Fermi liquids and non–Fermi liquids

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I. INTRODUCTION

Much of the current understanding of solid state physics is based on a picture of non–interacting electrons. This is clearly true at the elementary level, but in fact extends to many areas of current research, examples being the physics of disordered systems or mesoscopic physics. The most outstanding examples where the non–interacting electron picture fails are provided by electronic phase transitions like superconductivity or magnetism. However, more generally one clearly has to understand why the non–interacting approximation is successful, for example in understanding the physics of metals where one has a rather dense gas (or liquid) of electrons which certainly interact via their mutual Coulomb repulsion. A first answer is provided by Landau's theory of Fermi liquids [1–3] which, starting from the (reasonable but theoretically unproven) hypothesis of the existence of *quasiparticles* shows that the properties of an interacting system of fermions are qualitatively similar to that of a non–interacting system. A brief outline of Landau's theory in its most elementary aspects will be given in the following section, and a re–interpretation as a fixed point of a renormalization group will be discussed in sec.III.

A natural question to ask is whether Fermi liquid like behavior is universal in many–electron systems. The by far best studied example showing that this is not the case is the one–dimensional interacting electron gas. Starting with the early work of Mattis and Lieb [4], of Bychkov et al. [5] and of others is has become quite that in one dimension Landau type quasiparticles do not exist. The unusual one–dimensional behavior has now received the name "Luttinger liquid". This is still a very active area of research, and the rest of these notes is devoted to the discussion of various aspects of the physics of one–dimensional interacting fermions.

The initial plan for these lectures was considerably wider in scope. It was in particular considered to include a discussion of the Kondo effect and its non–Fermi–liquid derivatives, as well as possibly current theories of strongly correlated fermions in dimension larger than one. This plan however turned out to be overly ambitious and it was decided to limit the scope to the current subjects, allowing for sufficiently detailed lectures. Beyond this limitation on the scope of the lectures, length restrictions on the lecture notes imposed further cuts. In view of the fact that there is a considerable and easily accessible literature on Fermi liquid theory, it seemed best to remain at a rather elementary level at this point and to retain sufficient space for the discussion of the more unusual one–dimensional case. It is hoped that the references, especially in the next and the last section will allow the interested reader to find sources for further study.

II. FERMI LIQUIDS

Landau's Fermi liquid theory is concerned with the properties of a many–fermion system at low temperatures (much lower than the Fermi energy) in *the normal state*, i.e. in

the absence of or at least at temperatures above any symmetry breaking phase transition (superconducting, magnetic, or otherwise). The ideal example for Landau's theory is liquid helium 3, above its superfluid phase transition, however, the conceptual basis of Landau's theory is equally applicable to a variety of other systems, in particular electrons in metals. Quantitative applications are however more difficult because of a variety of complications, in particular the absence of translational and rotational invariance and the presence of electron–phonon interactions, which are not directly taken into account in Landau's theory. Subsequently, I will first briefly discuss the case of a noninteracting many–fermion system (the Fermi *gas*), and then turn to Landau's theory of the interacting case (the *liquid*), from a phenomenological point of view and limited to the equilibrium properties. A much more detailed and complete exposition of these subjects can be found in the literature [6–10].

A. The Fermi Gas

In a noninteracting translationally invariant systems, the single-particle eigenstates are plane waves with energy $\varepsilon_{\vec{k}} = \vec{k}^2/2m$. We will always use units so that $\hbar = 1$. The ground state of an N–particle system is the well–known Fermi sea: all states up to the *Fermi wavevector* k_{F} are filled, all the other states are empty. The energy of the last occupied state is the *Fermi energy* $E_{\mathrm{F}} = k_{\mathrm{F}}^2/(2m)$.

It is usually convenient to define the Hamiltonian in a way so that the absolute ground state has a well–defined fixed particle number. This is achieved simply by including the chemical potential in the definition of the Hamiltonian, i.e. by writing

$$H = \sum_{\vec{k}} \xi_{\vec{k}} n_{\vec{k}} \ , \tag{2.1}$$

where $n_{\vec{k}}$ is the usual number operator, $\xi_{\vec{k}} = \varepsilon - \mu$, and the spin summation is implicit. With this definition of the Hamiltonian, the elementary excitations of the Fermi gas are

- addition of a particle at wavevector $\vec{k}$ ($\delta n_{\vec{k}} = 1$). This requires $|k| > k_{\mathrm{F}}$, and thus the energy of this excitation is $\epsilon_{\vec{k}} - \mu > 0$.

- destruction of a particle at wavevector $\vec{k}$ ($\delta n_{\vec{k}} = -1$), i.e. creation of a hole. This requires $|k| < k_{\mathrm{F}}$, and thus the energy is $\mu - \epsilon_{\vec{k}} > 0$.

These excitations change the total number of particles. Construction of states at constant particle number is of course straightforward: one takes one particle from some state $\vec{k}$, with $|k| < k_{\mathrm{F}}$, and puts it into a state $\vec{k}'$, with $|\vec{k}'| > k_{\mathrm{F}}$. These *particle–hole excitations* are parameterized by the *two* quantum numbers $\vec{k}, \vec{k}'$ and thus form a continuum. The restriction on the allowed values of $\vec{k}, \vec{k}'$ insures that all particle–hole states have positive energy. Higher excited states, i.e. states with many particles and many holes, are straightforwardly constructed, the only restriction being imposed by the Pauli principle.

Thermodynamic quantities are easily obtained and are all determined by the density of states at the Fermi energy. For example, the specific heat obeys the well known linear law $C(T) = \gamma T$, with

$$\gamma = \frac{2\pi^2}{3} N(E_{\mathrm{F}}) k_{\mathrm{B}}^2 \tag{2.2}$$

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and similarly the (Pauli) spin susceptibility χ and the compressibility κ are given by

$$\chi = 2N(E_{\mathrm{F}})\mu_{\mathrm{B}}^2 \quad , \kappa = 2N(E_{\mathrm{F}})/\rho^2 \quad . \tag{2.3}$$

It should be emphasized that eqs. (2.2) to (2.3) are valid for an arbitrary density of states, in particular in solids where bandstructure effects can change the electronic dispersion relation quite drastically. Thus, for noninteracting electrons one expects the so–called "Wilson ratio" $R_W = \pi^2 k_{\mathrm{B}}^2 \chi/(3\mu_{\mathrm{B}}^2 \gamma)$ to be unity, independently of details of the bandstructure. Any deviation from unity is necessarily an indication of some form of interaction effect.

B. Landau's theory: phenomenological

1. Basic hypothesis

Landau's theory is to a large extent based on the idea of a continuous and one–to–one correspondence between the eigenstates (ground state *and* excited states) of the noninteracting and the interacting system. For this to be an acceptable hypothesis it is crucial that the interactions do not lead to any form of phase transition or symmetry–broken ground state. The standard way to represent this situation is the so–called "adiabatic switching on" of an interaction, parameterized by a variable V. On can easily imagine to numerically calculate eigenvalues of a Hamiltonian as a function of V (though this may be very difficult in practice). The basic hypothesis of Landau's theory then is that the ground state energy as well as the energy of the low–lying states as a function of V do not have any singularities due to some form of symmetry breaking. Moreover, it should be possible to label the low–lying states by the same quantum numbers as those of the noninteracting system, i.e. the number of excited particles and holes and their respective momenta.

In particular one can consider a state obtained by adding a particle (with momentum $|\vec{p}| > k_{\mathrm{F}}$) to the noninteracting ground state:

$$|\vec{p}, N+1\rangle = a_{\vec{p}}^+ |0, N\rangle \quad . \tag{2.4}$$

Here $a_{\vec{p}}^+$ is a fermion creation operator for momentum state $\vec{p}$, and $|0, N\rangle$ is the N–particle ground state of the noninteracting system. Now we add some form of particle–particle interaction. In a translationally invariant system, interactions conserve total momentum, and thus even after switching on the interaction the state still has total momentum $\vec{p}$. However, the interaction of the added particle with the filled Fermi sea, as well as the interaction of the particles in the sea amongst themselves, will change the distribution of particles in $\vec{k}$–space, and of course also modify the energy of our state. The complex formed by the particle added at $\vec{p}$ and the perturbed distribution of the other particles is called a *Landau quasiparticle*. The Pauli principle implied $|\vec{p}| > k_{\mathrm{F}}$ in the absence of interactions, and by the continuity hypothesis the same restriction remains valid in the interacting case. In particular, the value of k_{F}, which imposes a lower limit on the allowed momentum of the quasiparticle, is unchanged by the interactions.

Analogous considerations can be performed for a state obtained by destruction of a particle (e.g. creation of a hole):

$$|\vec{p}, N-1\rangle = a_{-\vec{p}} |0, N\rangle \quad . \tag{2.5}$$

Note that due to the momentum $-\vec{p}$ the total momentum of this state is indeed $\vec{p}$. The state obtained after switching on the interactions (and which of course still obeys $|\vec{p}| < k_{\mathrm{F}}$) could be called a quasi–hole, but one generally refers to both quasi–particles and quasi–holes as quasi–particles.

The quasi–particle concept has a certain number of limitations, mainly due to the fact that, as will be discussed below, the lifetime of a quasi–particle is finite. However, for excitations close to k_{F} one has $1/\tau \propto (\varepsilon - E_{\mathrm{F}})^2$, i.e. the lifetime becomes much longer than the inverse excitation energy, and the quasi–particles therefore are reasonably well defined in the vicinity of the Fermi energy. In practice, this means that Landau's theory is useful for phenomena at energy scales much smaller than the Fermi energy, but inapplicable otherwise. In metals, where $E_{\mathrm{F}} \approx 3 \ldots 5 eV$, this restriction is not too serious when one is concerned with thermodynamic or transport properties. One should also note that the ground state energy itself has important contributions from states well below E_{F}, and therefore is not accessible to Landau's theory. On the other, the *excitations above the ground state* are the fundamental objects of Landau's theory.

2. Equilibrium properties

To be able to derive physical quantities from the picture of the low–energy excitations, one needs some information about the energetics of the quasiparticles and of their interactions. To be specific, starting from the ground state quasiparticle distribution

$$
\begin{aligned}
n_0(k) &= 1 \text{ if } |\vec{k}| < k_{\mathrm{F}} \\
&= 0 \text{ if } |\vec{k}| > k_{\mathrm{F}}
\end{aligned} \tag{2.6}
$$

one considers changes in quasiparticle occupation number of the form $n_0(k) \to n_0(k) + \delta n(k)$, i.e. $\delta n(k) = 1$ represents an excited quasi–particle, $\delta n(k) = -1$ an excited quasi–hole (with the notation $k = (\vec{k}, \sigma)$, and $\sigma = \uparrow, \downarrow$ the spin index). The corresponding change in energy is

$$
\delta E = \sum_k \varepsilon_{\vec{k}}^0 \delta n(k) + \frac{1}{2\Omega} \sum_{kk'} f(k, k') \delta n(k) \delta n(k') \ , \tag{2.7}
$$

where the first and second term represent the energy of a single quasi–particle and the interaction between quasiparticles, respectively. To be more precise, we assume that the chemical potential is included in the Hamiltonian, as in eq.(2.1). Consequently, $\varepsilon_{\vec{k}}^0$ vanishes on the Fermi surface, and, given that we are mainly interested in phenomena in the vicinity of k_{F}, it is sufficient to retain the lowest order term in an expansion around $|\vec{k}| = k_{\mathrm{F}}$. One thus writes

$$
\varepsilon_{\vec{k}}^0 = \frac{k_{\mathrm{F}}}{m^*} (|\vec{k}| - k_{\mathrm{F}}) \ , \tag{2.8}
$$

thus defining the *effective mass m^**. The difference between the "bare mass m and m^* is due to interaction effects which in principle can be calculated from a microscopic theory of the system in question. At the present level m^* will be considered as a phenomenological parameter.

