Strongly correlated systems
in atomic and condensed matter physics

Lecture notes for Physics 284

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Chapter 10

Topological Phases of matter

10.1 Introduction

Exploration of different phases of matter has been the focus of condensed matter physics, because not only they are the origin of many intriguing phenomena in nature, but also the deep understanding of such phases led to tremendous advance in technologies. Cold atom systems further extended the understanding of the phases such as superfluid phase and Mott insulating phase through the unprecedented control of the systems. We have seen in the previous lectures how cold atom systems can study various phases with their unique probes such as quantum noise, that are often not easily available in condensed matter systems.

Many phases, including superconductivity/superfluidity and ferro/anti-ferromagnetism we have studied in this course, can be understood in terms of the theory of spontaneous symmetry breaking [1]. On the other hand, novel phases of matter that cannot be understood in the paradigm of spontaneous symmetry breaking are found through the experimental discoveries of integer and fractional quantum Hall effects in 1980s[2, 3]. These phases are characterized not by order parameters but rather by topology of ground state wave functions. There are two notable property of such topological phases. One is that topological phases are characterized by integer numbers called topological invariants, and therefore, direct manifestations of topological phases show a quantization of physical observables, corresponding to the topological invariants. We will see below one such phenomenon through the example of integer quantum Hall(IQH) phase where Hall conductivity is quantized to be an integer multiple of e^2/h. Another important property is the robustness of such topological properties due to the fact that topology is a characterization of "global shape," and it does not change under a small, local deformations. This robustness allows the observation of the quantization of the Hall conductivity in IQH systems independent of the shape of the materials as well as the amount of impurities present in the

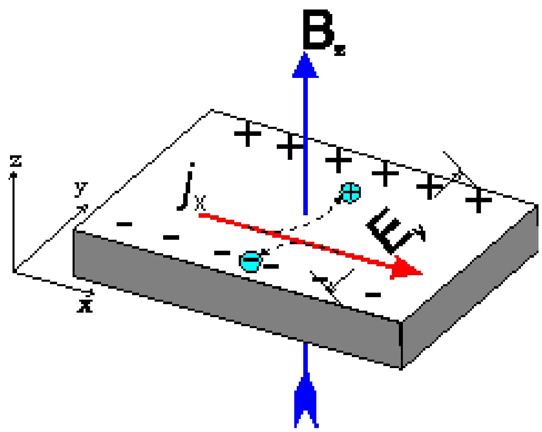

Figure 10.1: The configuration we consider in this lecture. Two dimensional electron gas is in $x-y$ plane and magnetic field is pointing in z direction. The Hall current runs along x axis. reference: http://www.warwick.ac.uk/ phsbm/qhe.htm

material. Topological robustness can also protect quantum information from environment, and it has been suggested topological phases can host quantum computations[4].

The study of topological phases of matter is still an early stage, but its interest as a new phase of matter has been quickly increasing. In this lecture, we will study an example of integer quantum Hall effects, and see the connection between the physical phenomena and its topological origin.

10.2 Integer quantum Hall effect

10.2.1 phenemenon

Integer quantum Hall effect is observed in two dimensional electron gas in the presence of strong transverse magnetic field(see Fig.10.1). Let the two dimensional plane of electrons to be $x-y$ plane, and the direction of magnetic field to be z. We apply a weak electric field in y direction, and observe the conduction of current in x and y direction, given by

$$\begin{pmatrix} J^x \\ J^y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E^x \\ E^y \end{pmatrix} \tag{10.1}$$

where $E^y = E$ and $E^x = 0$. Because the electric field is in y direction, J^y reflects the longitudinal conductivity, and J^x is Hall conductivity.

From classical consideration, it is easy to obtain the Hall conductivity. The electrons that run at velocity v in $+x$ direction feels the force evB/c in y di-

