^\nu \partial_\nu U^\mu + \Gamma^\mu_{\alpha\nu} U^\alpha U^\nu = 0$$

$$\frac{dU^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu} U^\alpha U^\nu = 0 = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\nu} \frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda}$$

Una propiedad importante de las geodésicas y el tranporte paralelo es que el producto interno entre vectores transportados paralelamente se mantiene constante a lo largo de una geodésica. En particular, la norma del vector tangente conserva su valor a lo largo de su geodésica correspondiente.

Se dice que una geodésica es tipo tiempo, nula o espacio según la norma de su vector tangente sea negativa, cero o positiva, respectivamente.

$$U = \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu = U^\mu \partial_\mu \quad , \quad U \cdot U = g(U,U) = g(U^\mu \partial_\mu, U^\nu \partial_\nu)$$

$$= U^\mu g(\partial_\mu, \partial_\nu) U^\nu = g_{\mu\nu} U^\mu U^\nu$$

$$U_\mu U^\mu = U \cdot U$$

$$\frac{d}{d\lambda} g(u,u) = u\,(g(u,u)) = \nabla_u g(u,u) = \underbrace{(\nabla_u g)}_{\substack{=0 \\ \text{conex comp}}}(u,u) + 2g(\underbrace{\nabla_u u}_{\substack{=0 \\ \text{geod}}}, u) = 0$$

▶ Vectores de Killing

Un vector de Killing es un campo vectorial que cumple

$$\mathcal{L}_\xi g = 0$$

$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu \xi^\alpha g_{\alpha\nu} + \partial_\nu \xi^\alpha g_{\alpha\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

▶ Leyes de Conservación y Vectores de Killing

$$S = \frac{1}{2}\int d\lambda \; L\left(\lambda, x(\lambda), \frac{dx}{d\lambda}(\lambda)\right) = \frac{1}{2}\int d\lambda \; g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

$$0 = \frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial\left(\frac{dx^\mu}{d\lambda}\right)}\right)$$

Si L no depende explícitamente de una coordenada x^β (β está fijo, por ejemplo β=1)

$$\Rightarrow \quad \frac{d}{d\lambda}\left(\frac{\partial L}{\partial\left(\frac{dx^\beta}{d\lambda}\right)}\right) = 0 \Rightarrow \boxed{\frac{\partial L}{\partial\left(\frac{dx^\beta}{d\lambda}\right)} = \text{const}}$$

La contracción de un vector de Killing con el vector tangente de la geodésica nos da una cantidad conservada

$$\frac{d}{d\lambda}(\xi\cdot\bar{v}) = \frac{d}{d\lambda} g(\xi,\bar{v}) = \nabla_{\bar{v}}\, g(\xi,\bar{v}) = \underbrace{(\nabla_{\bar{v}} g)}_{=0}(\xi,\bar{v}) + g(\nabla_{\bar{v}}\xi,\bar{v}) + g(\xi,\underbrace{\nabla_{\bar{v}}\bar{v}}_{=0})$$

$$= g_{\mu\nu} v^\rho (\xi^\mu{}_{;\rho}) v^\nu = \underbrace{v^\rho v^\nu}_{\text{sim en }(\rho,\nu)} \xi_{\nu;\rho} = v^\rho v^\nu \xi_{(\nu;\rho)} = 0 \Rightarrow \boxed{\xi \ \text{Killing}}$$

▲ Qué tiene que ver una cosa con la otra? Lo más sencillo es verlo con un ejemplo.

Ejemplo: si β=1, entonces $\partial_1 g_{\mu\nu} = 0$, $\xi = (0,1,0,0)$

$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \xi^\rho + g_{\rho\nu}\partial_\mu \xi^\rho$$

$$= \underbrace{\partial_1 g_{\mu\nu}}_{=0} + g_{\mu\rho}\underbrace{\partial_\nu \delta^\rho_1}_{=0} + g_{\rho\nu}\underbrace{\partial_\mu \delta^\rho_1}_{=0} = 0 \quad \checkmark$$

En este caso, notemos que la contracción del vector de Killing con el vector tangente nos da la componente covariante que se conserva

$$\bar{\xi} \cdot \bar{V} = g(\bar{\xi}, \bar{V}) = g_{\mu\nu}\, \xi^{\mu} V^{\nu} = g_{\mu\nu}\, \delta^{\mu}_{1}\, V^{\nu} = g_{1\nu} V^{\nu} = V_1 = \text{const}$$

En general resolver las ecuaciones de las geodésicas es bastante complicado. Si contamos con vectores de Killing tenemos automáticamente cantidades conservadas (constantes de integración).