The energy of a quasi–particle added to the system is easily obtained from eq.(2.7) by calculating the difference in δE between a state with $\delta n(k) = 1$ and a state with $\delta n(k) = 0$. One finds

$$\varepsilon_k = \varepsilon_{\vec{k}}^0 + \frac{1}{\Omega} \sum_{k'} f(k, k') \delta n(k') \ , \tag{2.9}$$

i.e. the energy of an added quasi–particle is not just the "bare" quasiparticle energy $\varepsilon_{\vec{k}}^0$ but also depends, via the interaction term, on the presence or not of other quasi–particles. Given that the non–interacting particles obey Fermi–Dirac statistics, the quasi–particles do so too, and consequently, the occupation probability of a quasi–particle state is given by

$$n(k) = \frac{1}{e^{\beta \varepsilon_k} + 1} \ . \tag{2.10}$$

Note that the full quasi–particle energy ε_k and not the bare ε_k^0 enters this expression. In principle, $n(k)$ thus has to be determined selfconsistently from eqs.(2.9) and (2.10).

For the subsequent calculations, it is convenient to transform the quasiparticle interaction $f(k, k')$. First, spin symmetric and antisymmetric f–functions are defined via

$$\begin{aligned}
f(\vec{k} \uparrow, \vec{k'} \uparrow) &= f^s(\vec{k}, \vec{k'}) + f^a(\vec{k}, \vec{k'}) \\
f(\vec{k} \uparrow, \vec{k'} \downarrow) &= f^s(\vec{k}, \vec{k'}) - f^a(\vec{k}, \vec{k'})
\end{aligned} \tag{2.11}$$

Moreover, given the implicit restrictions of the theory, one is only interested in processes where all involved particles are very close to the Fermi surface. Under the assumption that the interaction functions are slowly varying as a function of $\vec{k}$, one then can set $|\vec{k}| = |\vec{k'}| = k_{\mathrm{F}}$. Because of rotational symmetry, the f–functions then can only depend on the angle between $\vec{k}$ and $\vec{k'}$, called θ. One can then expand the f–function in a Legendre series as

$$f^{a,s}(\vec{k}, \vec{k'}) = \sum_{L=0}^{\infty} f_L^{a,s} P_L(\cos \theta) \quad , \cos \theta = \frac{\vec{k} \cdot \vec{k'}}{k_{\mathrm{F}}^2} \quad , \tag{2.12}$$

where the P_L are the Legendre polynomials. This formula can be inverted to give

$$f_L^{a,s} = \frac{2L + 1}{4\pi} \int \mathrm{d}^2 \Omega P_L(\cos \theta) f^{a,s}(\vec{k}, \vec{k'}) \ . \tag{2.13}$$

Finally, one usually puts these coefficients into dimensionless form by introducing

$$F_L^{a,s} = \frac{k_{\mathrm{F}} m^*}{\pi^2} f_L^{a,s} \ . \tag{2.14}$$

We are now in a position to calculate some equilibrium properties. The first one will be the specific heat at constant volume

$$C_\Omega = \frac{1}{\Omega} \frac{\partial U}{\partial T} \tag{2.15}$$

where U is the internal energy. The temperature–dependent part of U comes from thermally excited quasi–particles, as determined by the distribution (2.10). In principle, in this expression ε_k is itself temperature–dependent, because of the temperature dependent second term in eq.(2.9). However, one can easily see that this term only gives contributions of order T^2,

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and therefore can be neglected in the low–temperature limit. Consequently, one can indeed replace ε_k by $\varepsilon_{\vec{k}}^0$, and then one only has to replace the bare mass by m^* in the result for a non–interacting system to obtain

$$C_\Omega = \frac{m^* k_F}{3} k_B^2 T \quad . \tag{2.16}$$

The spin susceptibility (at $T = 0$) is related to the second derivative of the ground state energy with respect to the (spin) magnetization M:

$$\chi = \left[\Omega \frac{\partial^2 E_0}{\partial M^2} \right]^{-1} \quad . \tag{2.17}$$

At finite temperature the ground state energy E_0 has to be replaced by the free energy $F(T, N)$. Spin magnetization is created by increasing the number of $\uparrow$ spin particles and decreasing the number of $\downarrow$ spins ($M = \mu_B(N_\uparrow - N_\downarrow)$), i.e. by changing the Fermi wavevectors for up and down spins: $k_F \rightarrow k_F + \delta k_F$ for $\sigma = \uparrow$ and $k_F \rightarrow k_F - \delta k_F$ for $\sigma = \downarrow$. One has

$$\begin{aligned}
\delta n(\vec{k}, \uparrow) &= 1 \text{ if } k_F < |\vec{k}| < k_F + \delta k_F \\
\delta n(\vec{k}, \downarrow) &= -1 \text{ if } k_F - \delta k_F < |\vec{k}| < k_F
\end{aligned} \tag{2.18}$$

and the magnetization is

$$M = \Omega \frac{\mu_B k_F^2}{\pi^2} \delta k_F \quad . \tag{2.19}$$

Using the δn's from eq.(2.18), the contributions from the first and second term in eq.(2.7) are straightforwardly calculated as

$$\delta E_1 = \Omega \frac{k_F^3}{2\pi^2 m^*} \delta k_F^2 \quad , \quad \delta E_2 = \Omega \frac{k_F^3}{2\pi^2 m^*} F_0^a \delta k_F^2 \quad . \tag{2.20}$$

The coefficient F_0^a appears because the distortion (2.18) is antisymmetric in the spin index and has spherical spatial symmetry ($L = 0$). From eqs.(2.17) and (2.20) one obtains the susceptibility as

$$\chi = \frac{1}{1 + F_0^a} \frac{\mu_B^2 k_F m^*}{\pi^2} \quad , \tag{2.21}$$

Note that here (and contrary to the specific heat) interactions enter not only via m^* but also explicitly via the coefficient F_0^a. The Wilson ratio then is

$$R_W = \frac{1}{1 + F_0^a} \quad . \tag{2.22}$$

Beyond the concrete result obtained here, it should be noted that the contributions from the two terms in eq.(2.7) are both of order δk_F^2, even though formally they seem to be of order δn and δn^2, respectively. This is in fact due to the "extra smallness" coming from the vanishing of $\varepsilon_{\vec{k}}^0$ at the Fermi surface, whereas $f(k, k')$ remains of course finite. On the other hand, it should also be clear that adding terms of order δn^3 to eq.(2.7) would only lead to higher order corrections.

Following a similar reasoning, one can obtain a result for the compressibility of a Fermi liquid. The general definition of κ is

$$\kappa = -\frac{1}{\Omega}\frac{\partial \Omega}{\partial P} = \left[\Omega\frac{\partial^2 E_0}{\partial \Omega^2}\right]^{-1} \ . \tag{2.23}$$

where at finite temperature one again has to replace the ground state energy by the free energy. For the calculation it is convenient to replace the variation of the volume at constant particle number by a corresponding variation of the particle number at constant volume. One then obtains an equivalent expression:

$$\kappa = \left[\rho^2\frac{\partial^2 e_0}{\partial \rho^2}\right]^{-1} \ , \tag{2.24}$$

where ρ is the particle density and e_0 is the ground state energy density. The calculation then is very similar to the one for the spin susceptibility, only now k_F is changed in the same way for up and down spins, and consequently F_0^s appear in the final result

$$\kappa = \frac{m^*k_F}{\pi^2\rho^2(1 + F_0^s)} \ . \tag{2.25}$$

It is also interesting that in a translationally invariant system as we have considered here, the effective mass is not independent of the interaction coefficients. This can be seen the following way. Because of Galilean invariance a system moving at constant velocity v has an extra kinetic energy $Nmv^2/2$. On the other hand, such a system can be represented by taking some particles from the left side of the Fermi sea to the right side, i.e. by a $\delta n(k)$ of dipolar symmetry. One can then use this $\delta n(k)$ to calculate the corresponding change in energy and compare to $Nmv^2/2$. From the comparison one finds

$$\frac{m^*}{m} = 1 + F_1^s/3 \ . \tag{2.26}$$

In the present phenomenological framework, the F–coefficients are phenomenological parameters, to be determined from experiment. The standard example is liquid helium 3, for which one finds [11,12] $m^*/m \approx 3$, $F_0^s \approx 9$, $F_1^s \approx 5$, $F_0^a \approx -0.7$, $F_1^a \approx -0.55$. These values indicate rather strong interaction effects. Note the negative value of F_0^a, which corresponds to a strong enhancement of the spin susceptibility.

C. Conclusion

The Landau theory of equilibrium properties is of rather limited quantitative predictive power because there is no prediction about the actual values of the Landau parameters. Its principal importance is of conceptual nature: from the single hypothesis about the existence of quasiparticles it follows that the low temperature equilibrium properties are very similar to those of a noninteracting system. In particular, the temperature dependencies are unaffected, only prefactors are interaction–dependent. Actual quantitative predictions are obtained when one extends the theory to nonequilibrium properties, using a Boltzmann equation. [6] A new phenomenon predicted (and actually observed in ^{3}He [13]) is the existence of

collective excitations, called "zero sound". This approach also allows the calculation of the quasiparticle lifetime and its divergence as the Fermi energy is approached, as well as the treatment of a number of transport phenomena.

It does not seem generally possible to *derive* Landau's theory starting from some microscopic Hamiltonian, apart possibly in perturbation theory for small interactions. It is however possible to formulate the basic hypotheses in terms of microscopic quantities. In particular, in the microscopic language the existence of quasiparticles translates into the existence of the *quasiparticle pole* in the one particle Green function:

$$\mathcal{G}(\vec{k}, i\omega_n) = \frac{z_k}{i\omega_n - \varepsilon_{\vec{k}}^0 + i\mathrm{sgn}(\omega_n)\tau^{-1}} + \cdots \quad . \tag{2.27}$$

Here $z_{\vec{k}}$ is the so-called quasiparticle weight, and gives rise to a jump in the momentum distribution function at k_F of height $z_{\vec{k}}$, rather than unity in the noninteracting case. Similarly, the Landau interaction parameters can be related to two-particle vertex parts in the limit of vanishing momentum transfer. It should however be emphasized that this line of reasoning provides a microscopic interpretation of Landau's picture, rather than proving its correctness. Similar remarks apply to the calculated diverging quasiparticle lifetime: this at best shows that Landau's picture is internally consistent.

As already mentioned, the ideal system for the application of Landau's theory is ^{3}He, which has both short-range interaction and is isotropic. The application to electrons in metals is more problematic. First, the interactions are long-ranged (Coulombic). This can however be accommodated by properly including screening effects. More difficulties, at least at the quantitative level, arise because metals are naturally anisotropic. This problem is not of fundamental nature: even when the Fermi surface is highly anisotropic, an expansion like eq.(2.7) can still be written down and thus interaction parameters can be defined. However, a simple Legendre expansion like eq.(2.12) is not in general possible, i.e. the description of the quasiparticle interaction in terms of a few parameters becomes impossible. An exception cases with a very nearly spherical Fermi surface like the alkali metals, where a determination of Landau parameters can indeed be attempted. [6] It should be noticed that the difficulties with the Landau description of metals are not of conceptual nature and in particular do not invalidate the quasiparticle concept but are rather limitations on the usefulness of the theory for quantitative purposes.

III. RENORMALIZATION GROUP FOR INTERACTING FERMIONS

In this chapter, we will consider properties of interacting fermions in the framework of renormalization group theory. This will serve two purposes: first, the treatment one-dimensional interacting fermions, which will be considered in detail in the following chapters, gives rise to divergences which can only be handled by this approach. Results obtained in this way will be an essential ingredient in the subsequent discussion of "Luttinger liquids". More generally, the renormalization group method will clarify the status of both Landau's Fermi liquid theory and the Luttinger liquid picture as renormalization group fixed points, thus establishing a link with a number of other phenomena in condensed matter physics. We will formulate the problem in terms of fermion functional integrals, as done by Bourbonnais

in the one–dimensional case [14] and more recently for two and three dimensions by Shankar [15,16]. For the most part, I will closely follow Shankar's notation.