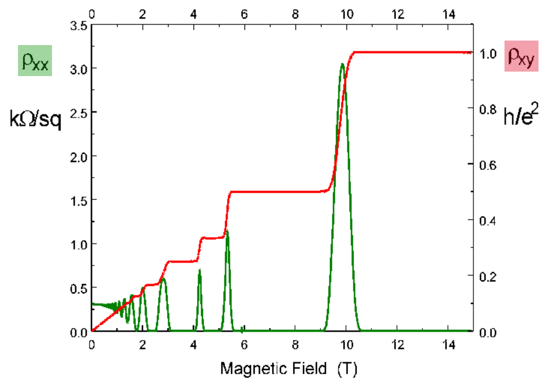

Figure 10.2: The Hall resistivity of integer quantum Hall system. The red line is ρ_{xy} and the green line is ρ_{xx}. Pronounced plateau is clearly observable. reference: http://www.warwick.ac.uk/ phsbm/qhe.htm

rection, and get deflected on the edges. The accumulation of charges on the edge continues until the electrostatic force coming from the accumulated charge equals the force coming from the magnetic field. Therefore, E in y direction is given by $E = -vB/c$. At this point, the current along x direction is given by $J^x = -env$ where n is the density of conduction electrons in the system. Therefore, the Hall conductivity is $\sigma_{xy} = \frac{enc}{B}$. In this classical, simple analysis, we see that the Hall conductivity is inversely proportional to B.

The actual experimental result of resistivity, which is the inverse of conductivity, is shown in Fig. 10.2. We see that for small magnetic field B, the Hall resistivity indeed linearly increases with B, in agreement with the classical Hall resistance we computed above. However for strong magnetic field, we see that the transverse resistance ρ_{xy} becomes flat for a range of values of B. Moreover, as it turns out, these plateaux occur precisely at values $\frac{h}{e^2 s}$ with integers s. The quantization of Hall resistance is so precise (more than 1 part in 10^7!) that it can be used as the standard of fundamental constant $\alpha = \frac{e^2}{hc}$. Because fundamental constant α can also be determined by the measurements of electron g factor (the ratio of spin angular momentum and magnetic moment) together with Quantum Electro Dynamics calculations, integer quantum Hall effect is even useful for the test of QED[5].

Because the longitudinal resistivity is zero at each plateau(see Fig.10.2), we find that the resistivity tensor and conductivity tensors at the plateau take the form

$$\begin{pmatrix} \rho_{xx} & \rho_{yx} \\ \rho_{xy} & \rho_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \frac{h}{e^2 s} \\ \frac{h}{e^2 s} & 0 \end{pmatrix} \tag{10.2}$$

$$\begin{pmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \frac{e^2 s}{h} \\ \frac{e^2 s}{h} & 0 \end{pmatrix} \tag{10.3}$$

where s is an integer which differs for each plateau. In this lecture, we will try to understand this quantization of conductance in two different ways. First, the quantizations of Hall conductivity can be simply explained by considering free electrons in the presence of strong magnetic field. While this explanation gives an explicit demonstration of quantized Hall conductance in this ideal situation, it does not explain why such a phenomenon should be robust against disorders and interactions. There are many different ways to account for this rich physics, but we will take the approach from topology, and show that a hidden topological structure of this phenomenon gives a generic explanation of the robustness.

10.2.2 Hall conductivity of translationally invariant system

Hall conductivity of infinite two dimensional electron system in the presence of magnetic field can be rather easily obtained by going to a moving frame[5]. As before, suppose that the electric field points in y direction with magnitude E. If we move along x direction with velocity $v = \frac{-cE}{B}$, this electric field disappears (in the lab frame, the force on electrons coming from the electric field is cancelled with the force from magnetic field). Since there is no preferred direction in this moving frame, it is clear that there is no current in the moving frame. This implies, in return, that the current in the lab frame must be given by $J^x = -env = \frac{encE}{B}$, where n is the density of electrons.

This expression of Hall current obtained in this fashion is valid not only in classical treatment but also in quantum treatment as well, because we only used the property of Galilean invariance to derive the result. Now from this expression, it is seen that the quantization of Hall conductance is in principle possible, if, for a range of magnetic fields, the density of electrons linearly scales with magnetic field, *i.e.* $n \propto B$. In the following sections, we will see how this occurs in

10.2.3 Quantum mechanical treatment of free electrons in the magnetic field

The Hamiltonian of free electrons in the presence of magnetic field can be written as

$$H = \frac{1}{2m}\left(-i\vec{\partial} - \frac{e}{c}\vec{A}\right)^2 \tag{10.4}$$

where we take the symmetric gauge for the vector potential and choose $\vec{A} = \frac{B}{2}(y, -x)$. This vector potential produces the magnetic field in z direction, *i.e.* $\vec{B} = \nabla \times \vec{A} = B\hat{z}$. Here we set $\hbar = 1$. (One can confirm that this is a correct Hamiltonian for the problem by, for example, writing the equation of motion and checking that it correctly reproduces the equation of motion of the particle in the presence of magnetic field).