En n dimensiones tenemos n ecuaciones diferenciales para resolver.

La norma del vector tangente de la geodésica siempre provee una "conservación" (una restricción).

- Geodésicas temporal (timelike)

$$V \cdot V > 0 \quad , \quad g(\bar{V}, \bar{V}) = g_{\mu\nu} V^{\mu} V^{\nu} = g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \overset{\lambda = \tau}{=} g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = +1 = \text{const}$$

Notemos que $\frac{dx^{\mu}}{d\tau} = u^{\mu}$ la cuadrivelocidad. Se suele usar como vector tangente para las geodésicas temporales, y se define el cuadrimomento como $p^{\mu} = m u^{\mu} \Rightarrow \bar{p} \cdot \bar{p} = +m^2$

- Geodésica espacial

$$V \cdot V < 0 \quad , \quad g(\bar{V}, \bar{V}) = g_{\mu\nu} V^{\mu} V^{\nu} = g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \overset{\lambda = s \text{ intervalo}}{=} g_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} = -1 = \text{const}$$

- Geodésica nula

En este caso no hay una elección obvia y conveniente para el parámetro afín. Simplemente se elije λ para que se cumpla

$$V \cdot V = 0, \; V^{\mu} = \frac{dx^{\mu}}{d\lambda} = (\omega, \vec{k}) = k^{\mu} \quad , \quad +\omega^2 - |\vec{k}|^2 = 0$$

Entonces tenemos (n-1) ecuaciones a resolver. Cuantos más vectores de Killing tengamos menos ecuaciones tendremos que resolver.

Problema 3

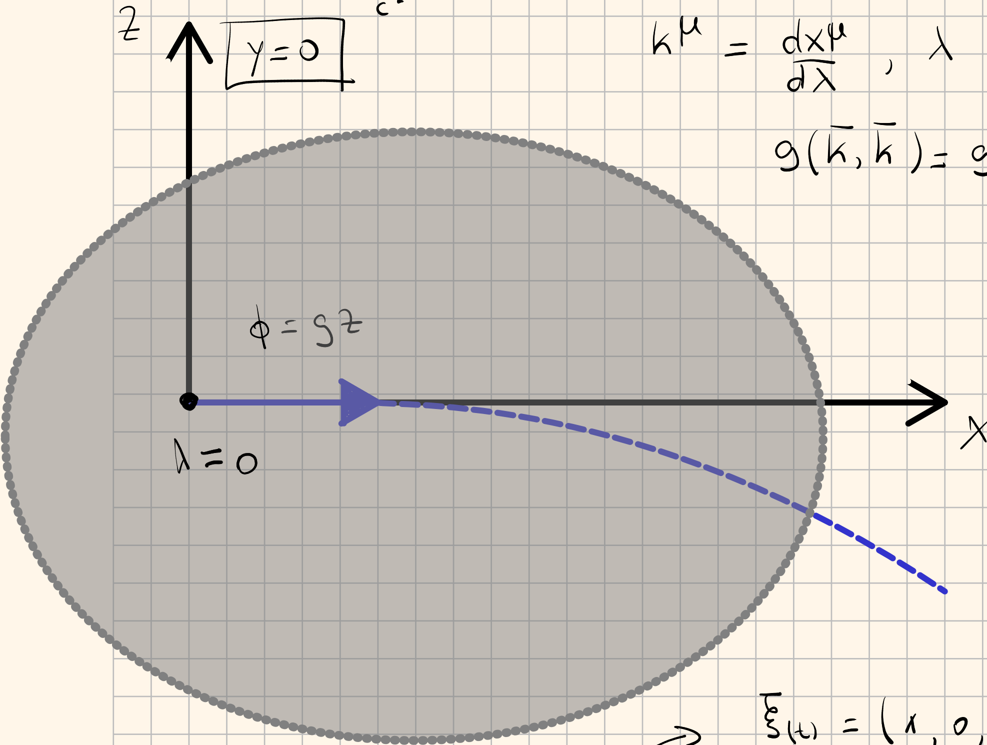
En una región del espacio la métrica es

$$ds^2 = +c^2\left(1+2\frac{\phi}{c^2}\right)dt^2 - \left(1-2\frac{\phi}{c^2}\right)\left(dx^2+dy^2+dz^2\right) \qquad (c=1)$$