Before considering the interacting fermion problem in detail, let us briefly recall the general idea behind the renormalization group, as formulated by Kadanoff and Wilson: one is interested in the statistical mechanics of a system described by some Hamiltonian H. Equilibrium properties then are determined by the partition function

$$Z = \sum_{\text{configurations}} e^{-\beta H} = \sum_{\text{configurations}} e^{-S} \ , \tag{3.1}$$

where the second equality defines the *action* $S = \beta H$. Typically, the action contains degrees of freedom at wavevectors up to some *cutoff* Λ, which is of the order of the dimensions of the Brillouin zone. One wishes to obtain an "effective action" containing only the physically most interesting degrees of freedom. In standard phase transition problems this is the vicinity of *the point* $\vec{k} = 0$, however, for the fermion problem at hand the *surface* $|\vec{k}| = k_{\mathrm{F}}$ is relevant, and the cutoff has to be defined with respect to this surface. In order to achieve this one proceeds as follows:

1. Starting from a cutoff–dependent action $S(\Lambda)$ one eliminates all degrees of freedom between Λ and Λ/s, where s is a factor larger than unity. This gives rise to a new action $S'(\Lambda' = \Lambda/s)$.

2. One performs a "scale change" $\vec{k} \to s\vec{k}$. This brings the cutoff back to its original value, i.e. one obtains a new action $S'(\Lambda)$. Because of the degrees of freedom integrated out, coupling constants (or functions) are changed.

3. One chooses a value of s infinitesimally close to unity: $s = 1 + \varepsilon$, and performs the first two steps iteratively. This then gives rise to differential equations for the couplings, which (in favorable circumstances) can be integrated until all non–interesting degrees of freedom have been eliminated.

A. One dimension

The one–dimensional case, which has interesting physical applications, will here be mainly used to clarify the procedure. Let us first consider a noninteracting problem, e.g. a one–dimensional tight–binding model defined by

$$H = \sum_k \xi_k a_k^\dagger a_k \ , \quad \xi_k = -2t\cos k - \mu \ , \tag{3.2}$$

where t is the nearest–neighbor hopping integral. We will consider the metallic case, e.g. the chemical potential is somewhere in the middle of the band. Concentrating on low–energy properties, only states close to the "Fermi points" $\pm k_{\mathrm{F}}$ are important, and one can then linearize the dispersion relation to obtain

$$H = \sum_{k,r=\pm} v_F (rk - k_{\mathrm{F}}) a_{kr}^\dagger a_{kr} \ , \tag{3.3}$$

where $v_F = 2t \sin k_F$ is the Fermi velocity, and the index r differentiates between right– and left–going particles, e.g. particles close to k_F and $-k_F$. To simplify subsequent notation, we (i) choose energy units so that $v_F = 1$, (ii) translate k-space so that zero energy is at $k = 0$, and (iii) replace the k–sum by an integral. Then

$$H = \sum_{r=\pm} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} rk a_r^\dagger(k) a_r(k) \quad . \tag{3.4}$$

For the subsequent renormalization group treatment we have to use a functional integral formulation of the problem in terms of Grassmann variables (a detailed explanation of this formalism is given by Negele and Orland [17]). The partition function becomes

$$Z(\Lambda) = \int \mathcal{D}\phi e^{-S(\Lambda)} \quad , \tag{3.5}$$

where $\mathcal{D}\phi$ indicates functional integration over a set of Grassmann variables. The action is

$$S(\Lambda) = \int_0^\beta d\tau \left\{ \sum_{r=\pm} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \phi_r^*(k,\tau) \partial_\tau \phi_r(k,\tau) + H(\phi^*, \phi) \right\} \quad , \tag{3.6}$$

where the zero–temperature limit $\beta \to \infty$ has to be taken, and $H(\phi^*, \phi)$ indicates the Hamiltonian, with each $a^\dagger$ replaced by a ϕ^*, and each a replaced by a ϕ. Fourier transforming with respect to the imaginary time variable

$$\phi_r(k,\tau) = T \sum_{\omega_n} \phi_r(k,\omega_n) e^{-i\omega_n \tau} \quad (\omega_n = 2\pi(n+1/2)T) \tag{3.7}$$

and passing to the limit $T \to 0$ one obtains the noninteracting action

$$S_0(\Lambda) = \sum_{r=\pm} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \phi_r^*(k,\omega)[-i\omega + rk]\phi_r(k,\omega) \quad . \tag{3.8}$$

We notice that this is diagonal in k and ω which will greatly simplify the subsequent treatment. Because of the units chosen, ω has units of $(\text{length})^{-1}$ (which we will abbreviate as L^{-1}), and then $\phi_r(k,\omega)$ has units $L^{3/2}$.

We now integrate out degrees of freedom. More precisely, we will integrate over the strip $\Lambda/s < |k| < \Lambda$, $-\infty < \omega < \infty$. The integration over *all* ω keeps the action local in time. One then has

$$Z(\Lambda) = Z(\Lambda, \Lambda/s)Z(\Lambda/s) \quad , \tag{3.9}$$

where $Z(\Lambda, \Lambda/s)$ contains the contributions from the integrated degrees of freedom, and $Z(\Lambda/s)$, $S_0(\Lambda/s)$ are of the form of eqs.(3.5) and (3.8). The diagonality of $S_0(\Lambda)$ leads to the factorized form of eq.(3.9). Introducing the scale change

$$k' = ks \quad , \quad \omega' = \omega s \quad , \quad \phi' = \phi s^{-3/2} \tag{3.10}$$

one easily finds that in fact $S_0(\Lambda/s) = S_0(\Lambda)$, i.e. the action does not change under scale change (or *renormalization*), we are at a *fixed point*. One should notice that the scale change of k implies that k' is quantized in units of $\Delta k' = 2\pi s/L$, i.e. eliminating degrees of freedom actually implies that we are considering a shorter system, with correspondingly less degrees of freedom. This means that even though the action is unchanged the new $Z(\Lambda)$ is the

partition function of a shorter system. To derive this in detail, one has to take into account the change in the functional integration measure intervening due to the scale change on ϕ.

Before turning to the problem of interactions, it is instructive to consider a quadratic but diagonal perturbation of the form

$$\delta S_2 = \sum_{r=\pm} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \mu(k,\omega)\phi_r^*(k,\omega)\phi_r(k,\omega) \ . \tag{3.11}$$

We assume that $\mu(k,\omega)$ can be expanded in a power series

$$\mu(k,\omega) = \mu_{00} + \mu_{10}k + \mu_{01}i\omega + \ldots \tag{3.12}$$

Under the scale change (3.10) one then has

$$\mu_{nm} \rightarrow s^{1-n-m}\mu_{nm} \ . \tag{3.13}$$

There now are three cases:

1. a parameter m_{nm} grows with increasing s. Such a parameter is called *relevant*. This is the case for μ_{00}.

2. a parameter remain unchanged (μ_{10},μ_{01}). Such a parameter is *marginal*.

3. Finally, all other parameter decrease with increasing s. These are called *irrelevant*.

Generally, one expects relevant parameter, which grow after elimination of high–energy degrees of freedom, to strongly modify the physics of the model. In the present case, the relevant parameter is simply a change in chemical potential and this doesn't change the physics much (the same is true for the marginal parameters). One can easily see that another relevant perturbation is a term coupling right– and left–going particles of the form $m(\phi_1^*\phi_2 + \phi_2^*\phi_1)$. This term in fact does lead to a basic change: it leads to the appearance of a gap at the Fermi level.

Let us now introduce fermion–fermion interactions. The general form of the interaction term in the action is

$$S_I = \int_{k\omega} u(1234)\phi^*(1)\phi^*(2)\phi(3)\phi(4) \ . \tag{3.14}$$

Here $\phi(3)$ is an abbreviation for $\phi_{r_3}(k_3,\omega_3)$, and similarly for the other factors. u is an interaction function to be specified. The integration measure is

$$\int_{k\omega} \ldots = \left(\prod_{i=1}^{4} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \right)$$
$$\times \delta(k_1 + k_2 - k_3 - k_4)\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)\ldots \ . \tag{3.15}$$

We now note that the dimension of the integration measure is L^{-6}, and the dimension of the product of fields is L^6. This in particular means that if we perform a series expansion of u in analogy to eq.(3.12) the constant term will be s–independent, i.e. marginal, and all other terms are irrelevant. In the following we will thus only consider the case of a constant (k– and ω–independent) u.

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These considerations are actually only the first step in the analysis: in fact it is quite clear that (unlike in the noninteracting case above) integrating out degrees of freedom will not in general leave the remaining action invariant. To investigate this effect, we use a more precise form of the interaction term:

$$
\begin{aligned}
S_I &= g_1 \int_{k\omega} \sum_{ss'} \phi^*_{s+}(1)\phi^*_{s'-}(2)\phi_{s'+}(3)\phi_{s-}(4) \\
&\quad + g_2 \int_{k\omega} \sum_{ss'} \phi^*_{s+}(1)\phi^*_{s'-}(2)\phi_{s'-}(3)\phi_{s+}(4) \ .
\end{aligned}
\tag{3.16}
$$

Here we have reintroduced spin, and the two coupling constants g_1 and g_2 denote, in the original language of eq.(3.3), backward $((k_F, -k_F) \to (-k_F, k_F))$ and forward $((k_F, -k_F) \to (k_F, -k_F))$ scattering. Note that in the absence of spin the two processes are actually identical.

Now, the Kadanoff–Wilson type mode elimination can be performed via

$$
e^{-S'} = \int \mathcal{D}\bar{\phi}e^{-S} \ ,
\tag{3.17}
$$

where $\mathcal{D}\bar{\phi}$ denotes integration only over degrees of freedom in the strip $\Lambda/s < |k| < \Lambda$. Dividing the field ϕ into $\bar{\phi}$ (to be eliminated) and ϕ' (to be kept), one easily sees that the noninteracting action can be written as $S_0 = S_0(\phi') + S_0(\bar{\phi})$. For the interaction part, things are a bit more involved: writing

$$
S_I = \sum_{i=0}^{4} S_{I,i} = S_{I,0} + \bar{S}_I \ ,
\tag{3.18}
$$

where $S_{I,i}$ contains i factors $\bar{\phi}$ we obtain

$$
e^{-S'} = e^{-S_0(\phi')-S_{I,0}} \int \mathcal{D}\bar{\phi}e^{-S_0(\bar{\phi})-\bar{S}_I} \ .
\tag{3.19}
$$

Because $\bar{S}_I$ contains up to four factors $\bar{\phi}$, the integration is not straightforward, and has to be done via a perturbative expansion, giving

$$
\int \mathcal{D}\bar{\phi}e^{-S_0(\bar{\phi})-\bar{S}_I} = Z_0(\Lambda, \Lambda/s) \exp\left[-\sum_{i=1}^{\infty} \frac{1}{n!}\langle \bar{S}_I^n \rangle_{\bar{0},con}\right] \ ,
\tag{3.20}
$$

where the notation $\langle \ldots \rangle_{\bar{0},con}$ indicates averaging over $\bar{\phi}$ and only the connected diagrams are to be counted. The first order cumulants only give corrections to the energy and the chemical potential and are thus of minor importance. The important contributions come from the second order term $\langle S_{I,2}^2 \rangle_{\bar{0},con}$ which after averaging leads to terms of the form $\phi'^*\phi'^*\phi'\phi'$, i.e. to corrections of the interaction constants $g_{1,2}$. The calculation is best done diagrammatically, and the four intervening diagram are shown in fig.1. One can easily see that not all of these diagrams contribute corrections to g_1 or g_2. Specifically, one has

$$
\begin{aligned}
\delta g_1 &\propto g_1 g_2[(a) + (c)] + 2g_1^2(d) \\
\delta g_2 &\propto (g_1^2 + g_2^2)(a) + g_2^2(b)
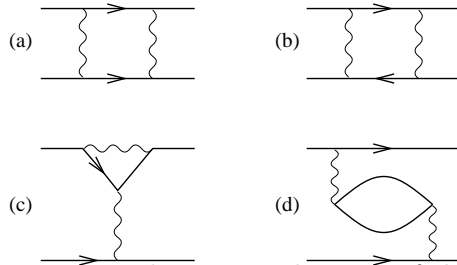\end{aligned}
\tag{3.21}
$$

FIG. 1. The diagrams intervening in the renormalization of the coupling constants g_1 and g_2. Note that in (b) the direction of one arrow is reversed with respect to (a), i.e. this is a particle–hole diagram.

where the factor 2 for diagram (d) comes from the spin summation over the closed loop. Because the only marginal term is the constant in $u(1234)$, one can set all external energies and momenta to zero. The integration over the internal lines in diagram (a) then gives

$$(a) = \int_s \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \frac{1}{i\omega - k} \frac{1}{-i\omega - k} = \int_{\Lambda/s}^{\Lambda} \frac{dk}{2\pi} \frac{1}{k} = \frac{1}{2\pi} d\ell \quad , \tag{3.22}$$

where $s = 1 + d\ell$, and similarly the particle–hole diagrams (b) to (d) give a contribution $-d\ell/(2\pi)$. Performing this procedure recursively, using at each step the renormalized couplings of the previous step, one obtains the renormalization group equations

$$\frac{dg_1}{d\ell} = -\frac{1}{\pi} g_1^2(\ell) \quad , \quad \frac{dg_2}{d\ell} = -\frac{1}{2\pi} g_1^2(\ell) \quad , \tag{3.23}$$

where $s = e^\ell$. These equations describe the effective coupling constants to be used after degrees of freedom between Λ and Λe^ℓ have been integrated out. As initial conditions one of course uses the bare coupling constants appearing in eq.(3.16).