We can find the kinetic momenta π_x and π_y through

$$\pi_x = m\frac{dx}{dt} = -i\partial_x - \frac{e}{c}A_x \tag{10.5}$$

$$\pi_y = m\frac{dy}{dt} = -i\partial_y - \frac{e}{c}A_y \tag{10.6}$$

It is important to note that the kinetic momenta for x and y do not commute because of the coordinate dependence of $\vec{A}$ and the commutation relation is given by

$$[\pi_x, \pi_y] = -i\frac{eB}{c} \tag{10.7}$$

Here we define the characteristic length scale of the system l through $l^2 = \frac{c\hbar}{eB}$.

Let us define the "raising" and "lowering" operator through the relation

$$a^\dagger = \frac{l}{\sqrt{2}}(\pi_x + i\pi_y) \quad a = \frac{l}{\sqrt{2}}(\pi_x - i\pi_y) \tag{10.8}$$

They indeed satisfy the correct commutation $[a, a^\dagger] = 1$ as one can explicitly check. Then it is easy to see that the Hamiltonian in Eq. (10.4) is given by

$$H = \frac{\hbar\omega_c}{2}(a^\dagger a + aa^\dagger) \tag{10.9}$$

where $\omega_c = \frac{eB}{cm}$ is again cyclotron angular frequency.

From this expression of Hamiltonian, it is clear that the energy of free electrons in the presence of magnetic field is quantized to be $\hbar\omega_c(n + 1/2)$ where n is an integer. These states are often referred to as states in nth Landau levels. The lowest energy state, or states in the first Landau level, is determined by the condition $a|G\rangle = 0$. A simple way to obtain the ground states is to use the complex variable $z = x + iy$ and its derivative $\partial_z = (\partial_x - i\partial_y)/2$. Notice that its conjugates $\bar{z} = x - iy$ and $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$ commute, such that $\partial_z \bar{z} = \partial_{\bar{z}} z = 0$. Using these variables, the annihilation operator a can be written as

$$a = \frac{-i}{\sqrt{2}}\left(\frac{\bar{z}}{2l} + 2l\partial_z\right) \tag{10.10}$$

and therefore, the wavefunctions of a ground state are given by

$$\psi_{0,0}(z, \bar{z}) = \frac{1}{\sqrt{2\pi l^2}}e^{-z\bar{z}/4l^2} \tag{10.11}$$

This wave function has the probability distribution that is peaked at the origin $(x, y) = (0, 0)$ and exponentially decays away from the origin within the length scale of l.

Notice that because a does not involve the derivative of $\partial_{\bar{z}}$, any powers of $\bar{z}$ can be multiplied to $\psi_{0,0}$ and still they represent ground states with the same energy $\hbar\omega_c/2$. Such degeneracy is what we have already seen in classical picture;

it is the degeneracy coming from the choice of center coordinates for cyclotron motion. All the ground state wave functions are then given by

$$\psi_{0,m}(z,\bar{z}) = \frac{1}{\sqrt{2\pi l^2 (2l^2)^m m!}} \bar{z}^m e^{-z\bar{z}/4l^2} \tag{10.12}$$

It is straightforward to work out that these states are concentrated at radius $r_m = l\sqrt{2(m+1)}$. These are quantum analogue of cyclotron motion obtained in classical picture above. One heuristic way to understand r_m is the quantization condition coming from Abrahanov-Bohm effect. When electrons makes a circle in the presence of magnetic field, it accumulates the phase corresponding to the magnetic flux enclosed by the motion, $\frac{e}{c}\Phi$. Because electron wave function has to be singled-valued, this has to be an integer multiple of 2π, so that we require $\frac{e\pi r_m^2 B}{c} = 2\pi(m+1)$, leading to the value $r_m^2 = 2(m+1)l^2$.

