Clase 23/8

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Más convenciones:
* índice arriba ($A^\mu$) → "contravariante"
* índice abajo ($A_\mu$) → "covariante"

* Los relacionamos con la métrica: $A_\mu = g_{\mu\nu} A^\nu$ → análogamente "subimos el índice": $A^\mu = g^{\mu\nu} A_\nu$

$g^{\mu\nu}$ → la inversa de la métrica ($g^{\mu\nu} g_{\nu\alpha} = \delta^\mu_\alpha$)

Ej 8 y 9

Recordemos las ecs. de Maxwell:

$\nabla \times \vec{E} + \partial_t \vec{B} = 0$ $\quad$ $\nabla \times \vec{B} - \partial_t \vec{E} = \vec{j}$

$\nabla \cdot \vec{B} = 0$ $\quad$ $\vec{\nabla} \cdot \vec{E} = \rho$

si defino $\phi$ y $\vec{A}$ tal que $\vec{B} = \vec{\nabla} \times \vec{A}$ y $\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}$

puedo formar $A_\alpha := (\phi, -\vec{A})$ ó $A^\alpha = (\phi, \vec{A})$

$\hookrightarrow A_1 = -A^x,\ A_2 = -A^y,\ A_3 = -A^z$

Obs. , $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ "invariancia de gauge"

$$\underset{\text{"Trans. de gauge"}}{A'_\alpha = A_\alpha + \partial_\alpha \lambda} = (\underbrace{\phi + \dot{\lambda}}_{\phi' = \phi + \dot{\lambda}},\ \underbrace{-\vec{A} + \vec{\nabla}\lambda}_{-\vec{A}' \Rightarrow \vec{A}' = \vec{A} - \vec{\nabla}\lambda})$$

Def. $F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$ $\quad \left(\partial_\alpha = \frac{\partial}{\partial x^\alpha}\right)$ $\Rightarrow$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -\partial_0 \vec{A} - \vec{\nabla}\phi & & \\ & 0 & -(\nabla\times\vec{A})^z & (\vec{\nabla}\times\vec{A})^y \\ & -F_{12} & 0 & -(\nabla\times\vec{A})^x \\ & -F_{13} & -F_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & & \vec{E} & \\ -\vec{E} & 0 & -B^z & B^y \\ & B^z & 0 & -B^x \\ & -B^y & B^x & 0 \end{pmatrix}$$

¡invariante de gauge!

$\hookrightarrow F'_{\alpha\beta} = \partial_\alpha A'_\beta - \partial_\beta A'_\alpha = \partial_\alpha (A_\beta + \partial_\beta \lambda) - \partial_\beta (A_\alpha + \partial_\alpha \lambda)$

$= \partial_\alpha A_\beta - \partial_\beta A_\alpha + \partial_\alpha \partial_\beta \lambda - \partial_\beta \partial_\alpha \lambda = F_{\alpha\beta}$ ✓

(Ej.) $F^{12} = g^{1\mu} g^{2\nu} F_{\mu\nu} = \underbrace{\eta^{1\mu}}_{-\delta^\mu_1} \underbrace{\eta^{2\nu}}_{-\delta^\nu_2} F_{\mu\nu} = + \delta_1^\mu \delta_2^\nu F_{\mu\nu} = F_{12} = -B^z$

F contravariante $\Rightarrow F^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} = g^{\alpha\mu} \left( \sum_{\nu=0}^{3} g^{\beta\nu} F_{\mu\nu} \right)$

$$F^{\alpha\beta} \underset{g=\eta}{=} \begin{pmatrix} 0 & & -\vec{E} & \\ \vec{E} & 0 & -B^z & B^y \\ & B^z & 0 & -B^x \\ & -B^y & B^x & 0 \end{pmatrix}$$

Obs: $g^{\beta\nu} F_{\mu\nu} \overset{(?)}{=} g^\beta{}_\nu F_\mu{}^\nu = \overbrace{g^{\beta\alpha} g_{\alpha\nu}}^{\delta^\beta_\nu} F_\mu{}^\nu$

$= \delta^\beta_\nu F_\mu{}^\nu = F_\mu{}^\beta = g^{\beta\nu} F_{\mu\nu}$ ✓

o sea: $A_\mu B^\mu = A^\mu B_\mu$ (da lo mismo cuál está arriba y cuál abajo si están siendo contraídos)

$i = 1, 2, 3$

Obs: $F_{ij} = -\epsilon_{ijk} B^k$ ($i=1,2,3$) con $\epsilon_{123}=+1$, o sea $\epsilon_{132}=-1$