Equations (3.23) are easily solved. The combination $g_1 - 2g_2$ is ℓ–independent, and one has further

$$g_1(\ell) = \frac{g_1}{1 + g_1 \ell} \quad . \tag{3.24}$$

There then are two cases:

1. Initially, $g_1 \geq 0$. One then renormalizes to the *fixed line* $g_1^* = g_1(\ell \to \infty = 0$, $g_2^* = g_2 - g_1/2$, i.e. one of the couplings has actually vanished from the problem, but there is still the free parameter g_2^*. A case like this, where perturbative corrections lead to irrelevancy, is called "marginally irrelevant".

2. Initially, $g_1 < 0$. Then g_1 diverges at some finite value of ℓ. One should however notice that well before the divergence one will have left the weak–coupling regime where the perturbative calculation leading to the eq.(3.23) is valid. One should thus not overinterpret the divergence and just remember the renormalization towards strong coupling. This type of behavior is called "marginally relevant".

We will discuss the physics of both cases in detail in the next section.

Two remarks are in order here: first, had we done a straightforward order–by–order perturbative calculation, integrals like eq.(3.22) would have been logarithmically divergent,

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both for particle–particle and particle–hole diagrams. This would have lead to inextricably complicated problem already at the next order. Secondly, for a spinless problem, the factor 2 in the equation for $g_1(\ell)$ is replaced by unity. Moreover, in this case only the combination $g_1 - g_2$ is physically meaningful. This combination then remains unrenormalized.

B. Two and three dimensions

We will now follow a similar logic as above to consider two and more dimensions. Most arguments will be made for the two–dimensional case, but the generalization to three dimensions is straightforward. The argument is again perturbative, and we thus start with free fermions with energy

$$\xi_{\vec{K}} = \frac{\vec{K}^2}{2m} - \mu = v_F k + O(k^2) \quad (v_F = k_{\mathrm{F}}/m) \ . \tag{3.25}$$

We use upper case momenta $\vec{K}$ to denote momenta measured from zero, and lower case to denote momenta measured from the Fermi surface: $k = |\vec{K}| - k_{\mathrm{F}}$. The Fermi surface geometry now is that of a circle as shown in fig.2. One notices in particular that states are
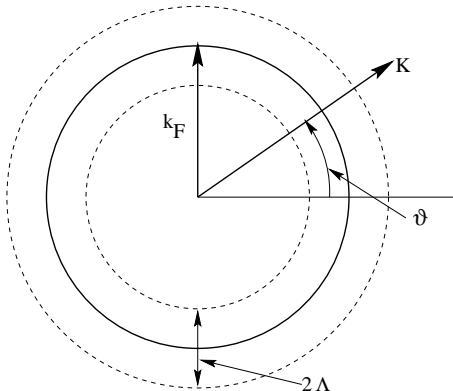

FIG. 2. Fermi surface geometry in two dimensions.

now labeled by two quantum numbers which one can take as radial (k) and angular (θ). Note that the cutoff is applied around the low–energy excitations at $|\vec{K}| - k_{\mathrm{F}}$, not around $\vec{K} = 0$. The noninteracting action then takes the form

$$S_0 = k_{\mathrm{F}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{\mathrm{d}k}{2\pi} [\phi^*(k\theta\omega)(-i\omega - k)\phi(k\theta\omega)] \ . \tag{3.26}$$

One notices that this is just a (continuous) collection of one–dimensional action functional, parameterized by the variable θ. The prefactor k_{F} comes from the two–dimensional integration measure $\mathrm{d}^2 K = (k_{\mathrm{F}} + k)\mathrm{d}k\mathrm{d}\theta$, where the extra factor k has been neglected because it is irrelevant, as discussed in the previous section.

The general form of the interaction term is the same as in the one–dimensional case

$$S_I = \int_{\vec{K}\omega} u(1234)\phi^*(1)\phi^*(2)\phi(3)\phi(4) \ , \tag{3.27}$$

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however, the integration measure is quite different because of two–dimensional $\vec{K}$–space. Performing the integration over $\vec{K}_4$ and ω_4 in the two–dimensional analogue of eq.(3.15), the measure becomes

$$\int_{\vec{K}\omega} \ldots =$$
$$\left(\frac{k_{\mathrm{F}}}{2\pi}\right)^3 \left(\prod_{i=1}^{3} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega_i}{2\pi} \int_{0}^{2\pi} \frac{\mathrm{d}\theta_i}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{\mathrm{d}k_i}{2\pi}\right) \Theta(\Lambda - |k_4|) \ldots \tag{3.28}$$

Here $\vec{K}_4 = \vec{K}_1 + \vec{K}_2 - \vec{K}_3$. Now the step function poses a problem because one easily convinces oneself that even when $\vec{K}_{1,2,3}$ are on the Fermi surface, in general $\vec{K}_4$ can be far away from it. This is quite different from the one–dimensional case, where everything could be (after a trivial transformation) brought back into the vicinity of $k = 0$.

To see the implications of this point, it is convenient to replace the sharp cutoff in eq.(3.28) by a soft cutoff, imposed by an exponential:

$$\Theta(\Lambda - |k_4|) \rightarrow \exp(-|k_4|/\Lambda) \ . \tag{3.29}$$

Introducing now unit vectors $\vec{\Omega}_i$ in the direction of $\vec{K}_i$ via $\vec{K}_i = (k_{\mathrm{F}} + k_i)\vec{\Omega}_i$ one obtains

$$k_4 = |k_{\mathrm{F}}(\vec{\Omega}_1 + \vec{\Omega}_2 - \vec{\Omega}_3) + k_1\vec{\Omega}_1 + \ldots| - k_{\mathrm{F}} \approx k_{\mathrm{F}}(|\vec{\Delta}| - 1)$$
$$\vec{\Delta} = \vec{\Omega}_1 + \vec{\Omega}_2 - \vec{\Omega}_3 \ . \tag{3.30}$$

Now, integrating out variables leaves us with $\Lambda \rightarrow \Lambda/s$ in eq.(3.28) everywhere, including the exponential cutoff factor for k_4. After the scale change (3.10) the same form of the action as before is recovered, with

$$u'(k_i', \omega_i', \theta_i') = \mathrm{e}^{-(s-1)(k_{\mathrm{F}}/\Lambda)||\vec{\Delta}|-1|} u(k/i_s, \omega_i/s, \theta_i) \ . \tag{3.31}$$

We notice first that nothing has happened to the angular variable, as expected because it parameterizes the Fermi surface which is not affected. Secondly, as in the one–dimensional case, the k and ω dependence of u is scaled out, i.e. *only the values $u(0, 0, \theta_i)$ on the Fermi surface are of potential interest (i.e. marginal)*. Thirdly, the exponential prefactor in eq.(3.31) suppresses couplings for which $|\vec{\Delta}| \neq 1$. This is the most important difference with the one–dimensional case.

A first type of solution to $|\vec{\Delta}| = 1$ is

$$\vec{\Omega}_1 = \vec{\Omega}_3 \ \Rightarrow \ \vec{\Omega}_2 = \vec{\Omega}_4 \ , \text{or}$$
$$\vec{\Omega}_1 = \vec{\Omega}_4 \ \Rightarrow \ \vec{\Omega}_2 = \vec{\Omega}_3 \ . \tag{3.32}$$

These two cases only differ by an exchange of the two outgoing particles, and consequently there is a minus sign in the respective matrix element. Both process depend only on the angle θ_{12} between $\vec{\Omega}_1$ and $\vec{\Omega}_2$, and we will write

$$u(0, 0, \theta_1, \theta_2, \theta_1, \theta_2) = -u(0, 0, \theta_1, \theta_2, \theta_2, \theta_1) = F(\theta_1 - \theta_2) \ . \tag{3.33}$$

We can now consider the perturbative contributions to the renormalization of F. To lowest nontrivial (second) order the relevant diagrams are those of Fermi liquid theory and are reproduced in fig.3. Consider diagram (a). To obtain a contribution to the renormalization
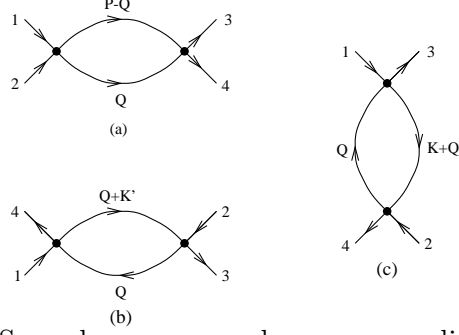
FIG. 3. Second order diagrams renormalizing the coupling function. Here $\mathbf{P} = \mathbf{K}_1 + \mathbf{K}_2, \mathbf{K}_3 = \mathbf{K}_1 - \mathbf{K}$, and $\mathbf{K}_4 = \mathbf{K}_1 - \mathbf{K}'$. $\mathbf{Q}$ is the loop integration variable.
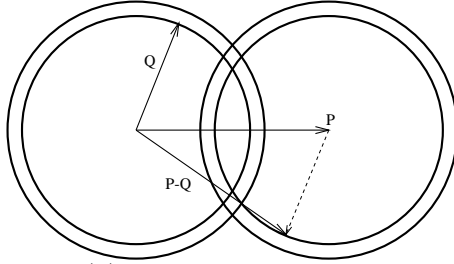

FIG. 4. Phase space for diagram (a). The rings are the degrees of freedom to be integrates out between Λ and Λ/s. Note that *only* if $\mathbf{P} = 0$ are $\mathbf{Q}$ and $\mathbf{P} - \mathbf{Q}$ simultaneously in the area to be integrates, giving a contribution of order $d\ell$.

of F, both $\vec{Q}$ and $\vec{P} - \vec{Q}$ have to lie in the annuli to be integrates out. As can be seen from fig.4, this will give a contribution of order $d\ell^2$ and therefore does not contribute to a renormalization of F. The same is true (if we consider the first case in eq.(3.32)) for diagram (b). Finally, for diagram (c), because $\vec{K}$ is small, the poles of both intervening Green's functions are on the same side of the real axis, and here then the frequency integration gives a zero result. For the second process in eq.(3.32) the same considerations apply, with the roles of diagrams (b) and (c) interchanged. The conclusion then is that F is not renormalized and remains marginal:

$$\frac{\mathrm{d}F}{\mathrm{d}\ell} = 0 \ . \tag{3.34}$$

The third possibility is to have $\vec{\Omega}_1 = -\vec{\Omega}_2$, $\vec{\Omega}_3 = -\vec{\Omega}_4$. Then the angle between $\vec{\Omega}_1$ and $\vec{\Omega}_3$ can be used to parameterize u:

$$u(0, 0, \theta_1, -\theta_1, \theta_3, -\theta_3) = V(\theta_1 - \theta_3) \ . \tag{3.35}$$

In this case $\vec{P} = 0$, and therefore in diagram (a) if $\vec{Q}$ is to be eliminated, so is $-\vec{Q}$. Consequently, one has a contribution of order $d\ell$. For the other two diagrams, one finds again negligible contributions of order $d\ell^2$. Thus, one obtains

$$\frac{\mathrm{d}V(\theta_1 - \theta_3)}{\mathrm{d}\ell} = -\frac{1}{8\pi^2} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} V(\theta_1 - \theta) V(\theta - \theta_3) \ . \tag{3.36}$$

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This is a renormalization equation for a function, rather than for a constant, i.e. one here has an example of a "functional renormalization group". Nevertheless, a Fourier transform

$$V_\lambda = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \mathrm{e}^{i\lambda\theta} V(\theta) \tag{3.37}$$

brings this into a more standard form:

$$\frac{\mathrm{d}V_\lambda(\ell)}{\mathrm{d}\ell} = -\frac{V_\lambda(\ell)^2}{4\pi} \quad . \tag{3.38}$$

This has the straightforward solution

$$V_\lambda(\ell) = \frac{V_\lambda}{1 + V_\lambda \ell/(4\pi)} \quad . \tag{3.39}$$

From eqs.(3.34) and eq.(3.39) there are now two possibilities:

1. At least one of the V_λ is negative. Then one has a divergence of $V_\lambda(\ell)$ at some finite energy scale. Given that this equation only receives contributions from BCS–like particle–particle diagrams, the interpretation of this as a superconducting pairing instability is straightforward. The index λ determines the relative angular momentum of the particles involved.

