Clase 18/8

martes, 17 de agosto de 2021 14:37

(contiene 2 pequeñas correcciones)

Relatividad General

* Convenciones

i) Índices → griegos: $\underbrace{\mu = 0,1,2,3}_{\text{espacio-temporales}}$
→ latinos: $\underbrace{i = 1,2,3}_{\text{espaciales}}$

ii) Signatura de la métrica:

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad (\text{"mostly minus"})$$

Guía 1 : Relatividad especial

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\eta_{ij} = -\delta_{ij}$

→ métrica de Minkowski

→ $\eta_{00} = +1$, $\eta_{0i} = 0$, $\eta_{ii} = -1$, $\eta_{ij} = 0$ $(i \neq j)$

* Transf. de Lorentz $\Lambda^{\mu}{}_{\nu}$ tal que $\boxed{\Lambda^{\mu}{}_{\nu}\Lambda^{\alpha}{}_{\beta}\,\eta_{\mu\alpha} = \eta_{\nu\beta}}$ ←

$$\text{"}\Lambda^{\mu}{}_{\nu}\Lambda^{\alpha}{}_{\beta}\,\eta_{\mu\alpha}\text{"} = \sum_{\sigma=0}^{3}\sum_{\alpha=0}^{3}\Lambda^{\sigma}{}_{\nu}\Lambda^{\alpha}{}_{\beta}\,\eta_{\sigma\alpha} = \sum_{\sigma=0}^{3}\left(\Lambda^{\sigma}{}_{\nu}\Lambda^{0}{}_{\beta}\,\eta_{\sigma 0} + \Lambda^{\sigma}{}_{\nu}\Lambda^{1}{}_{\beta}\,\eta_{\sigma 1} + \Lambda^{\sigma}{}_{\nu}\Lambda^{2}{}_{\beta}\,\eta_{\sigma 2} + \Lambda^{\sigma}{}_{\nu}\Lambda^{3}{}_{\beta}\,\eta_{\sigma 3}\right)$$

* Cuadrivector V : objeto con 4 componentes V^{μ} ; $\mu = 0,1,2,3$ asociadas a cierto S

$\Rightarrow$ En S': $V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu}$

Ejemplo: x, con $x^{\mu} = (ct, x, y, z)$

y una transf. de Lorentz manda $x^{\mu} \mapsto x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

* Transf. de Lorentz → rotaciones

y una Trans. de Lorentz manda $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$

* Trans de Lorentz:
- rotaciones
- boosts $\longrightarrow$ Ej:

$$ct' = \gamma\left(ct - \frac{v}{c}x\right) \;;\; y' = y$$
$$x' = \gamma\left(x - vt\right) \;;\; z' = z$$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} \Lambda^0{}_0 &= \gamma \\ \Lambda^1{}_1 &= \gamma \\ \Lambda^0{}_1 &= -\gamma v/c \end{aligned}\right\}$$

* ¿ nos da un producto invariante de Lorentz:

$$V \cdot W := V^\mu W^\nu \eta_{\mu\nu} = V^0 W^0 - V^1 W^1 - V^2 W^2 - V^3 W^3 =$$
$$= V^0 W^0 - \vec{V} \cdot \vec{W}$$

Veamos que es invariante:

$$V'^\mu W'^\nu \eta_{\mu\nu} \overset{\text{def. de 4-vector}}{=} \Lambda^\mu{}_\beta V^\beta \Lambda^\nu{}_\alpha W^\alpha \eta_{\mu\nu}$$
$$= V^\beta W^\alpha \left(\Lambda^\mu{}_\beta \Lambda^\nu{}_\alpha \eta_{\mu\nu}\right) \underset{\text{def. } \Lambda}{=} V^\beta W^\alpha \eta_{\beta\alpha} = V \cdot W \quad \checkmark$$

* Separamos en 3 casos según la norma de V, $\|V\|^2 = V \cdot V$:

i) $V \cdot V > 0 \rightarrow$ V es time-like o tipo-tiempo

ii) $V \cdot V = 0 \rightarrow$ V es null-like o tipo-nulo o nulo

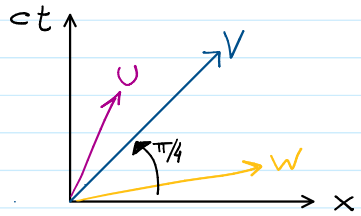
iii) $V \cdot V < 0 \rightarrow$ V es space-like o tipo-espacio

* Diagrama espacio-temporal:

V nulo

U time-like

W space-like



$V \cdot V = (V^0)^2 - (V^1)^2 = 0$
$|V^0| = |V^1|$

Obs: $U \cdot U = (U^0)^2 - \vec{U}^2 > 0 \Rightarrow$ una buena descripción de U