With this explicit solutions in hand, it is now possible to work out many properties of electrons in magnetic fields. Here, we contend with calculating the density of electrons in the ground states. If we fill the ground states, the probability to find an electron at a given point becomes

$$\sum_m^\infty |\psi_{0,m}(z,\bar{z})|^2 = \sum_m^\infty \frac{1}{2\pi l^2 (2l^2)^m m!} |z|^{2m} e^{-|z|^2/2l^2} \tag{10.13}$$

$$= \frac{1}{2\pi l^2} \sum_m^\infty \frac{r^m}{m!} e^{-r} \quad r = \frac{|z|^2}{2l^2} \tag{10.14}$$

$$= \frac{1}{2\pi l^2} \tag{10.15}$$

Therefore, the density of electron in the ground state is given by $\frac{1}{2\pi l^2} = \frac{Be}{2\pi c}$. The density in the ground state is proportional to the magnetic field. It is straightforward to show that each filled Landau level contributes the density of electron $\frac{1}{2\pi l^2}$.

Now in order to account for the integer quantum Hall effect, we note that experiments are often done at a constant chemical potential. Suppose we start from the system which has N filled Landau levels, and the chemical potential lies between $\hbar\omega_c(N+1/2)$ and $\hbar\omega_c(N+1+1/2)$(see Fig.10.3). The density of electrons in the thermodynamic limit is then given by $\frac{N}{2\pi l^2}$. According to the argument given in the previous section, this implies Hall conductance is Ne^2/h! As we increase the magnetic field, the energies of Landau levels increase as magnetic field is increased($\omega_c \propto B$). However, since chemical potential lies in the gap, the number of filled Landau levels does not change and therefore, the Hall conductance is constant at the value Ne^2/h. Moreover, notice that when chemical potential lies inbetween the Landau levels, the system is "band insulator" and therefore, the longitudinal conductivity must be zero. When the magnetic field is increased so that the chemical potential goes below $\hbar\omega_c(N+1/2)$, Nth Landau level is emptied, and the Hall conductance jumps from Ne^2/h to $(N-1)e^2/h$. This behavior qualitatively agrees with the experimental result presented in Fig.10.2 and the equation Eq. (10.3).

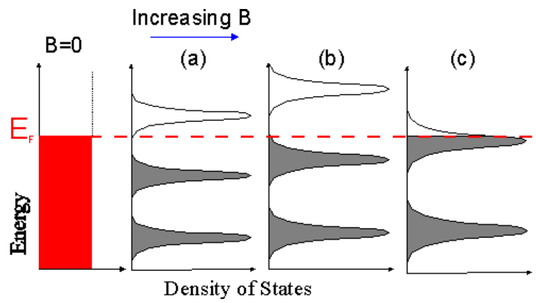

Figure 10.3: The Landau levels in the presence of disorder get broadened, but for weak disorder, the levels are still well-separated. (a), (b)The Landau levels increase its energy as magnetic field becomes strong. As long as the chemical potential lies in the gap of Landau levels, the Hall conductivity does not change and it takes the quantized value. (c) The Hall conductivity changes when the chemical potential crosses the Landau levels. reference: http://www.warwick.ac.uk/ phsbm/qhe.htm

10.2.4 Topological aspect of integer quantum Hall effect

In the previous section, we have shown the quantization of Hall conductivity for the ideal case of infinite two dimensional electron gas in the presence of magnetic field. In the real material, where the experiments were carried out, this ideal situation is far from the reality. There are, for example, effects of lattices, disorder and interactions to consider. Yet the experiments show that the quantization of Hall conductivity survives in the presence of these effects as long as their effects are not too strong.

One way to understand such robustness was provided by Thouless, Kohmoto, Niu and Nightingale. They calculated the Hall conductivity of electron gas in two dimensional lattice, and related it to the topological property of ground wave function. The topology of the ground state wavefunction is characterized by a topological invariant called Chern number, and they showed, through TKNN formula, that Hall conductivity is simply given by a fundamental constant e^2/h times the Chern number. The topological characterization of the ground state wavefunction is, as it turns out, possible even in the presence of disorder and interactions, and Hall conductivity is related in the same way to the Hall conductivity. Because Chern number only takes integer values, it is now clear that Hall conductivity has to be quantized even in the presence of disorder and interactions.