Las ecs. de Maxwell se reescriben como

$$\partial_\alpha F^{\alpha\beta} = j^\beta \quad , \quad \partial_{[\alpha} F_{\beta\mu]} = 0$$

con $j^\beta = (\rho, \vec{j})$

$$\partial_{[\alpha} F_{\beta\mu]} \to \frac{1}{3!}[\partial_\alpha F_{\beta\mu} - \partial_\beta \underbrace{F_{\alpha\mu}}_{-F_{\mu\alpha}} + \partial_\beta F_{\mu\alpha} - \partial_\mu \underbrace{F_{\beta\alpha}}_{-F_{\alpha\beta}} + \partial_\mu F_{\alpha\beta} - \partial_\alpha F_{\mu\beta}] \overset{F_{\alpha\beta}=-F_{\beta\alpha}}{=} \frac{2}{3!}(\partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} + \partial_\mu F_{\alpha\beta})$$

* Fuerza de Lorentz: $\vec{f} = q(\vec{E} + \vec{u} \times \vec{B})$, $\vec{u} = \frac{d\vec{r}}{dt}$

8 a) Cuadrifuerza: $K^\alpha := q F^\alpha{}_\beta U^\beta = q F^{\alpha\beta} U_\beta$

Entonces, i) $\alpha = 0$: $K^0 = q F^0{}_\beta U^\beta = q\, \eta_{\beta\alpha} \overbrace{F^{0\alpha}}^{-\delta^\alpha_i E^i} U^\beta$

$$q\,\eta_{\beta\alpha} F^{0\alpha} U^\beta = q(\overbrace{F^{00}}^{0} U^0 - \overbrace{F^{0i}}^{-E^i} U^i)$$

$$= -q\,\eta_{\beta\alpha}\,\delta^\alpha_i E^i U^\beta = -q\,\eta_{\beta i} E^i U^\beta = +q\,\delta^i_\beta E^i U^\beta$$

$$= q E^i U^i = q \vec{E}\cdot\vec{U} = q\gamma\, \vec{E}\cdot\vec{u}$$

ii) $\alpha = i$: $K^i = q F^i{}_\beta U^\beta = q\, \eta_{\alpha\beta} F^{i\alpha} U^\beta$

$$= q(F^{i0}U^0 - F^{ij}U^j) \underset{\substack{F^{i0}=E^i \\ F^{ij}=-\epsilon^{ij}{}_k B^k}}{=} q(E^i\gamma - (-\epsilon^{ij}{}_k B^k)\gamma u^j)$$

$$= q\gamma(E^i + \epsilon^{ij}{}_k u^j B^k) = q\gamma(\vec{E} + \vec{u}\times\vec{B})^i = \gamma f^i$$

$$\boxed{\vec{K} = \gamma \vec{f}}$$

Obs: $\boxed{K^\alpha = \frac{dp^\alpha}{d\tau}} =$

$$\begin{cases} q\gamma\vec{E}\cdot\vec{u} = \frac{d(m\gamma)}{d\tau} \overset{\frac{dt}{d\tau}=\gamma}{\Rightarrow} q\vec{E}\cdot\vec{u} = m\frac{d\gamma}{dt} = \frac{d}{dt}(m\gamma - \overset{\text{reposo}}{m}) = \frac{dT}{dt} \\ \gamma\vec{f} = \frac{d(\gamma\vec{u})}{d\tau} = \frac{dt}{d\tau}\frac{d(\gamma\vec{u})}{dt} = \gamma\frac{d(\gamma m\vec{u})}{dt} \end{cases}$$

$$\to \boxed{\frac{d\vec{p}}{dt} = \vec{f}}$$

8b) Cómo transforma $\vec{f}$? $\boxed{K'^\mu = \Lambda^\mu{}_\nu K^\nu}$

$$\Lambda_{(\text{en } x)} = \begin{pmatrix} \gamma & -\gamma v^x & 0 & 0 \\ -\gamma v^x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$f'^i = \gamma(v)^{-1} K'^i = \gamma^{-1}\Lambda^i{}_\mu K^\mu = \gamma^{-1}[\Lambda^i{}_0 K^0 + \Lambda^i{}_j K^j]$$

$$\underset{\substack{\Lambda\ \text{boost} \\ \text{con } \vec{v}\parallel\vec{f}}}{=} \gamma^{-1}[-\gamma v^i q\gamma\vec{E}\cdot\vec{u} + \gamma \underbrace{K^i}_{\gamma f^i}] = \gamma[v^i q\vec{E}\cdot\vec{u} + f^i]$$

o sea $\vec{f}' = \gamma(\vec{f} + q\vec{v}(\vec{E}\cdot\vec{u}))$

8c) Relación con $\frac{d\vec{p}}{dt}$:

$$\frac{d\vec{p}}{dt} = \gamma^{-1}\frac{d\vec{p}}{d\tau} \to \underbrace{\frac{d\vec{p}}{d\tau}}_{\text{Trans. como } \vec{K}=\gamma\vec{f}} = \gamma \underbrace{\frac{d\vec{p}}{dt}}_{\text{Trans. igual}}$$