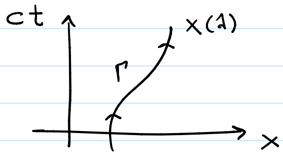
Obs: $U\cdot U = (U^0)^2 - \vec{U}^2 > 0 \Rightarrow$ una buena descripción de U

$$= \|U\|^2 (ch^2\beta - sh^2\beta) = \|U\|^2$$

es:
$$U^0 = \|U\|\, ch\beta$$
$$U^x = \|U\|\, sh\beta \,\sin\theta \cos\varphi$$
$$U^y = \|U\|\, sh\beta \,\sin\theta \sin\varphi$$
$$U^z = \|U\|\, sh\beta \,\cos\theta$$

ojo: Para W intercambio $ch\beta \leftrightarrow sh\beta \Rightarrow \|W\|^2 < 0$

Problema 1



$$\frac{dx^\alpha(\lambda)}{d\lambda} = \gamma(v)\left(c\, ch\frac{a\lambda}{c},\ c\, sh\frac{a\lambda}{c},\ 0,\ v\right)$$
$$\frac{d(ct)}{d\lambda} = \gamma c\, ch\frac{a\lambda}{c}$$

a) Veamos q' λ es τ (tiempo propio): $ds^2 = c^2dt^2 - d\vec{x}^2 \overset{t=\tau \Leftrightarrow d\vec{x}=0}{=} c^2 d\tau^2$

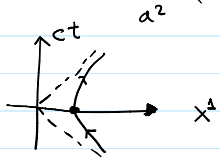
$$\Rightarrow ds^2\Big|_\Gamma = c^2dt^2 - d\vec{x}^2\Big|_\Gamma = c^2\left(\frac{dt}{d\lambda}d\lambda\right)^2 - \left(\frac{d\vec{x}}{d\lambda}\right)^2 d\lambda^2 = d\lambda^2\left(\frac{d(ct)}{d\lambda}^2 - \frac{d\vec{x}}{d\lambda}^2\right)$$

$\lambda = \tau$ ✓

$$= d\lambda^2\left[\gamma^2c^2ch^2\frac{a\lambda}{c} - \gamma^2c^2sh^2\frac{a\lambda}{c} - \gamma^2v^2\right] = d\lambda^2\gamma^2(c^2-v^2)$$
$$= d\lambda^2\gamma^2c^2(1-v^2/c^2) = c^2 d\lambda^2$$

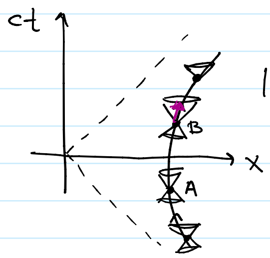
b) $$x^\alpha(\lambda) = \gamma\frac{c^2}{a}\left(sh\frac{a\lambda}{c},\ ch\frac{a\lambda}{c},\ cte,\ \frac{v}{c}\cdot\frac{a\lambda}{c}\right)$$

$$(x^0)^2 - (x^1)^2 = \gamma^2\frac{c^4}{a^2}\left(sh^2\frac{a\lambda}{c} - ch^2\frac{a\lambda}{c}\right) = -\gamma^2\frac{c^4}{a^2} < 0$$



$$\frac{dt}{dx} = \frac{dt/d\lambda}{dx/d\lambda} = \frac{1}{tgh\frac{a\lambda}{c}} > 0$$ (ojo, si $\lambda > 0$)

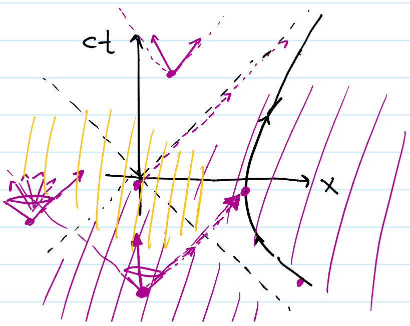
c)

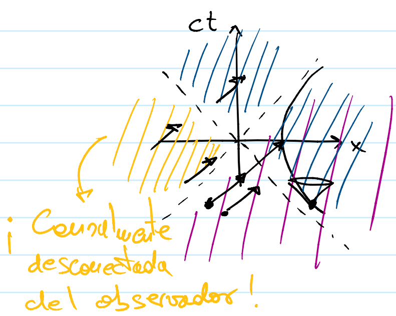


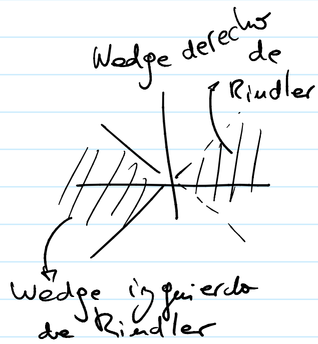
$$\left\|\frac{dx^\alpha}{d\lambda}\right\|^2 = c^2 > 0 \to \frac{dx^\alpha}{d\lambda} \text{ es time-like}$$

