Teoría BCS

Lecture Notes: BCS theory of superconductivity

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1-Una intreacción atractiva entre fermiones promueve la formación de pares
2-Origen de la interacción
3-Formalismo de egunda cuantificación
4-Teoría BCS

$$\left[-\frac{\hbar^2\nabla^2_{\mathbf{r}_1}}{2m}-\frac{\hbar^2\nabla^2_{\mathbf{r}_2}}{2m}+V\left(\mathbf{r}_1-\mathbf{r}_2\right)\right]\Psi\left(\mathbf{r}_1,\mathbf{r}_2\right)=E\Psi\left(\mathbf{r}_1,\mathbf{r}_2\right)$$

Donde V es atractivo

$$\left[-\frac{\hbar^2\nabla^2_{\mathbf{R}}}{2m^*}-\frac{\hbar^2\nabla^2_{\mathbf{r}}}{2\mu}+V\left(\mathbf{r}\right)\right]\Psi\left(\mathbf{r},\mathbf{R}\right)=E\Psi\left(\mathbf{r},\mathbf{R}\right)$$

$$\tilde{E}=E-\frac{\hbar^2K^2}{2m^*}$$

En el centro de masas

$$\psi(\mathbf{k}) = \int d^3r\, \psi(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}$$

$$\int \frac{d^3k'}{(2\pi)^3} V(\mathbf{k}-\mathbf{k}')\,\psi(\mathbf{k}') \;=\; (E - 2\varepsilon_{\mathbf{k}})\,\psi(\mathbf{k})$$

Estado ligado=Una solución con $E < 2\varepsilon_{\mathbf{k}}$

$$\Delta(\mathbf{k}) = (E - 2\varepsilon_{\mathbf{k}})\,\psi(\mathbf{k})$$

$$\Delta(\mathbf{k}) = -\int \frac{d^3k'}{(2\pi)^3} \frac{V(\mathbf{k}-\mathbf{k}')}{2\varepsilon_{\mathbf{k}'} - E}\,\Delta(\mathbf{k}')$$

Suposiciones sobre el potencial

$$V(\mathbf{k}-\mathbf{k}') = -V_0$$

$$\varepsilon_{\mathbf{k}'} - \varepsilon_F, \varepsilon_{\mathbf{k}} - \varepsilon_F < \hbar\omega_D.$$

$$\hbar\omega_D \ll \varepsilon_F$$

$$
\begin{aligned}
\Delta &= V_0 \rho(\varepsilon_F) \Delta \int_{\varepsilon_F}^{\varepsilon_F + \omega_D} \frac{d\varepsilon}{2\varepsilon - E} \\
\frac{2}{V_0 \rho(\varepsilon_F)} &= \ln\left(\frac{2\varepsilon_F - E + 2\omega_D}{2\varepsilon_F - E}\right)
\end{aligned}
$$

$$E_b = 2\omega_D e^{-\frac{2}{V_0 \rho(\varepsilon_F)}}$$

NB

1-El resultado es NO perturbativo en el potencial

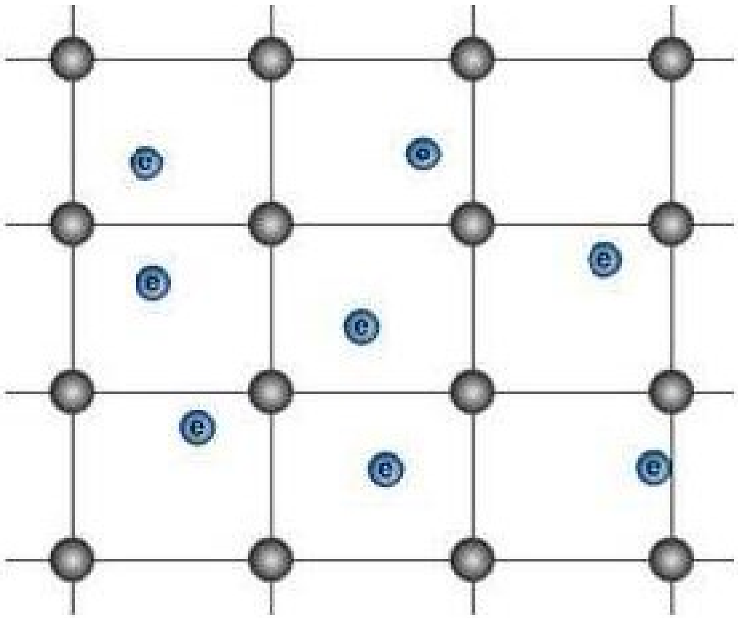
2-El estado ligado existe aún si V es débil