TKNN formula demonstrated that integer quantum Hall system presents a very first example where a phase is not captured by an order parameter of spontaneous symmetry breaking, but it is characterized by a topological number of a ground state. Moreover, because Chern number captures the topological structure of the ground state wave function, and topology is a global property that cannot be changed by a small perturbations to the system, a generic robustness of such Hall conductivity could be argued based on the general property of topology. Hall conductivity is, in a sense, a "peek" into the topological structure of the integer quantum Hall phase. Apart from the explanation of integer quantum Hall effect, this understanding of quantum Hall phase as a phase characterized by a topological invariant called a Chern number allows us to generalize the concept of topological phases. From this more general point of view, magnetic field is not a necessary condition to have topological phases; in fact recent study has predicted and realized other topological phases called topological insulators in the absence of magnetic field.

In the following, we explain TKNN formula. The aim here is to give a flavor of how some "topological invariant" can be defined for a given ground state. We will give a simpler example where Chern number can be understood in an intuitive fashion.

A short introduction to topology

Before we describe the topology of two dimensional electron gas, let's first get a feeling for what topology is, and how physics could have anything to do with it.

For simplicity, here we consider the topology of a closed line or a loop in two dimensional plane, with a hole at the origin. The topology of the closed line can be characterized by an integer number given by the winding number of the line around the origin(Fig.10.4). This winding number is invariant under the continuous deformation of the loop, provided that it does not cross the hole at the origin. Because of this property, the winding number is called "topological invariant." Topology, in a general sense, is a property of an object or shape which is invariant under continuous transformations. As we have seen in the example of the winding number, topology is often characterized by discrete values such as integers, and the change of the topology is usually "a sudden change."

The connection of the concept of topology described above with physics is made by identifying the loop as a ground state wave function described by some periodic parameter(see Fig.10.5). For example, we can think of non-interacting one dimensional lattice system, whose ground state is the filled lowest band. Then the ground state is made up of eigenstates labeled by quasi-momentum k, which is periodic. The two dimensional plane is an abstract space in which eigenstates of the system resides. The hole at the origin represents a "topological" phase transition point. The winding number of the loop can be related to some physical observables such as Hall conductivity. Through this relation, the observable becomes a direct consequence of topological structure of the ground state wave function.

The ground state wave function can be continuously deformed by some perturbations such as interactions or disorder. However, the topology of a loop is robust against such continuous change, and the winding number does not change for weak perturbations. For strong perturbations, the ground state "loop" can cross the topological phase transition point, and change its topology.

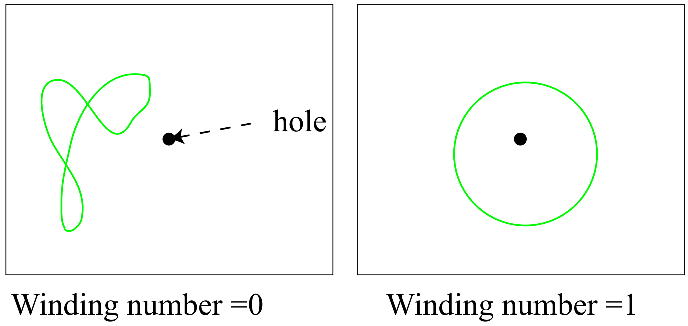

Figure 10.4: Illustration of the topology of a loop in two dimensional plane with a hole at the origin. Its topological structure is captured by the winding number of the loop around the circle. Such winding number is invariant under the continuous deformations of the loop, provided that the loop does not cross the origin.

topology of two dimensional system

Here, we give an explicit expression of a topological number called Chern number for two dimensional system, and show how the structure we outlined above appear.

In order to define the topological invariant in the simplest form, we consider non-interacting electrons in two dimensional lattice. The construction of the Chern number can be extended to the case of the presence of interactions and disorder. In the presence of the lattice, eigenstates of the system are given by Bloch wave functions. Now we suppose that the chemical potential lies in the band gap, and the ground state of the system is the filled lowest m bands. The existence of an excitation gap is essential for the definition of topology. In the case of integer quantum Hall phase, we can consider, for example, filled lowest m Landau levels. Each state $|\psi_{\vec{k}},\alpha\rangle$ in the filled bands can be indexed by quasi-momentum $\vec{k}$ and band or Landau level index α. Because each state is Bloch wave function, we can write it as $|\psi_{\vec{k}},\alpha\rangle = e^{-i\vec{k}\cdot\vec{r}}|u_{\vec{k}}\alpha\rangle$ where the wave function $|u_{\vec{k}}\alpha\rangle$ is periodic in real space with period of lattice vectors. Now using this periodic part of the filled eigenstates, we can write the Chern number as