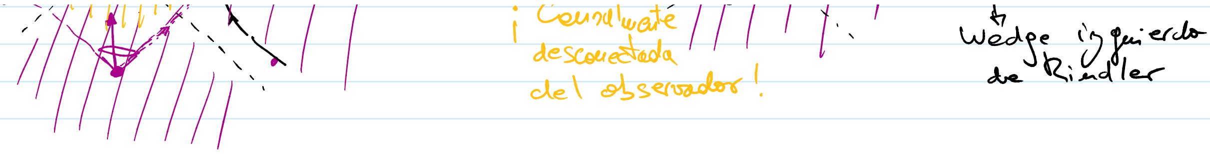
$$d(A,B) = \int_A^B ds = \cancel{c}\int_{\lambda_A}^{\lambda_B}\sqrt{\left\|\frac{dx^\alpha}{d\lambda}\right\|^2}\, d\lambda > 0$$

d)









Problema 3 $(c=1)$

a) $U^\alpha = \frac{dx^\alpha}{d\tau} \rightarrow$ cuadrivelocidad $\Rightarrow U^\alpha = \overset{\gamma(u)}{\gamma}(1, \vec{u})$ ya que

$$U^\alpha = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right) = \left(\gamma, \overbrace{\frac{dt}{d\tau}}^{\gamma} \frac{d\vec{x}}{dt}\right) = \gamma(1, \underset{\frac{d\vec{x}}{dt}}{\vec{u}})$$

$$dt^2 - d\vec{x}^2 = d\tau^2 = dt^2\left(1 - \frac{d\vec{x}}{dt}^2\right) = dt^2\gamma^{-2} \rightarrow \frac{dt}{d\tau} = \gamma$$

$$a^\alpha = \frac{dU^\alpha}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt}\left[\gamma(1,\vec{u})\right] = \gamma\left(\dot{\gamma}, \dot{\gamma}\vec{u} + \gamma\vec{a}\right)$$

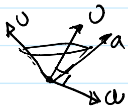
$$= \gamma\left(\vec{u}\cdot\vec{a}\,\gamma^3, \gamma\vec{a} + \vec{u}(\vec{u}\cdot\vec{a})\gamma^3\right)$$

b) $U \cdot U = \gamma^2(1 - \vec{u}^2) = 1$

c) $U \cdot a = U^\alpha a^\beta \eta_{\alpha\beta} = U^0 a^0 - \vec{U}\cdot\vec{a} = \gamma^2\left(\vec{u}\cdot\vec{a}\,\gamma^3 - \vec{u}\cdot\left(\gamma\vec{a} + \vec{u}(\vec{u}\cdot\vec{a})\gamma^3\right)\right)$

$$= \gamma^5\left[\vec{u}\cdot\vec{a} - \gamma^{-2}\vec{u}\cdot\vec{a} - \vec{u}^2(\vec{u}\cdot\vec{a})\right] = \gamma^5(\vec{u}\cdot\vec{a})\left[1 - \vec{u}^2 - \gamma^{-2}\right] = 0 \Rightarrow a\cdot a < 0$$

a es tipo-espacio



$$\cos\theta_{Ua} = \frac{U\cdot a}{\|U\|\,\|a\|}$$

d) $a\cdot a = a^\alpha a^\beta \eta_{\alpha\beta} = (a^0)^2 - \vec{a}^2 = \dots = -\gamma^4(\vec{u}\cdot\vec{a})^2\left[\gamma^2 + \frac{\vec{a}^2}{(\vec{u}\cdot\vec{a})^2}\right]$

$$\overset{u_\parallel = \vec{u}\cdot\vec{a}/|\vec{a}|}{=} -\gamma^4\vec{a}^2\left[1 + \frac{u_\parallel^2}{1-\vec{u}^2}\right] = -\frac{\gamma^6}{\gamma_\perp^2}\vec{a}^2 \quad \text{con} \quad \gamma_\perp = \left[1 - (\vec{u}^2 - u_\parallel^2)\right]^{-1/2}$$

$\vec{\alpha} := \frac{\gamma^3}{\gamma_\perp}\vec{a}$ aceleración propia $\rightarrow a\cdot a = -\vec{\alpha}^2$ invariante

obs. haciendo un boost con velocidad $\vec{u}$: $\vec{\alpha} = \frac{\gamma^3}{\gamma_\perp}\overbrace{\vec{a}'\gamma^{-3}}^{\gamma^3\vec{a}} = \frac{\vec{a}'}{\gamma_\perp}$

y ahora si $u_\perp = 0$: $\vec{\alpha} = \vec{a}'$ $\Rightarrow \vec{\alpha}$ es la aceleración (instantánea) en el sistema del objeto (asumiendo $u_\perp = 0$)