$$\left|\frac{\phi}{c^2}\right| << 1, \quad \partial_t \phi = 0, \quad \phi = gz, \quad g = \text{const}, \quad [g] = m/s^2$$

$k^\mu = \frac{dx^\mu}{d\lambda}$, λ parámetro de la geodésica nula

$$g(\bar{k},\bar{k}) = g_{\mu\nu}k^\mu k^\nu = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$



$$\xi_{(t)} = (1,0,0,0) = \partial_t$$

$$k^\mu = \frac{dx^\mu}{d\lambda}$$

- Vectores de Killing $\{\partial_t, \partial_x, \partial_y\}$

Cantidades conservadas

▶ $E \equiv g(k,\partial_t) = g_{tt}k^t = k_t = \text{const}$

▶ $K \equiv -g(k,\partial_x) = -g_{xx}k^x = -k_x = \text{const}$

▶ $K_y \equiv -g(k,\partial_y) = -g_{yy}k^y = -k_y = \text{const}$ ✗

$$= -g_{yy}\frac{dy}{d\lambda} = \text{const}$$

Notemos que debido a que $\left.\frac{dy}{d\lambda}\right|_{\lambda=0} = 0 = k^y$ ✓

- Ecuación de la geodésica

$$\nabla_{\bar{k}}\bar{k} = 0 \quad \Rightarrow \quad \frac{dk^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}k^\nu k^\rho = 0$$

▶ Símbolos de Christoffel

$$\Gamma^0_{0i} \simeq \partial_i \phi \Rightarrow \Gamma^0_{0z} \simeq g \qquad \Gamma^i_{ij} \simeq -\partial_j \phi \Rightarrow \Gamma^i_{ij} = -g$$

$$\Gamma^i_{00} = \Gamma^i_{jj} \simeq \partial_i \phi \Rightarrow \Gamma^z_{00} = \Gamma^z_{jj} = g \qquad \Gamma^i_{ii} \simeq -\partial_i \phi \Rightarrow \Gamma^i_{ii} = -g$$

Las ecuaciones de la geodésica son $(k^y=0)$

$$\begin{cases} \dfrac{dk^t}{d\lambda} + 2g\,k^t k^z = 0 \\ \dfrac{dk^x}{d\lambda} - 2g\,k^x k^z = 0 \\ \dfrac{dk^z}{d\lambda} + g\left[(k^t)^2 + (k^x)^2 + (k^z)^2\right] = 0 \end{cases}$$

$$\frac{dk^y}{d\lambda} - 2g\,k^y k^z = 0 \quad \checkmark$$

Con la restricción $k\cdot k = 0$

Resolver integrando directamente es un quilombo en general. Si hay vectores de Killing conviene tratar de utilizarlos para resolver el problema.

Comencemos por ver que información podemos obtener de restricción $k\cdot k = 0$

$$0 = g_{\mu\nu}k^\mu k^\nu = g_{tt}(k^t)^2 + g_{xx}(k^x)^2 + g_{zz}(k^z)^2$$

Notemos que $k_t = g_{tt}k^t = E = \text{const} \Rightarrow k^t = E/g_{tt}$

$k_x = g_{xx}k^x = K = \text{const} \Rightarrow k^x = K/g_{xx}$

$$\Rightarrow \quad 0 = \frac{E^2}{g_{tt}} + \frac{K^2}{g_{xx}} + g_{zz}\dot{z}^2 = \frac{E^2}{1+2\phi} - \frac{K^2}{1-2\phi} - (1-2\phi)\dot{z}^2$$

$$0 \simeq (1-2\phi)E^2 - (1+2\phi)K^2 - (1-2\phi)\dot{z}^2$$

La ecuación anterior vale para cualquier punto de la geodésica (el transporte paralelo preserva el producto interno entre vectores transportados paralelamente), en particular vale para z=0, donde tenemos que

$$\phi = gz \qquad \phi(z=0) = 0 \Rightarrow 0 = E^2 - K^2 \Rightarrow E^2 = K^2$$