2. All $V_\lambda > 0$. Then on has the fixed point $V_\lambda = 0$, $F(\theta_1 - \theta_2)$ arbitrary. What is the underlying physics of this fixed point? One notices that here one has $\theta_3 = \theta_1$, $\theta_4 = \theta_2$, i.e. the marginal term in the action is $\phi_{\theta_1}^* \phi_{\theta_2}^* \phi_{\theta_1} \phi_{\theta_2}$. In the operator language, this translates into

$$H_{int} \approx \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 n_{\theta_1} n_{\theta_2} \quad . \tag{3.40}$$

   We now can recognize this as an operator version of Landau's energy functional, eq.(2.7). The fixed point theory is thus identified as *Landau's Fermi liquid theory*.

One can notice that the fixed point Hamiltonian, eq.(3.40, has a very large symmetry group: given that $n_k = a_k^\dagger a_k$, this term (as well as the kinetic energy term that only contains n_k) is invariant under the U(1) transformation

$$a_k \rightarrow \mathrm{e}^{i\varphi_k} a_k \tag{3.41}$$

with arbitrarily k–dependent φ_k, i.e. the symmetry group is U(1)^N, with N the number of points on the Fermi surface. On the other hand, a standard interaction Hamiltonian has the form

$$H_{int} = \frac{1}{2\Omega} \sum_{kk'q} V(q) a_{k+q}^\dagger a_{k'-q}^\dagger a_{k'} a_k \quad , \tag{3.42}$$

and this is only invariant under a global U(1) transformation.

The generalization of the above to three dimensions is rather straightforward. In addition to the forward scattering amplitudes F, scattering where there is an angle $\phi_{12;34}$ spanned by the planes $(\vec{\Omega}_1, \vec{\Omega}_2)$ and $(\vec{\Omega}_3, \vec{\Omega}_4)$ is also marginal. For $\phi_{12;34} \neq 0$ these processes give rise to a finite quasiparticle lifetime, however they do not affect equilibrium properties. The (zero temperature) fixed point properties thus still only depend on amplitudes for $\phi_{12;34} = 0$, i.e. the Landau f–function.

IV. LUTTINGER LIQUIDS

The Fermi liquid picture described in the preceding two sections is believed to be relevant for most three–dimensional itinerant electron systems, ranging from simple metals like sodium to heavy–electron materials. The most studied example of non–Fermi liquid properties is that of interacting fermions in one dimension. This subject will be discussed in the remainder of these lecture notes. We have already started this discussion in sec.III A, where we used a perturbative renormalization group to find the existence of one marginal coupling, the combination $g_1 - 2g_2$. This approach, pioneered by Sólyom and collaborators in the early 70's, [18] can be extended to stronger coupling by going to second or even third order [19] in perturbation theory. A principal limitation remains however the reliance on perturbation theory, which excludes the treatment of strong–coupling problems. An alternative method, which allows one to handle, to a certain extent, strong–interaction problems as well, is provided by the bosonization approach, which will be discussed now and which form the basis of the so–called Luttinger liquid description. It should be pointed out, however, that frequently entirely analogous results can be obtained by many–body techniques. [20–22]

A. Bosonization for spinless electrons

The bosonization procedure can be formulated precisely, in the form of operator identities, for fermions with a linear energy–momentum relation, as discussed in section III A. To clarify notation, we will use a–(b–)operators for right–(left–)moving fermions. The linearized noninteracting Hamiltonian, eq.(3.3) then becomes

$$H_0 = v_F \sum_k \{(k - k_F)a_k^\dagger a_k + (-k - k_F)b_k^\dagger b_k\} \ , \tag{4.1}$$

and the density of states is $N(E_F) = 1/(\pi v_F)$. In the *Luttinger model*, [23,4] one generalizes this kinetic energy by letting the momentum cutoff Λ tend to infinity. There then are two branches of particles, "right movers" and "left movers", both with unconstrained momentum and energy, as shown in fig.5. At least for weak interaction, this addition of extra states
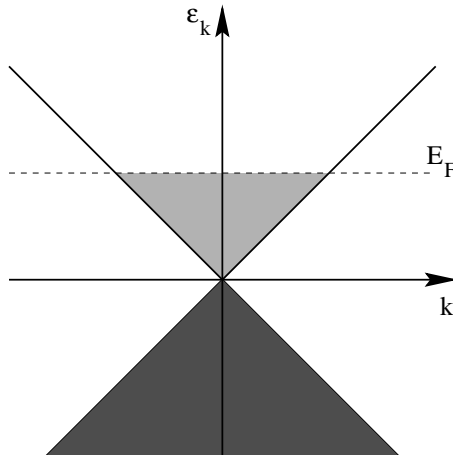

FIG. 5. Single–particle energy spectrum of the Luttinger model. Occupied states are shown in grey, the dark grey area represents the states added to make the model solvable.

far from the Fermi energy is not expected to change the physics much. However, this modification makes the model exactly solvable even in the presence of nontrivial and possibly strong interactions. Moreover, and most importantly, many of the features of this model carry over even to strongly interacting fermions on a lattice.

We now introduce the Fourier components of the particle density operator for right and left movers:

$$\rho_+(q) = \sum_k a^\dagger_{k+q} a_k \quad , \quad \rho_-(q) = \sum_k b^\dagger_{k+q} b_k \quad , \tag{4.2}$$

The noninteracting Hamiltonian (and a more general model including interactions, see below) can be written in terms of these operators in a rather simple form and then be solved exactly. This is based on the following facts:

1. the density fluctuation operators $\rho_\pm$ obey Bose type commutation relations:

$$[\rho_+(-q), \rho_+(q')] = [\rho_-(q), \rho_-(-q')] = \delta_{qq'} \frac{qL}{2\pi} \quad , \tag{4.3}$$

$$[\rho_+(q), \rho_-(q')] = 0 \quad . \tag{4.4}$$

The relation (4.4) as well as eq.(4.3) for $q \neq q'$ can be derived by straightforward operator algebra. The slightly delicate part is eq.(4.3) for $q = q'$. One easily finds

$$[\rho_+(-q), \rho_+(q)] = \sum_k (\hat{n}_{k-q} - \hat{n}_k) \quad , \tag{4.5}$$

where $\hat{n}_k$ is an occupation number *operator*. In a usual system with a finite interval of states between $-k_F$ and k_F occupied, the summation index of one of the $\hat{n}$ operators could be shifted, giving a zero answer in eq.(4.5). In the present situation, with an infinity of states occupied below k_F (see fig.6), this is not so. Consider for example the ground state and $q > 0$. Then each term in eq.(4.5) with $k_F < k < k_F + q$ contributes
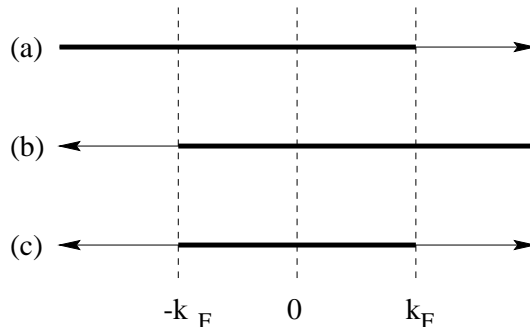

FIG. 6. Occupied states (heavy line) for right moving fermions (a), left moving fermions (b), and in a standard model without extra states at negative energy (c).

unity to the sum, all other terms vanish, thus establishing the result (4.3). More generally, consider a state with all levels below a certain value k_0 ($< k_F$) occupied, but

an arbitrary number of particle hole pairs excited otherwise. One then has, assuming again $q > 0$,

$$\sum_k (\hat{n}_{k-q} - \hat{n}_k) = \left(\sum_{k \geq k_0} + \sum_{k < k_0} \right) (\hat{n}_{k-q} - \hat{n}_k) \tag{4.6}$$

$$= \sum_{k \geq k_0} (\hat{n}_{k-q} - \hat{n}_k) \tag{4.7}$$

$$= \sum_{k \geq k_0 - q} \hat{n}_k - \sum_{k \geq k_0} \hat{n}_k \tag{4.8}$$

$$= \sum_{k_0 - q \leq k < k_0} \hat{n}_k = \frac{Lq}{2\pi} \quad . \tag{4.9}$$

The result is independent of k_0, and one thus can take the limit $k_0 \to -\infty$. Together with an entirely parallel argument for ρ_-, this then proves eq.(4.3). Moreover, for $q > 0$ both $\rho_+(-q)$ and $\rho_-(q)$ annihilate the noninteracting groundstate. One can easily recover canonical Bose commutation relations by introducing normalized operators, e.g. for $q > 0$ $b_q^\dagger = \sqrt{2\pi/(qL)}\rho_+(q)$ would be a canonical creation operator, but we won't use this type of operators in the following.

2. The noninteracting Hamiltonian obeys a simple commutation relation with the density operators. For example

$$[H_0, \rho_+(q)] = v_F q \rho_+(q) \quad , \tag{4.10}$$

i.e. states created by $\rho_+(q)$ are eigenstates of H_0, with energy $v_F q$. Consequently, the kinetic part of the Hamiltonian can be re–written as a term bilinear in boson operators, i.e. quartic in fermion operators:

$$H_0 = \frac{2\pi v_F}{L} \sum_{q>0} [\rho_+(q)\rho_+(-q) + \rho_-(-q)\rho_-(q)] \quad . \tag{4.11}$$

This equivalence may be made more apparent noting that $\rho_+(q)$ creates particle–hole pairs that all have total momentum q. Their energy is $\varepsilon_{k+q} - \varepsilon_k$, which, because of the linearity of the spectrum equals $v_F q$, *independently of k*. Thus, states created by $\rho_+(q)$ are linear combinations of individual electron–hole excitations all with the same energy, and therefore are also eigenstates of (4.1).

3. The above point shows that the spectra of the bosonic and fermionic representations of H_0 are the same. To show complete equivalence, one also has to show that the degeneracies of all the levels are identical. This can be achieved calculating the partition function in the two representations and demonstrating that they are equal. This then shows that the states created by repeated application of $\rho_\pm$ on the ground state form a complete set of basis states [24,25].