9) Ecuaciones de movimiento a partir de la acción

$$S = -m \int \sqrt{g_{\alpha\beta}\, dx^\alpha dx^\beta} - q \int A_\alpha dx^\alpha$$

$$= -m \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\, dt - q \int A_\alpha \frac{dx^\alpha}{dt} dt$$

$$= -m \int \sqrt{1-u^2}\, dt - q \int (\phi - \vec{A}\cdot\vec{u})\, dt \quad \rightarrow \text{ en 3D }, \; q^i = x^i$$

$$\mathcal{E}\text{-}\mathcal{L}: \frac{d}{dt}\frac{\delta L}{\delta u_i} - \frac{\delta L}{\delta x_i} = 0 = \frac{d}{dt}\left[-\frac{m}{2}\frac{(-2u^i)}{\sqrt{1-u^2}} + q A^i\right] - \left(-q\partial^i\phi + q\partial^i(\vec{A}\cdot\vec{u})\right)$$

$$0 = m\frac{d}{dt}(\gamma u^i) + q\frac{d}{dt}A^i + q\partial^i\phi - q\partial^i(\vec{A}\cdot\vec{u})$$

$$0 = \frac{d}{dt}(\vec{p}) + q(\partial_t\vec{A} + (\vec{u}\cdot\vec{\nabla})\vec{A}) + q\vec{\nabla}\phi - q\vec{\nabla}(\vec{A}\cdot\vec{u})$$

$$\vec{\nabla}(\vec{u}\cdot\vec{A}) = (\vec{u}\cdot\vec{\nabla})\vec{A} + \underbrace{(A\cdot\vec{\nabla})\vec{u}}_{0} + \vec{u}\times(\vec{\nabla}\times\vec{A}) + \vec{A}\times\underbrace{(\vec{\nabla}\times\vec{u})}_{0} = (\vec{u}\cdot\vec{\nabla})\vec{A} + \vec{u}\times(\vec{\nabla}\times\vec{A})$$

$$\vec{\nabla}(\vec{u}\cdot\vec{A})\Big|_{\partial u=0} - (\vec{u}\cdot\vec{\nabla})\vec{A} = \vec{u}\times(\vec{\nabla}\times\vec{A})$$

$$0 = \frac{d\vec{p}}{dt} + q(\partial_t\vec{A} - \vec{u}\times\overbrace{(\vec{\nabla}\times\vec{A})}^{\vec{B}} + \vec{\nabla}\phi) = \frac{d\vec{p}}{dt} + q(-\vec{E} - \vec{u}\times\vec{B})$$

$$\boxed{\frac{d\vec{p}}{dt} = \vec{f}}$$

Alternativamente, de manera covariante:

$$S = -m \int \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\, d\tau - q\int A_\alpha \frac{dx^\alpha}{d\tau} d\tau \qquad ; \quad U\cdot U = 1$$

$$\mathcal{E}\text{-}\mathcal{L}: \quad \frac{d}{d\tau}\frac{\delta L}{\delta U^\alpha} - \frac{\delta L}{\delta x^\alpha} = 0$$

$$\frac{d}{d\tau}\left[-\frac{m}{2}\frac{2U_\alpha}{\sqrt{U\cdot U}} - qA_\alpha\right] + q\partial_\alpha A_\beta U^\beta = 0$$

$$-m\frac{d}{d\tau}\left(\frac{U_\alpha}{\sqrt{U\cdot U}}\right) - q\frac{d}{d\tau}A_\alpha(x) + q\partial_\alpha A_\beta U^\beta = 0$$

$$-m\left(\frac{dU_\alpha}{d\tau}\frac{1}{\sqrt{U\cdot U}} + \left(-\frac{1}{2}\right)\frac{U_\alpha\, 2U_\beta \frac{dU^\beta}{d\tau}}{(U\cdot U)^{3/2}}\right) - qU^\beta\partial_\beta A_\alpha + q\partial_\alpha A_\beta U^\beta = 0$$

$$-m\left(\frac{a_\alpha}{\sqrt{U\cdot U}} - U_\alpha\frac{U_\beta a^\beta}{(U\cdot U)^{3/2}}\right) + q\underbrace{(\partial_\alpha A_\beta - \partial_\beta A_\alpha)}_{F_{\alpha\beta}}U^\beta = 0$$

Y ahora, $\underset{(a\cdot U=0)}{\overset{(\tau=\text{tiempo propio})}{U\cdot U = 1}}$ : $\quad -m a_\alpha + qF_{\alpha\beta}U^\beta = 0 \quad \Rightarrow m a^\alpha = qF^\alpha{}_\beta U^\beta =: K^\alpha$

$$U\cdot U = \left(\frac{dx^0}{dt'}\right)^2 - \left(\frac{d\vec{x}}{dt'}\right)^2 = \left(\frac{dt}{dt'}\right)^2 - \left|\frac{d\vec{x}}{dt}\right|^2\left(\frac{dt}{dt'}\right)^2 = \left(\frac{dt}{dt'}\right)^2(1-u^2) \overset{t'=\tau}{=} \gamma^2(1-u^2) = 1$$