3-El resultado depende de haber asumido que V actuaba solo cerca de la energá de Fermi

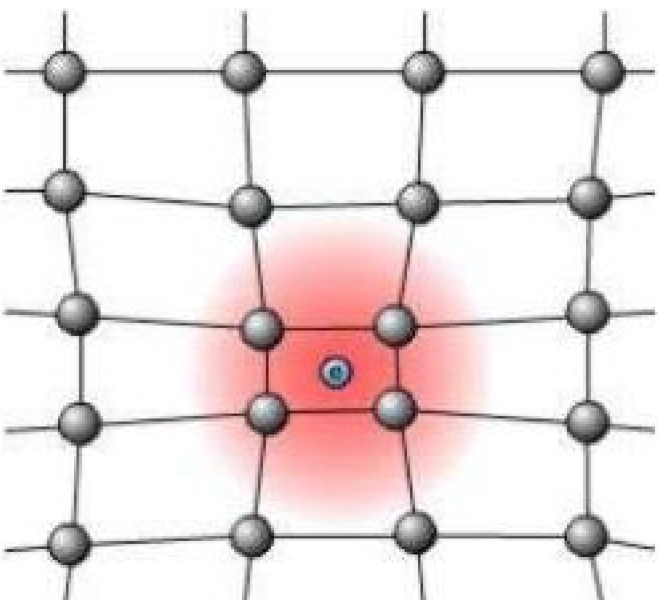
Para tener una teoría microscópica es necesario explicar

1-Cuál es origen del potencial atractivo entre los electrones?

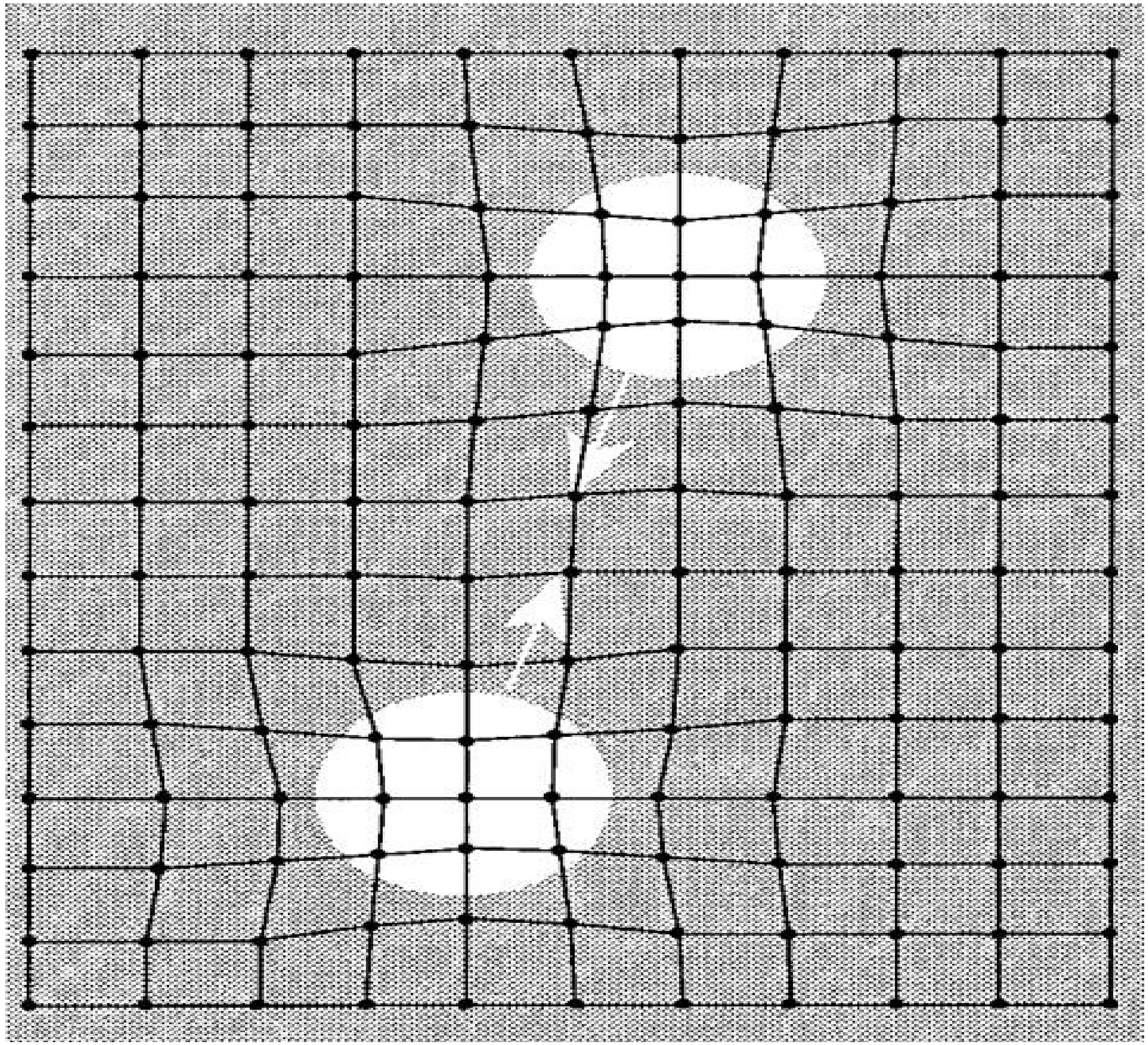
2-Cómo es la situación en el full many body picture?