We now introduce interactions between the fermions. As long as only forward scattering of the type $(k_F; -k_F) \rightarrow (k_F; -k_F)$ or $(k_F; k_F) \rightarrow (k_F; k_F)$ is introduced, the model remains exactly solvable. The interaction Hamiltonian describing these processes takes the form

$$
\begin{aligned}
H_{int} = \frac{1}{2L} \sum_q \{ & 2g_2(q)\rho_+(q)\rho_-(-q) \\
& + g_4(q)[\rho_+(q)\rho_+(-q) + \rho_-(-q)\rho_-(q)] \} \ .
\end{aligned} \tag{4.12}
$$

Here, $g_2(q)$ and $g_4(q)$ are the Fourier transforms of a real space interaction potential, and in a realistic case one would of course have $g_2(q) = g_4(q)$, but it is useful to allow for differences between g_2 and g_4. For Coulomb interactions one expects $g_2, g_4 > 0$. In principle, the long–range part of the Coulomb repulsion leads to a singular q–dependence. Such singularities in the g_i can be handled rather straightforwardly and can lead to interesting physical effects as will be discussed below. Here I shall limit myself to nonsingular g_2, g_4. Electron–phonon interactions can lead to effectively attractive interactions between electrons, and therefore in the following I will not make any restrictive assumptions about the sign of the constants. One should however notice that a proper treatment of the phonon dynamics and of the resulting retardation effects requires more care [26].

Putting together (4.11) and (4.12), the complete interacting Hamiltonian, the *Tomonaga–Luttinger model*, then becomes a bilinear form in boson operators that is easily diagonalized by a Bogolyubov transformation. A first consequence is the expression for the excitation spectrum

$$
\omega(q) = |q|[(v_F + g_4(q)/(2\pi))^2 - (g_2(q)/(2\pi))^2]^{1/2} \ . \tag{4.13}
$$

The diagonal boson operators are linear combinations of the original ρ operators, and consequently, these elementary excitations are collective density oscillations, their energy being determined both by the kinetic energy term and the interactions.

We note here that in order for the Bogolyubov transformation to be a well–defined unitary transformation, $g_2(q)$ has to decrease at large q at least as $|q|^{-1/2}$. on the other hand, the large–q behavior of g_2 is unimportant for the low–energy properties of the model. We therefore in the following will almost always use a q–independent g_2 and g_4. An approximate and frequently used way to cure the divergences arising due to this procedure is to keep the parameter α in subsequent formulae as a finite short–distance cutoff, of the order of a lattice spacing. One can then also include the "backward scattering" $(k_F; -k_F) \rightarrow (-k_F; k_F)$, because for spinless electron this is just the exchange analogue of forward scattering and does not constitute a new type of interaction. It is worthwhile emphasizing here that the solution is valid for arbitrarily strong interactions, no perturbative expansion is needed!

Up to this point, the construction does not allow for a direct calculation of correlation functions like the one–particle Green function or more generally any function involving individual creation or destruction operators. This type of correlation function becomes tractable by representing single particle operators in terms of the boson operators. To this end, we introduce the field operators

$$
\phi(x) = -\frac{i\pi}{L} \sum_{p \neq 0} \frac{1}{p} e^{-\alpha|p|/2 - ipx} [\rho_+(p) + \rho_-(p)] - N\frac{\pi x}{L} \ , \tag{4.14}
$$

$$\Pi(x) = \frac{1}{L} \sum_{p \neq 0} e^{-\alpha|p|/2 - ipx} [\rho_+(p) - \rho_-(p)] + J/L \tag{4.15}$$

$$N = N_+ + N_- \quad, \quad J = N_+ - N_- \quad. \tag{4.16}$$

Here $N_\pm$ is the number of particles added to the ground state on the right– and left–moving branch. Because addition of a particle changes both N and J, one has the "selection rule" $(-1)^N = (-1)^J$, and α is a cutoff parameter which (at least in principle, see the discussion above) has to be set to zero in the end of any calculation. ϕ and Π then obey canonical boson commutation relations:

$$[\phi(x), \Pi(y)] = i\delta(x - y) \quad, \tag{4.17}$$

and ϕ is related to the local particle density via

$$\partial \phi / \partial x = -\pi(\rho(x) - \rho_0) \quad, \tag{4.18}$$

where ρ_0 is the average particle density in the ground state. More precisely, in a lattice model this would represent the slowly varying components ($q \approx 0$) of the density, whereas components with $q \approx 2k_\mathrm{F}$ would correspond to crossproducts between $\psi_\pm$.

The expression for the single fermion operators then is

$$\psi_\pm(x) = \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi\alpha}} U_\pm \exp\left[\pm i k_F x \mp i\phi(x) + i\theta(x)\right] \quad, \tag{4.19}$$

where the upper and lower sign refer to right– and left–moving electrons respectively, and

$$\begin{aligned}
\theta(x) &= \pi \int^x \Pi(x') \mathrm{d}x' \\
&= \frac{i\pi}{L} \sum_{p \neq 0} \frac{1}{p} e^{-\alpha|p|/2 - ipx} [\rho_+(p) - \rho_-(p)] + J\frac{\pi x}{L} \quad.
\end{aligned} \tag{4.20}$$

The U–operators decrease the total particle number on one of the branches by unity and are necessary because the boson fields all conserve the total particle number. A detailed derivation of the important eq.(4.19) as an operator identity is given in the literature [24,25]. However, a simple plausibility argument can be given: creating a particle at site x means introducing a kink of height π in ϕ, i.e. at points on the left of x ϕ has to be shifted by π. Displacement operators are exponentials of momentum operators, and therefore a first guess would be $\psi(x) \approx \exp(i\pi \int_{-\infty}^x \Pi(x')dx')$. However, this operator commutes with itself, instead of satisfying canonical anticommutation relations. Anticommutation is achieved by multiplying with an operator, acting at site x, that changes sign each time a particle passes through x. Such an operator is $\exp(\pm i\phi(x))$. The product of these two factors then produces (4.19).

The full Hamiltonian can also be simply expressed in terms of ϕ and Π. Neglecting the momentum dependence of the g_i, one easily finds

$$H = H_0 + H_{int} = \int dx \left[\frac{\pi u K}{2}\Pi(x)^2 + \frac{u}{2\pi K}(\partial_x \phi)^2\right] \quad. \tag{4.21}$$

This is obviously just the Hamiltonian of an elastic string, with the eigenmodes corresponding to the collective density fluctuations of the fermion liquid. It is important to notice that

these collective modes are the only (low–energy) excited states, and that in particular *there are no well–defined single particle excitations*, nor are there the incoherent particle–hole pair excitations typical of a Fermi gas. The parameters in (4.21) are given by

$$u = [(v_F + g_4/(2\pi))^2 - g_2^2/(2\pi)^2]^{1/2} \ , \ K = \left[\frac{2\pi v_F + g_4 - g_2}{2\pi v_F + g_4 + g_2}\right]^{1/2} . \tag{4.22}$$

The energies of the eigenstates are $\omega(q) = u|q|$, in agreement with eq. (4.13).

From the continuity equation, the expression (4.18) for the local particle density and the equation of motion of ϕ the (number) current density is

$$j(x) = uK\Pi(x) \ . \tag{4.23}$$

Note in particular that for $g_2 = g_4$ one has $uK = v_F$, i.e. the expression for the current density is interaction–independent. The relation $uK = v_F$ holds in particular for systems with full (Galilean) translational invariance. On the other hand, in the continuum limit of lattice systems this relation is in general not true.

The most remarkable result here is the "collectivization" of the dynamics: there are no quasiparticle–like excitations. In fact there is a rather simple physical picture explaining this: imagine accelerating one particle a little bit in one direction. Very soon it will hit its neighbor and transmit its momentum to it, and the neighbor will in turn transmit its momentum to a further neighbor, and so on. Quite quickly, the initial localized motion will have spread coherently through the whole system. This picture can be formalized noting that in one dimension the difference between a solid and a fluid is not well–defined: whereas is higher dimensions solids and fluids are differentiated by the presence or absence of long–wavelength transverse modes, no transverse modes can exist in a system with movement along only one direction. The long–wavelength modes thus can equally well be considered as the phonons of a one–dimensional crystal. [27,28] Note that on the contrary in dimensions larger than one the neighbors of any given particle can be pushed aside, giving rise to a backflow that allows the particle to move trough the system more or less freely.

Rather than discussing the physics of the spinless case in detail, we will turn now to the more interesting case of fermions with spin.

B. Spin–1/2 fermions

In the case of spin–1/2 fermions, all the fermion operators acquire an additional spin index s. Following the same logic as above, the kinetic energy then takes the form

$$\begin{aligned} H_0 &= v_F \sum_{k,s} \{(k - k_F)a_{k,s}^\dagger a_{k,s} + (-k - k_F)b_{k,s}^\dagger b_{k,s}\} \\ &= \frac{2\pi v_F}{L} \sum_{q>0,s} [\rho_{+,s}(q)\rho_{+,s}(-q) + \rho_{-,s}(-q)\rho_{-,s}(q)] \ , \end{aligned} \tag{4.24}$$

where density operators for spin projections $s = \uparrow, \downarrow$ have been introduced:

$$\rho_{+,s}(q) = \sum_k a_{k+q,s}^\dagger a_{k,s} \ , \ \ \rho_{-,s}(q) = \sum_k b_{k+q,s}^\dagger b_{k,s} \ . \tag{4.25}$$

There are now two types of interaction. First, the "backward scattering" $(k_F, s; -k_F, t)$ $\rightarrow (-k_F, s; k_F, t)$ which for $s \neq t$ cannot be re–written as an effective forward scattering (contrary to the spinless case). The corresponding Hamiltonian is

$$H_{int,1} = \frac{1}{L} \sum_{kpqst} g_1 a_{k,s}^\dagger b_{p,t}^\dagger a_{p+2k_F+q,t} b_{k-2k_F-q,s} \quad . \tag{4.26}$$

And, of course, there is also the forward scattering, of a form similar to the spinless case

$$H_{int,2} = \frac{1}{2L} \sum_{qst} \{2g_2(q)\rho_{+,s}(q)\rho_{-,t}(-q) \tag{4.27}$$

$$+g_4(q)[\rho_{+,s}(q)\rho_{+,t}(-q) + \rho_{-,s}(-q)\rho_{-,t}(q)]\} \quad . \tag{4.28}$$

To go to the bosonic description, one introduces ϕ and Π fields for the two spin projections separately, and then transforms to charge and spin bosons via $\phi_{\rho,\sigma} = (\phi_\uparrow \pm \phi_\downarrow)/\sqrt{2}$, $\Pi_{\rho,\sigma} = (\Pi_\uparrow \pm \Pi_\downarrow)/\sqrt{2}$. The operators ϕ_ν and Π_ν obey Bose–like commutation relations:

$$[\phi_\nu(x), \Pi_\mu(y)] = i\delta_{\nu\mu}\delta(x-y) \quad ,$$

and single fermion operators can be written in a form analogous to (4.19):

$$\psi_{\pm,s}(x) =$$
$$\lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi\alpha}} \exp\left[\pm ik_F x - i(\pm(\phi_\rho + s\phi_\sigma) + (\theta_\rho + s\theta_\sigma))/\sqrt{2}\right] \quad , \tag{4.29}$$

where $\theta_\nu(x) = \pi \int^x \Pi_\nu(x')dx'$.

The full Hamiltonian $H = H_0 + H_{int,1} + H_{int,2}$ then takes the form

$$H = H_\rho + H_\sigma + \frac{2g_1}{(2\pi\alpha)^2} \int dx \cos(\sqrt{8}\phi_\sigma) \quad . \tag{4.30}$$

Here α is a short-distance cutoff, and for $\nu = \rho, \sigma$

$$H_\nu = \int dx \left[\frac{\pi u_\nu K_\nu}{2}\Pi_\nu^2 + \frac{u_\nu}{2\pi K_\nu}(\partial_x \phi_\nu)^2\right] \quad , \tag{4.31}$$

with

$$u_\nu = [(v_F + g_{4,\nu}/\pi)^2 - g_\nu^2/(2\pi)^2]^{1/2} \quad ,$$
$$K_\nu = \left[\frac{2\pi v_F + 2g_{4,\nu} + g_\nu}{2\pi v_F + 2g_{4,\nu} - g_\nu}\right]^{1/2} \quad , \tag{4.32}$$

and $g_\rho = g_1 - 2g_2$, $g_\sigma = g_1$, $g_{4,\rho} = g_4$, $g_{4,\sigma} = 0$. For a noninteracting system one thus has $u_\nu = v_F$ (charge and spin velocities equal!) and $K_\nu = 1$. For $g_1 = 0$, (4.30) describes independent long-wavelength oscillations of the charge and spin density, with linear dispersion relation $\omega_\nu(k) = u_\nu|k|$, and the system is conducting. As in the spinless case, there are no single–particle or single particle–hole pair excited states. This model (no backscattering), usually called the Tomonaga–Luttinger model, is the one to which the bosonization method was originally applied [23,4,29].