$$C = \int_{FBZ} d\vec{k}\left(\frac{\partial a_{k_y}}{\partial k_x} - \frac{\partial a_{k_x}}{\partial k_y}\right) \tag{10.16}$$

$$a_j = -i\sum_{\alpha}\langle u_{\vec{k}}\alpha|\partial_i|u_{\vec{k}}\alpha\rangle, \quad j = k_x, k_y \tag{10.17}$$

where the integration over crystal momentum runs over the first Brillouin zone, and summation over α runs over the filled bands. This expression is applicable whenever one has an insulator with filled bands, and not restricted to systems with magnetic fields. In the case of the system with N filled Landau levels, we

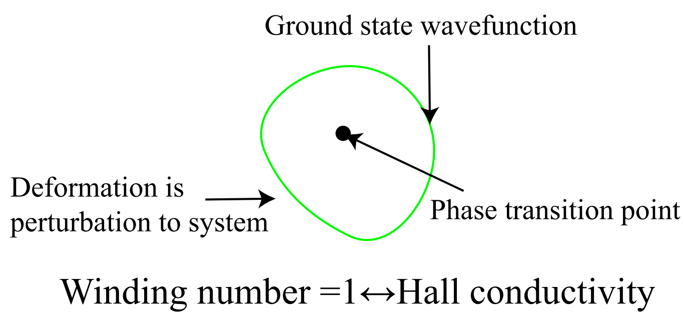

Figure 10.5: Description of how topology of a circle in a two dimensional plane is related to physics. The loop represents a ground state wave function, such as the set of single-particle eigenstates labeled by some periodic parameter. The two dimensional plane is an abstract space in which these eigenstates reside, and the hole represents a topological phase transition point. The winding number, which is the topological invariant of the ground state wave function, can be related to a physical observable such as Hall conductivity. The introduction of perturbations to Hamiltonian such as disorder and interactions lead to the deformation of the ground state "loop," but the topology is robust against such weak and continuous introduction of perturbations.

get $C = N$.

An intuitive understanding of Chern number comes from considering a two band model. We consider the following Hamiltonian in quasi-momentum space for spin 1/2 in square lattice[6]

$$H(\vec{k}) = (\sin k_x)\sigma_x + (\sin k_y)\sigma_y + (m + \cos k_x + \cos k_y)\sigma_z \tag{10.18}$$

where σ_a are the Pauli matrixes, and the lattice constant is set to 1. This model has two bands coming from the spin degrees of freedom. We consider filling the lower band, and calculate the Chern number of the lower band. Now, the eigenstate of lower band for a given quasi-momentum $\vec{k}$ can be represented as a point on a Bloch sphere. If we map the eigenstates of lower band for the whole first Brillouin zone, the eigenstates cover a certain area of Bloch sphere (see Fig. 10.6). Because of the periodicity condition of Brillouin zone, such area has to cover the whole Bloch sphere for integer number of times. This integer number is exactly the Chern number. If we write the Hamiltonian above as $H(\vec{k}) = E(\vec{k})\vec{n}(\vec{k}) \cdot \sigma$, where $\vec{n}(\vec{k})$ is a unit vector, the Chern number in Eq.(10.16) can be rewritten as

$$C = \frac{1}{4\pi} \int_{BZ} d^2k [\mathbf{n} \cdot (\partial_{k_x} \mathbf{n} \times \partial_{k_y} \mathbf{n})] \tag{10.19}$$

We urge the reader to confirm from the expression in Eq.(10.19) does measure the area covered by the eigenstates of the lower band in the first Brillouin zone.

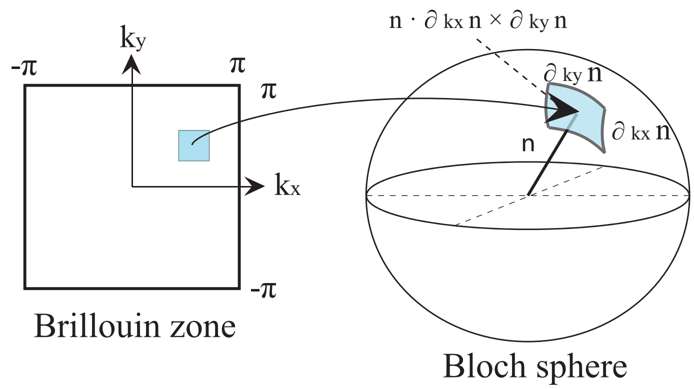

Figure 10.6: Chern number of two band model can be easily visualized as the area on the Bloch sphere covered by the lower band in the first Brillouin zone.