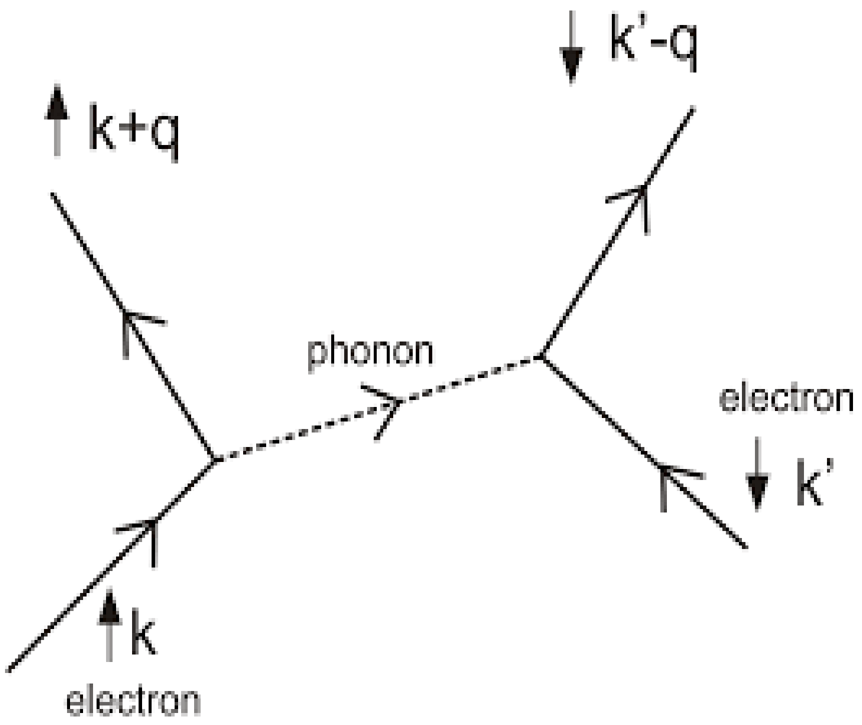
$$V(\mathbf{r}-\mathbf{r}') = \frac{e^2}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}$$



$$V(\mathbf{r}-\mathbf{r}') = \frac{e^2}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|} e^{-|\mathbf{r}-\mathbf{r}'|/r_{TF}}$$



INTREACCION FONON-ELECTRON



$$V_{eff}(\omega) = \left|g_{eff}\right|^2 \frac{1}{\omega^2 - \omega_D^2}$$

Hamiltoniano BCS

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$

$$\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu.$$

Mean field decoupling

$$\left\langle c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle \approx \left\langle c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \right\rangle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} + c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \left\langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle - \left\langle c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \right\rangle \left\langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle$$