For $g_1 \neq 0$ the cosine term has to be treated perturbatively. We have already obtained the corresponding renormalization group equations in the previous section (eq.(3.23)). In particular, for repulsive interactions ($g_1 > 0$), g_1 is renormalized to zero in the long-wavelength limit, and at the fixed point one has $K_\sigma^* = 1$. The three remaining parameters in (4.30) then completely determine the long-distance and low–energy properties of the system.

It should be emphasized that (4.30) has been derived here for fermions with linear energy–momentum relation. For more general (e.g. lattice) models, there are additional operators arising from band curvature and the absence of high–energy single–particle states. [24] One can however show that all these effects are, at least for not very strong interaction, irrelevant in the renormalization group sense, i.e. they do not affect the low–energy physics. Thus, *(4.30) is still the correct effective Hamiltonian for low–energy excitations*. The lattice effects however intervene to give rise to "higher harmonics" in the expression for the single–fermion operators, i.e. there are low energy contributions at wavenumbers $q \approx (2m + 1)k_F$ for arbitrary integer m. [27]

The Hamiltonian (4.30) also provides an explanation for the physics of the case of negative g_1, where the renormalization group scales to strong coupling (eq.(3.23)). In fact, if $|g_1|$ is large in (4.30), it is quite clear that the elementary excitations of ϕ_σ will be small oscillations around one of the minima of the cos term, or possibly soliton–like objects where ϕ_σ goes from one of the minima to the other. Both types of excitations have a gap, i.e. for $g_1 < 0$ one has a *gap in the spin excitation spectrum*, whereas the charge excitations remain massless. This can actually investigated in more detail in an exactly solvable case. [30]

1. Spin–charge separation

One of the more spectacular consequences of the Hamiltonian (4.30) is the complete separation of the dynamics of the spin and charge degrees of freedom. For example, in general one has $u_\sigma \neq u_\rho$, i.e. the charge and spin oscillations propagate with different velocities. Only in a noninteracting system or if some accidental degeneracy occurs does one have $u_\sigma = u_\rho = v_F$. To make the meaning of this fact more transparent, let us create an extra particle in the ground state, at $t = 0$ and spatial coordinate x_0. The charge and spin densities then are easily found, using $\rho(x) = -(\sqrt{2}/\pi)\partial\phi_\rho/\partial x$ (note that $\rho(x)$ is the deviation of the density from its average value) and $\sigma_z(x) = -(\sqrt{2}/\pi)\partial\phi_\sigma/\partial x$:

$$
\begin{aligned}
\langle 0|\psi_+(x_0)\rho(x)\psi_+^\dagger(x_0)|0\rangle &= \delta(x - x_0) \ , \\
\langle 0|\psi_+(x_0)\sigma_z(x)\psi_+^\dagger(x_0)|0\rangle &= \delta(x - x_0) \ .
\end{aligned}
\tag{4.33}
$$

Now, consider the time development of the charge and spin distributions. The time–dependence of the charge and spin density operators is easily obtained from (4.30) (using the fixed point value $g_1 = 0$), and one obtains

$$
\begin{aligned}
\langle 0|\psi_+(x_0)\rho(x,t)\psi_+^\dagger(x_0)|0\rangle &= \delta(x - x_0 - u_\rho t) \ , \\
\langle 0|\psi_+(x_0)\sigma_z(x,t)\psi_+^\dagger(x_0)|0\rangle &= \delta(x - x_0 - u_\sigma t) \ .
\end{aligned}
\tag{4.34}
$$

Because in general $u_\sigma \neq u_\rho$, after some time charge and spin will be localized at completely different points in space, i.e. *charge and spin have separated completely*. An interpretation of this surprising phenomenon in terms of the Hubbard model will be given in sec.(VI B). For

simplicity we have taken here $K_\rho = 1$, otherwise there would also have been a left–moving part in the time–dependent density.

Here a linear energy–momentum relation has been assumed for the electrons, and consequently the shape of the charge and spin distributions is time–independent. If the energy–momentum relation has some curvature (as is necessarily the case in lattice systems) the distributions will widen with time. However this widening is proportional to $\sqrt{t}$, and therefore much smaller than the distance between charge and spin. Thus, the qualitative picture of spin-charge separation is unchanged.

2. Physical properties

The simple form of the Hamiltonian (4.30) at the fixed point $g_1^* = 0$ makes the calculation of physical properties rather straightforward. The specific heat now is determined both by the charge and spin modes, and consequently the specific heat coefficient γ is given by

$$\gamma/\gamma_0 = \frac{1}{2}(v_F/u_\rho + v_F/u_\sigma) \ . \tag{4.35}$$

Here γ_0 is the specific heat coefficient of noninteracting electrons of Fermi velocity v_F.

The spin susceptibility and the compressibility are equally easy to obtain. Note that in (4.30) the coefficient u_σ/K_σ determines the energy necessary to create a nonzero spin polarization, and, as in the spinless case, u_ρ/K_ρ fixes the energy needed to change the particle density. Given the fixed point value $K_\sigma^* = 1$, one finds

$$\chi/\chi_0 = v_F/u_\sigma \ , \quad \kappa/\kappa_0 = v_F K_\rho/u_\rho \ , \tag{4.36}$$

where χ_0 and κ_0 are the susceptibility and compressibility of the noninteracting case.

The quantity $\Pi_\rho(x)$ is proportional to the current density. As before, the Hamiltonian commutes with the total current, one thus has

$$\sigma(\omega) = 2K_\rho u_\rho \delta(\omega) + \sigma_{reg}(\omega) \ , \tag{4.37}$$

i.e. the product $K_\rho u_\rho$ determines the weight of the dc peak in the conductivity. If the total current commutes with the Hamiltonian σ_{reg} vanishes, however more generally this part of the conductivity varies as ω^3 at low frequencies. [31]

The above properties, linear specific heat, finite spin susceptibility, and dc conductivity are those of an ordinary Fermi liquid, the coefficients u_ρ, u_σ, and K_ρ determining renormalizations with respect to noninteracting quantities. However, the present system is *not a Fermi liquid*. This is in fact already obvious from the preceding discussion on charge–spin separation, and can be made more precise calculating the single–particle Green function using the representation (4.29) of fermion operators. One then obtains the momentum distribution function in the vicinity of k_F

$$n_k \approx n_{k_F} - \text{const.}\,\text{sign}(k - k_F)|k - k_F|^\delta \ , \tag{4.38}$$

and the single-particle density of states (i.e. the momentum–integrated spectral density):

$$N(\omega) \approx |\omega|^\delta \ . \tag{4.39}$$

27

In both cases $\delta = (K_\rho + 1/K_\rho - 2)/4$. Note that for any $K_\rho \neq 1$, i.e. *for any nonvanishing interaction*, the momentum distribution function and the density of states have power–law singularities at the Fermi level, with a vanishing single particle density of states at E_F. This behavior is obviously quite different from a standard Fermi liquid which would have a finite density of states and a step–like singularity in n_k. The absence of a step at k_F in the momentum distribution function implies the *absence of a quasiparticle pole* in the one–particle Green function. In fact, a direct calculation of the spectral function $A(k, \omega)$ [32,33] shows that the usual quasiparticle pole is replaced by a continuum, with a lower threshold at $\min(u_\nu)(k - k_F)$ and branch cut singularities at $\omega = u_\rho p$ and $\omega = u_\sigma p$:

The coefficient K_ρ also determines the long-distance decay of all other correlation functions of the system: Using the representation (4.29) the charge and spin density operators at $2k_F$ are

$$O_{CDW}(x) = \sum_s \psi_{-,s}^\dagger \psi_{+,s} = \lim_{\alpha \to 0} \frac{e^{2ik_F x}}{\pi \alpha} e^{-i\sqrt{2}\phi_\rho} \cos[\sqrt{2}\phi_\sigma] \ , \tag{4.40}$$

$$O_{SDW_x}(x) = \sum_s \psi_{-,s}^\dagger \psi_{+,-s} = \lim_{\alpha \to 0} \frac{e^{2ik_F x}}{\pi \alpha} e^{-i\sqrt{2}\phi_\rho} \cos[\sqrt{2}\theta_\sigma] \ . \tag{4.41}$$

Similar relations are also found for other operators. It is important to note here that all these operators decompose into a product of one factor depending on the charge variable only by another factor depending only on the spin field. Using the Hamiltonian (4.30) at the fixed point $g_1^* = 0$ one finds for example for the charge and spin correlation functions[1]

$$\langle n(x)n(0) \rangle = K_\rho/(\pi x)^2 + A_1 \cos(2k_F x) x^{-1-K_\rho} \ln^{-3/2}(x)$$
$$+ A_2 \cos(4k_F x) x^{-4K_\rho} + \dots \ , \tag{4.42}$$

$$\langle \vec{S}(x) \cdot \vec{S}(0) \rangle = 1/(\pi x)^2 + B_1 \cos(2k_F x) x^{-1-K_\rho} \ln^{1/2}(x) + \dots \ , \tag{4.43}$$

with model dependent constants A_i, B_i. The ellipses in (4.42) and (4.43) indicate higher harmonics of $\cos(2k_F x)$ which are present but decay faster than the terms shown here. Similarly, correlation functions for singlet (SS) and triplet (TS) superconducting pairing are

$$\langle O_{SS}^\dagger(x) O_{SS}(0) \rangle = C x^{-1-1/K_\rho} \ln^{-3/2}(x) + \dots \ ,$$
$$\langle O_{TS_\alpha}^\dagger(x) O_{TS_\alpha}(0) \rangle = D x^{-1-1/K_\rho} \ln^{1/2}(x) + \dots \ . \tag{4.44}$$

The logarithmic corrections in these functions [35] have been studied in detail recently [36–39]. The corresponding susceptibilities (i.e. the Fourier transforms of the above correlation functions) behave at low temperatures as

$$\chi_{CDW}(T) \approx T^{K_\rho - 1} |\ln(T)|^{-3/2} \ , \chi_{SDW}(T) \approx T^{K_\rho - 1} |\ln(T)|^{1/2} \ ,$$
$$\chi_{SS}(T) \approx T^{1/K_\rho - 1} |\ln(T)|^{-3/2} \ , \chi_{TS}(T) \approx T^{1/K_\rho - 1} |\ln(T)|^{1/2} \ , \tag{4.45}$$

i.e. for $K_\rho < 1$ (spin or charge) density fluctuations at $2k_F$ are enhanced and diverge at low temperatures, whereas for $K_\rho > 1$ pairing fluctuations dominate. The "phase diagram", showing in which part of parameter space which type of correlation diverges for $T \to 0$ is shown in fig.7.

---

[1]The time- and temperature dependence is also easily obtained, see [34].
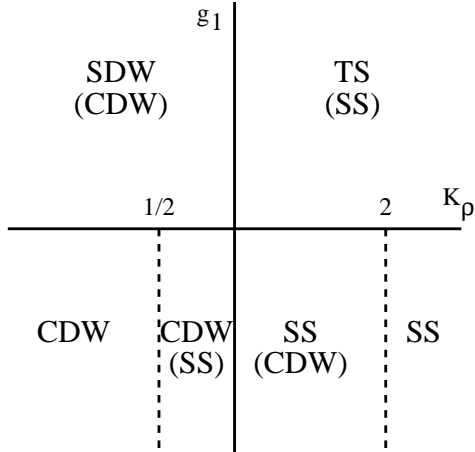
FIG. 7. Phase diagram for interacting spin–1/2 fermions.