At $m = 1$, the lower band of the Hamiltonian in Eq.(10.18) has Chern number $= 1$. This Chern number of the lower band can change if and only if the lower band mix with the upper band through the closing of the gap. These closing of gaps signals topological (quantum) phase transition point. This occurs, for example, at $m = 0$ and $m = 2$.

10.3 Realizing topological phases with cold atoms

10.3.1 overview

As we have seen in the previous section, a strong magnetic field suppress the kinetic energy and macroscopic number of states become degenerate. This implies that the physics of electrons in a strong magnetic field is dominated by other effects such as disorder and interactions, which we ignored in the previous section. In particular, when interactions dominate the phase, it is known that a very intriguing phase of matter called fractional quantum Hall states appears[3]. This phase is argued to have excitations with non-abelian statistics, which, in return, can host topological quantum computations. Both integer quantum Hall phase as well as fractional quantum Hall phase can be categorized as "topological phases" in a sense that the phases can be characterized by topological properties. In order to probe the physics of such topological phases, it is desirable to realize the phases with a well-controlled system, such as cold atoms.

There are few different approaches to realize topological phases in cold atom systems. One obvious option is to imitate the electrons in a strong magnetic field. Particles used in cold atom systems are not charged, and therefore, they do not couple with electromagnetic fields. Mathematically equivalent effects can be obtained by rotating the systems[7, 8] or coupling the atoms with Raman

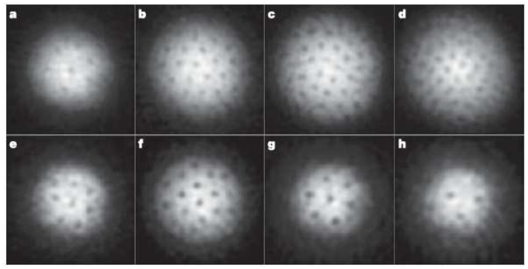

Figure 10.7: Creation of vortices in the rotating condensates. From the Wolfgang Ketterle's experiment[7]

lasers[9]. In the following section, we briefly describe these experiments.

On the other hand, the physics of topological phases can also be studied by effectively simulating the Hamiltonians with non-trivial topology by dynamically driving the system. This approach is quite unique to cold atom systems because (coherent) dynamical drive of systems with condensed matter materials is much harder. We give an example of such dynamics with topological property in the last section through a dynamical protocol called quantum walk.

10.3.2 creating effective magnetic field

rotating the condensates

Many groups have created an "effective magnetic field" by rotating a condensate. In the frame that rotates at constant speed, the particles feel both centrifugal force and Coriolis force

$$F = F_{ext} - 2m\vec{\Omega} \times \vec{v} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{10.20}$$

where the velocity is written in the rotating frame. It is clear that the Coriolis force imitates the magnetic field through the identification $\vec{B} = 2\vec{\Omega}$. In cold atoms, they rotated a Bose-Einstein condensate, and as a result, many vortices were created in analogy with the vortices of superconductors in the presence of magnetic field (see Fig.10.7[7])

Recently, using the effective magnetic field of rotations, the group of Steven Chu created a strongly correlated phase called fractional quantum Hall phase[8].

Coupling with lasers

Originally the creation of the effective magnetic field through rotations encountered a technical problem which led to the instability of the system (rotation was not perfect and the system was easily heated), and fast rotation could not be achieved. The group of Ian Spielman used a more ingenious method to create the effective magnetic field which avoids the problem of heating due the the rotations.