El término cuártico, se convierte en cuadrático en operadores

DEFINIMOs LA FUNCION DEL GAP

$$\Delta_{\mathbf{k}} = -\frac{1}{N} \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left\langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle$$

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + \Delta^{*}_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right) + \sum_{\mathbf{k}} \Delta_{\mathbf{k}} \left\langle c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \right\rangle$$

El Hamiltoniano es cuadrático pero no es diagonal!

$$\begin{aligned} c_{\mathbf{k}\uparrow} &= u^{*}_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \gamma^{\dagger}_{-\mathbf{k}\downarrow} \\ c^{\dagger}_{-\mathbf{k}\downarrow} &= u_{\mathbf{k}} \gamma^{\dagger}_{-\mathbf{k}\downarrow} - v^{*}_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} \end{aligned}$$

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$$

Un potencial atractivo entre electrones que actúa en las cercanías de la energía de fermi da origen a un estado ligado

PAR DE COOPER

$$H = H_0 + H_1 + H_2$$

$$\begin{aligned}
H_0 &= \sum_{\mathbf{k}} \left[2\xi_{\mathbf{k}} \left| v_{\mathbf{k}} \right|^2 - \Delta_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}^* - \Delta_{\mathbf{k}}^* u_{\mathbf{k}}^* v_{\mathbf{k}} + \Delta_{\mathbf{k}} \left\langle c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger \right\rangle \right] \\
H_1 &= \sum_{\mathbf{k}} \left[\xi_{\mathbf{k}} \left(\left| u_{\mathbf{k}} \right|^2 - \left| v_{\mathbf{k}} \right|^2 \right) + \Delta_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}^* + \Delta_{\mathbf{k}}^* u_{\mathbf{k}}^* v_{\mathbf{k}} \right] \left(\gamma_{\mathbf{k}\uparrow}^\dagger \gamma_{\mathbf{k}\uparrow} + \gamma_{-\mathbf{k}\downarrow}^\dagger \gamma_{-\mathbf{k}\downarrow} \right) \\
H_2 &= \sum_{\mathbf{k}} \left[\left(2\xi_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} - \Delta_{\mathbf{k}} u_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^* v_{\mathbf{k}}^2 \right) \right] \left(\gamma_{\mathbf{k}\uparrow}^\dagger \gamma_{-\mathbf{k}\downarrow}^\dagger \right) + \text{h.c.}
\end{aligned}$$

Pedimos entonces que el
Coeficiente se anule

$$2\xi_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} - \Delta_{\mathbf{k}} u_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^* v_{\mathbf{k}}^2 = 0$$

$$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}}\right)$$

$$|v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}}\right)$$

$$H = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \gamma^{\dagger}_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} + E_0$$

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$$

$$E_0 = \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - E_{\mathbf{k}} + \Delta_{\mathbf{k}} \left\langle c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \right\rangle \right)$$

Todo parece resuelto, salvo que NO sabemos quien es $\Delta_{\mathbf{k}}$

$$
\begin{aligned}
\gamma_{\mathbf{k}\uparrow} &= u_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} \\
\gamma^{\dagger}_{-\mathbf{k}\downarrow} &= u^{*}_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} + v^{*}_{\mathbf{k}} c_{\mathbf{k}\uparrow}
\end{aligned}
$$

$\Delta_{\mathbf{k}} \to 0$ $\xi_{\mathbf{k}} > 0$ $|u_{\mathbf{k}}|^2 \to 1$

$\xi_{\mathbf{k}} < 0$ a $|v_{\mathbf{k}}|^2 \to 1$

ESTADO FUNDAMENTAL

$$\gamma_{\mathbf{k}\sigma}\,|\Psi_{\mathrm{BCS}}\rangle = 0$$

$$u_{\mathbf{k}} c_{\mathbf{k}\uparrow}\,|\Psi_{\mathrm{BCS}}\rangle = v_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow}\,|\Psi_{\mathrm{BCS}}\rangle$$

$$|\Psi_{\mathrm{BCS}}\rangle = \mathcal{N}\prod_{\mathbf{q}} \mathrm{e}^{\alpha_{\mathbf{q}} c^{\dagger}_{\mathbf{q}\uparrow} c^{\dagger}_{-\mathbf{q}\downarrow}}\,|0\rangle$$