Reemplazando esto en $k\cdot k = 0$

$$0 = +(1-2\phi)E^2 - (1+2\phi)E^2 - (1-2\phi)\left(\frac{dz}{d\lambda}\right)^2$$

$$= -4\phi E^2 - (1-2\phi)\left(\frac{dz}{d\lambda}\right)^2 \Rightarrow \left(\frac{dz}{d\lambda}\right)^2 = \frac{-4\phi E^2}{1-2\phi} \simeq -4\phi E^2$$

$$\Rightarrow \frac{dz}{d\lambda} = \pm 2\sqrt{-\phi}\,E \qquad \phi = gz$$

$$\frac{dz}{d\lambda} = -2\sqrt{-gz}\,E \qquad \frac{dz}{dx} = \frac{\frac{dz}{d\lambda}}{\frac{dx}{d\lambda}} = \frac{-2\sqrt{-gz}\,E}{k^x}$$

$k^x \to E$

$$E = K = g_{xx}\frac{dx}{d\lambda}\Big|_{\lambda=0} = \frac{dx}{d\lambda} = k^x = E$$

Nos va a resultar conveniente hallar la trayectoria z=z(x)

$$\Rightarrow \quad \frac{dz}{dx} = -2\sqrt{-gz} \quad \Rightarrow \quad dx = -\frac{1}{2}\frac{dz}{\sqrt{-gz}} \underset{u=-z}{=} \frac{1}{2}\frac{du}{\sqrt{gu}} = \frac{1}{\sqrt{g}} d(\sqrt{u})$$

$$\Rightarrow \quad x = \frac{1}{\sqrt{g}}\sqrt{u} = \frac{1}{\sqrt{g}}\sqrt{-z} \quad \Rightarrow \quad z(x) = -gx^2$$

Si en física Newtoniana postulamos que la luz también es afectada por la gravedad del mismo modo que una partícula masiva entonces

$$\left.\begin{array}{l} x(t) = t \\ z(t) = \frac{1}{2}gt^2 \end{array}\right\} \quad \boxed{z(x) = -\frac{1}{2}gx^2}$$

Relatividad General predice una deflexión de la luz debido a efectos de gravedad y en un factor 2 respecto del que "predice" la física Newtoniana.