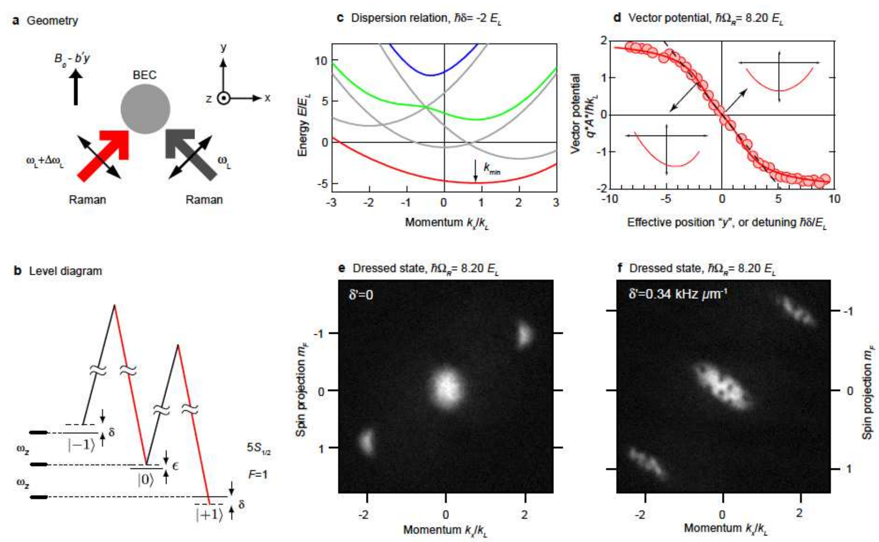

Figure 10.8: Creation of effective magnetic field through Raman coupling of atoms. The figure from the paper of Spielman's group[9]

HIs approach is to couple three different hyperfine states through Raman coupling. By using the two lasers with x-momentum difference, they achieved the dispersion $H = \frac{(\vec{k}-k_{min}\hat{x})^2}{2m}$ whose shift k_{min} is determined by detuning of the Raman laser. By changing the detuning in y spatial direction through Zeeman effect, they successfully created the magnetic field $\vec{A} = k_{min}(y)\hat{x}$ which linearly changes in y direction. This effectively creates the magnetic field. While they could not achieve the degenerate states of Landau levels, they obtained the signature of magnetic field through the observation of vortices (see Fig.10.8)[9].

10.3.3 Dynamical simulation of topological phases

In cold atom systems, coherence is maintained for a long time, and quantum dynamics is possible. Because of this long coherence, it is possible to use dynamics to simulate Hamiltonians with non-trivial topology.

The idea is the following. Suppose we drive the system in a periodic fashion and obtain the evolution operator after one period

$$U(T) = \mathcal{T}e^{-i\int_0^T H(t)dt}, \tag{10.21}$$

where $H(t)$ is time-dependent Hamiltonian and T is one period of the dynamics. Now we define a (time-independent) effective Hamiltonian as follows

$$U(T) \equiv e^{-iH_{eff}T} \tag{10.22}$$

Through this definition of H_{eff}, one can interpret the dynamics as stroboscopically realizing the effective Hamiltonian at integer multiple times of T. In this fashion, one can realize the dynamics with topological properties.

One simple example of such dynamical simulations of topological phases is given by so-called quantum walk[10].

Bibliography

[1] For example, see Henrik Bruus & Karsten Flensberg, Many-body quantum theory in condensed matter physics: an introduction. Oxford University Press, 2004.

[2] Klitzing, K. von; Dorda, G.; Pepper, M. (1980). "New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance". Phys. Rev. Lett. 45 (6): 494497.

[3] D.C. Tsui, H.L. Stormer, A.C. Gossard. "Two-Dimensional Magnetotransport in the Extreme Quantum Limit". Physical Review Letters 48: 1559; A.H. MacDonald, arxiv: cond-mat/9410047; S. M. Girvin, arxiv: cond-mat/9907002.

[4] A.Y. Kitaev, Annals Phys. 303 (2003) 2-30

[5] A. Karlhede, S. A. Kivelson and S. L. Sondhi, "The quantum Hall effect", in "Correlated electron systems", ed. V. J. Emery (World Scientific 1993)

[6] X-L. Qi, T. L. Hughes, & S-C. Zhang. Phys. Rev. B 78, 195424(2008).

[7] M. W. Zwierlein, et.al. Nature 435, 1047-1051 (23 June 2005)

[8] Gemelke, Nathan; Sarajlic, Edina; Chu, Steven, arXiv:1007.267

[9] Yu-Ju Lin, et.al., Nature 462, 628 (2009)

[10] T. Kitagawa, et al, Phys. Rev. A 82, 033429 (2010)