Algebra-------------------------------->>>>>>>>>

$$|\Psi_{\mathrm{BCS}}\rangle = \mathcal{N}\prod_{\mathbf{k}} \mathrm{e}^{\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow}}\,|0\rangle$$

$$|\Psi_{\mathrm{BCS}}\rangle = \mathcal{N}\prod_{\mathbf{k}} \left(1 + \frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow}\right)|0\rangle$$

$$|\Psi_{\mathrm{BCS}}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow}\right)|0\rangle$$

THE GAP EQUATION

$$\Delta_{\mathbf{k}} = -\frac{1}{N}\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left\langle \left(u^*_{\mathbf{k}'} \gamma_{-\mathbf{k}'\downarrow} - v_{\mathbf{k}'} \gamma^\dagger_{\mathbf{k}'\uparrow} \right) \left(u^*_{\mathbf{k}'} \gamma_{\mathbf{k}'\uparrow} + \gamma^\dagger_{-\mathbf{k}'\downarrow} \right) \right\rangle$$

$$\Delta_{\mathbf{k}} = -\frac{1}{N}\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} u^*_{\mathbf{k}'} v_{\mathbf{k}'} \left(\left\langle \gamma_{-\mathbf{k}'\downarrow} \gamma^\dagger_{-\mathbf{k}'\downarrow} \right\rangle - \left\langle \gamma^\dagger_{\mathbf{k}'\uparrow} \gamma_{\mathbf{k}'\uparrow} \right\rangle \right)$$

Como los Bololiubones siguen una estadística Fermi Dirac

$$\left\langle \gamma^\dagger_{\mathbf{k}'\uparrow} \gamma_{\mathbf{k}'\uparrow} \right\rangle = \left\langle \gamma^\dagger_{-\mathbf{k}'\downarrow} \gamma_{-\mathbf{k}'\downarrow} \right\rangle = \frac{1}{e^{\beta E_{\mathbf{k}'}} + 1}$$

$$V_{\mathbf{k}\mathbf{k}'} = -V_0$$

$$\Delta_{\mathbf{k}} = -\frac{1}{N}\sum_{\mathbf{k}'}\frac{V_{\mathbf{k}\mathbf{k}'}\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}}\tanh\left(\frac{E_{\mathbf{k}'}}{2k_BT}\right)$$

$$|\xi_{\mathbf{k}}|\,,|\xi_{\mathbf{k}'}| < \hbar\omega_D$$

Asumimos que $\Delta_{\mathbf{k}} = \Delta$

S-wave gap equation

$$1 = \frac{V_0}{N}\sum_{k<k_D}\frac{1}{2E_{\mathbf{k}}}\tanh\left(\frac{E_{\mathbf{k}}}{2k_BT}\right)$$

$$1 = V_0\int_{-\hbar\omega_D}^{\hbar\omega_D}\frac{\rho(\varepsilon)\,d\varepsilon}{2\sqrt{\varepsilon^2+\Delta^2}}\tanh\left(\frac{\sqrt{\varepsilon^2+\Delta^2}}{2k_BT}\right)$$

$$1 = V_0\rho_F\int_{0}^{\hbar\omega_D}\frac{d\varepsilon}{\sqrt{\varepsilon^2+\Delta^2}}\tanh\left(\frac{\sqrt{\varepsilon^2+\Delta^2}}{2k_BT}\right)$$

$T = 0$

$$\frac{1}{V_0 \rho_F} = \text{arcsinh}\left(\frac{\hbar\omega_D}{\Delta_0}\right)$$

$$\Delta_0 \quad = \quad 2\hbar\omega_D \, \mathrm{e}^{-\frac{1}{V_0 \rho_F}}$$

T_c $\Delta \to 0,$

$$T_c = \frac{2\mathrm{e}^{\gamma_E}}{\pi} \frac{\hbar\omega_D}{k_B} \, \mathrm{e}^{-\frac{1}{V_0 \rho_F}}$$

$$\frac{\Delta_0}{k_B T_c} \approx 1.76$$

$$T_c \propto \omega_D \propto M^{-1/2},$$