Problema 4

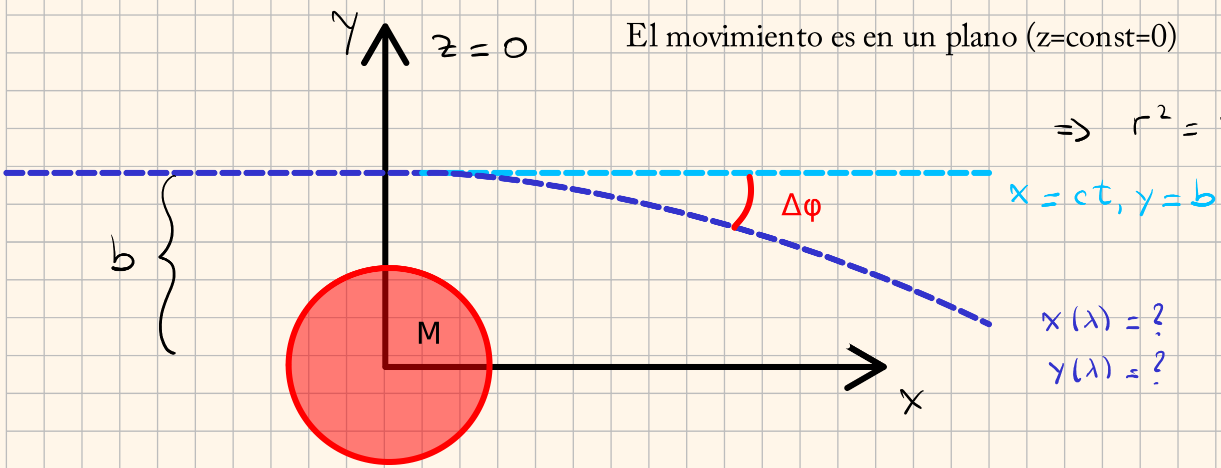
En una región del espacio la métrica es

$$ds^2 = -(1+2\phi)\,dt^2 + (1-2\phi)\,(dx^2+dy^2+dz^2) \qquad (c=1)$$

$$|\phi| \ll 1, \quad \partial_t \phi = 0, \qquad \phi = -\frac{GM}{r}, \qquad r^2 = x^2+y^2+z^2$$

El movimiento es en un plano (z=const=0)

$$\Rightarrow \quad r^2 = x^2+y^2$$



Del constraint de geodésica nula tenemos

$$0 = \bar{k}\cdot\bar{k} = g_{\mu\nu}k^\mu k^\nu = (k^t)^2 - |\vec{k}|^2 + \mathcal{O}(\phi) \quad \Rightarrow \quad (k^t)^2 \simeq |\vec{k}|^2$$

Símbolos de Christoffel

$$\Gamma^0_{\ 0i} \simeq \Gamma^i_{\ 00} = \Gamma^i_{\ jj} \simeq \partial_i\phi \quad , \quad \Gamma^j_{\ ji} = \Gamma^i_{\ ii} \simeq -\partial_i\phi$$

Ecuaciones de la geodésica

$$\mu = t \qquad \underbrace{\frac{dk^t}{d\lambda}} + \Gamma^t_{\ \rho\sigma}k^\rho k^\sigma = \frac{dk^t}{d\lambda} + 2\,\partial_i\phi\, k^i k^t = -2\partial_i\phi\, k^i k^t + 2\,\partial_i\phi\, k^i k^t = 0$$

$$\frac{d}{d\lambda}\left(g^{tt}k_t\right) = \frac{d}{d\lambda}\left(\frac{1}{1+2\phi}\right)k_t \simeq \frac{d}{d\lambda}(1-2\phi)\,k_t = -2\,\partial_i\phi\, k^i k_t$$

$$\simeq -2\partial_i\phi\, k^i k^t$$

$$\mu = i \qquad \frac{dk^i}{d\lambda} = -\Gamma^i_{\ \rho\sigma}k^\rho k^\sigma = -\Gamma^i_{\ 00}(k^t)^2 - \Gamma^i_{\ jj}|\vec{k}|^2 - \Gamma^i_{\ ij}k^i k^j$$

$$= -\partial_i\phi\,(k^t)^2 - \partial_i\phi\,|\vec{k}|^2 + 2\,\partial_j\phi\, k^i k^j$$

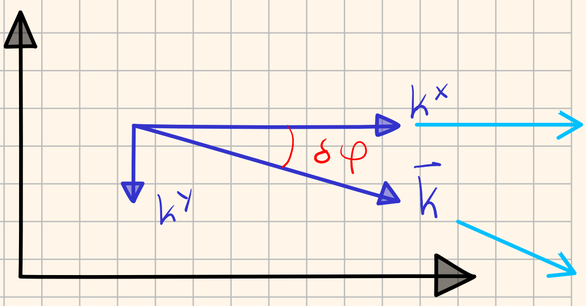
$$= -\partial_i\phi\left((k^t)^2 + |\vec{k}|^2\right) + 2\,\partial_j\phi\, k^i k^j$$

$$\boxed{\frac{dk^i}{d\lambda} = -2\,\partial_i\phi\,|\vec{k}|^2 + 2\,\partial_j\phi\, k^i k^j}$$

Asumiendo que la desviación del rayo de luz es pequeña, es decir,

$$|k^y| << |k^x|$$

$$\Rightarrow \quad |\vec{k}| \simeq |k^x|$$



▶ $i = x \qquad \frac{dk^x}{d\lambda} = -2(k^x)^2\,\partial_x\phi + 2\underbrace{(\partial_x\phi\, k^x + \partial_y\phi\, k^y)}_{\simeq\, \partial_x\phi\, k^x}k^x \simeq 0 \Rightarrow k^x \simeq const$

▶ $i = y \qquad \frac{dk^y}{d\lambda} = -2(k^x)^2\,\partial_y\phi + 2\underbrace{(\partial_x\phi\, k^x + \partial_y\phi\, k^y)}_{\simeq\, \partial_x\phi\, k^x}k^y \simeq -2(k^x)^2\partial_y\phi$

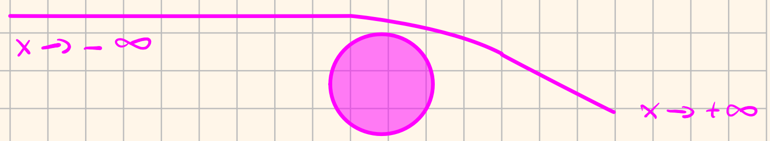
$$\Rightarrow \quad \boxed{\frac{dk^y}{d\lambda} \simeq -2(k^x)^2\,\partial_y\phi} \quad (\star)$$

Tenemos que resolver $(\star)$ al igual que en el problema anterior nos va a convenir hallar la ecuación de la trayectoria, es decir, hallar y(x)

$$\frac{dk^y}{dx} = \frac{\frac{dk^y}{d\lambda}}{\frac{dx}{d\lambda}} = \frac{-2(k^x)^2\partial_y\phi}{k^x} \simeq \boxed{-2k^x\,\partial_y\phi = \frac{dk^y}{dx}}$$

También podemos aproximar y=b en toda la trayectoria (deflexión pequeña)

$$\Rightarrow \quad \frac{dk^y}{dx} = -\frac{2k^x GMb}{(x^2+b^2)^{3/2}}$$



Asumiendo que muy lejos de la fuente $x \sim -\infty$, $y = b$ la trayectoria es la misma que en espacio plano, es decir, $k^y(x \sim -\infty) = 0$

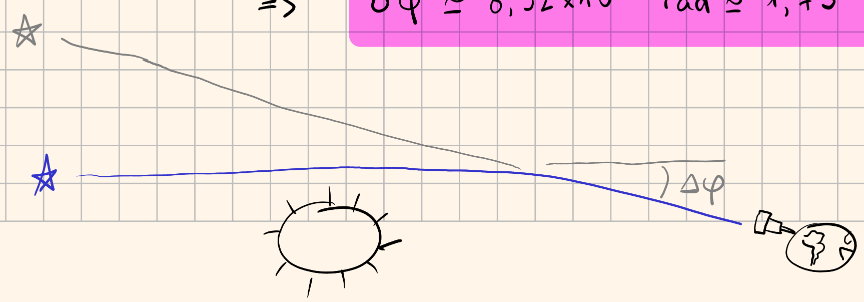
$$\int_{k^y(-\infty)=0}^{k^y(x=+\infty)} dk^y = -2k^x GMb \int_{-\infty}^{+\infty} \frac{dx}{(x^2+b^2)^{3/2}} = \boxed{-\frac{4GMk^x}{b} = k^y(x\sim+\infty)}$$

La deflexión se obtiene de $\quad \boxed{\tan(\delta\varphi) = \left|\frac{k^y}{k^x}\right| = \frac{4GM}{c^2 b} \simeq \delta\varphi}$

Para el Sol: $M_\odot \simeq 2{,}00\times10^{30}\,kg$, $b = R_\odot \simeq 6{,}96\times10^{8}\,m$, $G \simeq 6{,}67\times10^{-11}\,\frac{m^3}{kg\,s^2}$

$$\Rightarrow \quad \boxed{\delta\varphi \simeq 8{,}52\times10^{-6}\,rad \simeq 1{,}75''}$$

Eddington 1919



LIGHTS ALL ASKEW IN THE HEAVENS
Special Cable to THE NEW YORK TIMES.
New York Times (1857-1922); Nov 10, 1919;
ProQuest Historical Newspapers The New York Times (1851 - 2007)
pg. 17

LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

$$k^\mu \equiv \frac{dx^\mu}{d\lambda}$$

$$k^\mu = (\omega, \bar{k})$$

$$E = \underbrace{g_{tt} \frac{dx^t}{d\lambda}}_{k_t} = g_{tt} k^t = g_{tt}\omega = E$$