A remarkable fact in all the above results is that there is only *one coefficient*, K_ρ, which determines all the asymptotic power laws, i.e. there are scaling relations between the exponents of different correlation functions. These relations would have been impossible to obtain within a purely perturbative approach. [18] The correlation functions with their power law decay determine experimentally accessible quantities: the $2k_F$ and $4k_F$ charge correlations lead to X–ray scattering intensities $I_{2k_F} \approx T^{K_\rho}$, $I_{4k_F} \approx T^{4K_\rho-1}$, and similarly the NMR relaxation rate due to $2k_F$ spin fluctuations varies as $1/T_1 \approx T^{K_\rho}$. Experimental observations of the NMR relaxation rate [40] and of photoemission spectra [41] on the quasi–one–dimensional organic compound $(\mathrm{TMTSF})_2\mathrm{PF}_6$ suggest $K_\rho \approx 0.25$, i.e. rather strong non–Fermi liquid behavior. It is currently unclear whether this interpretation is compatible with the transport properties.

We here re–emphasize the two important properties of spin–1/2 interacting fermions in one dimension: (i) correlation functions show power–law decay, with interaction–dependent powers determined by one coefficient, K_ρ; and (ii) "spin–charge separation": spin and charge degrees of freedom propagate with different velocities. Both these properties invalidate the Landau quasiparticle concept in one dimension. Rather, the name "Luttinger liquid" has been coined to characterize the properties of one–dimensional interacting fermions. [24]

3. Long–range interactions

The above calculations can be straightforwardly generalized to the case of long–range interactions. [42] Of interest is the case of unscreened Coulomb interactions ($V(r) = e^2/r$) which for example is of relevance for the physics of an isolated quantum wire. The short–distance singularity has to be cut off, and for example in a wire of diameter d an approximate from would be $V(r) = e^2/\sqrt{x^2 + d^2}$, leading to a Fourier transform $V(q) = g_2(q) = g_4(q) = 2e^2 K_0(qd)$. The long–range nature of the interaction is only of importance for the forward–scattering processes, and these appear only in the charge part of the Hamiltonian which remains diagonal. The elementary excitations then are found to be charge oscillations (plasmons), with energy–momentum relation

$$\omega_\rho(q) = v_F|q|[(1 + \tilde{g}_1)(1 - \tilde{g}_1 + 2\tilde{V}(q))]^{1/2} \tag{4.46}$$

29

where $\tilde{V}(q) = V(q)/(\pi v_F)$ and $\tilde{g}_1 = g_1/(2\pi v_F)$. The long–wavelength form, $\omega_\rho(q) \approx |q^2 \ln q|^{1/2}$, agrees with RPA calculations [43,44], however, the effect of g_1, which is a short–range exchange contribution, is usually neglected in those calculations. The spin modes are still given by $\omega_\sigma(q) = u_\sigma|q|$, with $u_\sigma = v_F\sqrt{1 - \tilde{g}_1^2}$. Distinct charge– and spin modes have indeed been observed in Raman scattering from quantum wires. [45]

Correlation functions can be similarly calculated and one obtains for example

$$
\begin{aligned}
\langle \rho(x)\rho(0) \rangle =\ & A_1 \cos(2k_F x) \exp(-c_2\sqrt{\ln x})/x \\
& + A_2 \cos(4k_F x) \exp(-4c_2\sqrt{\ln x}) + ... \ ,
\end{aligned}
\tag{4.47}
$$

where $A_{1,2}$ are interaction dependent constants, $c_2 = \sqrt{(1+\tilde{g}_1)\pi v_F/e^2}$, and only the most slowly decaying Fourier components are exhibited. The most interesting point here is the extremely slow decay (much slower than any power law!) of the $4k_F$ component, showing an incipient charge density wave at wavevector $4k_F$. This slow decay should be compared with the power–law decay found for short–range interactions (eq.(4.42)). The $4k_F$ oscillation period is exactly the average interparticle spacing, i.e. the structure is that expected for a one–dimensional *Wigner crystal*. Of course, because of the one–dimensional nature of the model, there is no true long–range order, however, the extremely slow decay of the $4k_F$ oscillation would produce strong quasi–Bragg peaks in a scattering experiment. It is worthwhile to point out that this $4k_F$ contribution arises even if the Coulomb interaction is extremely weak and depends only on the long–range character of the interaction. On the other hand, any $2k_F$ scattering is considerably weaker, due to the $1/x$ prefactor in (4.47) which has its origin in the contribution of spin fluctuations. On the other hand, correlation functions that involve operators changing the total number of particles (e.g. the single particle Green function and pairing correlation functions) decay like $\exp[-cst.(\ln x)^{-3/2}]$, i.e. *faster* than any power law. This in particular means that the momentum distribution function n_k and all its derivatives are continuous at k_F, and there is only an essential singularity at k_F.

It is instructive to notice that the above result (4.47), obtained in the limit of weak Coulomb interactions, can equally be derived assuming strong long–range repulsion [42] by expanding around the equilibrium configuration of an equally spaced crystal. We thus are lead to the rather remarkable conclusion that the long–distance behavior of correlation functions is independent of the strength of the Coulomb repulsion, provided the interaction is truly long–ranged. We note that an interpretation of Luttinger liquid properties in terms of a one–dimensional harmonic chain can also be given for short–range interactions [27,28,46].

4. Persistent current

The Luttinger model description can be used straightforwardly to obtain the current induced in a strictly one–dimensional ring threaded by a magnetic flux Φ. [47,48] The argument can in fact be made very simply: in the one–dimensional geometry, the vector field can be removed entirely from the Hamiltonian via a gauge transformation, which then leads to the boundary condition $\psi(L) = \exp(2\pi i\Phi/\Phi_0)\psi(0)$ for the fermion field operator. Here L is the perimeter of the ring, and $\Phi_0 = hc/e$. For spinless fermions, this is achieved by

replacing

$$\Pi(x) \to \Pi(x) + \frac{2\Phi}{L\Phi_0} \tag{4.48}$$

in the bosonization formula, eq.(4.19). The total J–dependent part of the Hamiltonian then becomes

$$H_J = \frac{\pi u K}{2L}(J + 2\Phi/\Phi_0)^2 \quad, \tag{4.49}$$

giving rise to a number current

$$j = \frac{\Phi_0}{2\pi}\frac{\partial E}{\partial \Phi} = \frac{uK}{L}\left(J + \frac{2\Phi}{\Phi_0}\right) \quad . \tag{4.50}$$

At equilibrium, J is chosen so as to minimize the energy. Given that at constant total particle number J can only change by two units, one easily sees that the equilibrium (persistent) current has periodicity Φ_0, and reaches is maximum value uK/L at $\Phi = \Phi_0/2$, giving rise to the familiar sawtooth curve for the current as a function of flux.

For fermions with spin, as long as there is no spin gap ($g_1 > 0$), the above results can be taken over, with the simple replacement $uK \to 2u_\rho K_\rho$, the factor 2 coming from the spin degeneracy. Note in particular that the persistent current, an equilibrium property, is given by the same combination of parameters as the Drude weight in the conductivity. This is an illustration of Kohn's result [49] relating the Drude weight to the effect of a magnetic flux through a ring.

In the case of negative g_1, electrons can be transfered from the right to the left–going branch only by pairs, Consequently, the periodicity of the current and the ground state energy is doubled to $2\Phi_0$, and the maximum current is equally doubled. This behavior has actually been found in numerical calculations. [50,51]

V. TRANSPORT IN A LUTTINGER LIQUID

In the previous section we were concerned with equilibrium properties and correlation functions, in order to characterize the different phases possible in a one–dimensional system of interacting fermions. Here, we will investigate transport, in particular the dc conductivity. Finite–frequency effects have also been investigated, and the reader is referred to the literature. [31,52]

A. Conductance and conductivity

To clarify some of the basic notions, let us first consider a Luttinger model in the presence of a weak space– and time–dependent external potential φ. The interaction of the fermions with φ is described by the extra term

$$H_{ext} = -e \int dx \hat{\rho}(x)\varphi(x,t) \tag{5.1}$$

in the total Hamiltonian. We will assume that the external field is slowly varying in space, so that in the particle–density operator $\hat{\rho}$ only products of either two right– or two left–going

fermion operators appear but no cross terms. Standard linear response theory tells us that the current induced by the potential is given by

$$j(x,t) = -\frac{e^2}{\hbar} \int_{-\infty}^{t} dt' \int dx' D_{j\rho}(x-x', t-t')\varphi(x', t') \ , \tag{5.2}$$

where the *retarded current–density correlation function* is given by

$$\begin{aligned} D_{j\rho}(x,t) &= -i\theta(t)\langle [j(x,t), \rho(0,0)]\rangle \\ &= -\frac{u_\rho K_\rho}{\pi}\theta(t)[\delta'(x-u_\rho t) + \delta'(x+u_\rho t)] \ . \end{aligned} \tag{5.3}$$

The second line is the result for spin–1/2 electrons. For spinless fermions one has to make the replacement $u_\rho K_\rho \to uK/2$.

Let us now first consider the situation where we adiabatically switch on a potential of frequency ω and wavenumber q along the whole length of the system. From eq.(5.3) one then straightforwardly obtains the q– and ω–dependent conductivity as

$$\sigma(q,\omega) = \frac{4e^2}{\hbar} u_\rho K_\rho \frac{i(\omega+i\delta)}{(\omega+i\delta)^2 - u_\rho^2 q^2} \ . \tag{5.4}$$

In particular, the real part of the conductivity for constant applied field is

$$\sigma(0,\omega) = \frac{2e^2}{\hbar} u_\rho K_\rho \delta(\omega) \ , \tag{5.5}$$

in agreement with eq.(4.37) (where units with $e^2 = \hbar = 1$ were used).

Applying on the other hand a static field over a finite part of the sample, one obtains a current $j = 2e^2 K_\rho U/h$, where U is the applied tension. The conductance thus is

$$G = \frac{2e^2}{h} K_\rho \ , \tag{5.6}$$

and depends on K_ρ only, not on u_ρ. For the noninteracting case $K_\rho = 1$ this is Landauer's well–known result. [53] Note that interactions affect the value of the conductance. The conductance here is independent of the length over which the field is applied. Noting that in dimension d the conductance is related to the dc conductivity via $G = L^{d-2}\sigma$, a length–independent conductivity implies an infinite conductivity in one dimension, in agreement with eq.(5.3). The fact that u_ρ does not appear in the expression for G can be understood noting that applying a static field over a finite (but large) part of the sample, one is essentially studying the wavenumber–dependent conductivity at strictly zero frequency, which from eq.(5.15) is given by $\sigma(q,0) = 2e^2 K_\rho \delta(q)/\hbar$, indeed independent of u_ρ. On the other hand, applying a field of finite frequency over a finite length ℓ, one can see that one measures the conductivity $\sigma(q \to 0, \omega)$ if $\omega > u_\rho/\ell$.

B. Edge states in the quantum Hall effect

The Luttinger liquid picture has an interesting application to the physics of the fractional quantum Hall effect, as discovered and discussed by Wen. [54,55] To see how this comes
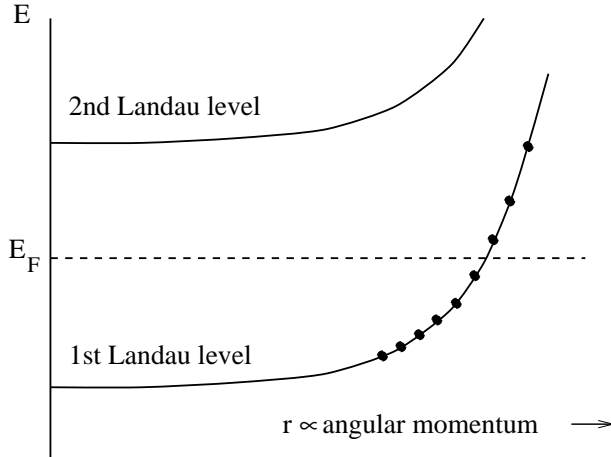
FIG. 8. Quantum states in the lowest Landau level in the vicinity of the edge of a quantum hall device. The spatial variation of the confining potential is assumed to be slow on the scale of the magnetic length, so that the energies of the different quantum states are determined by the local value of the confining potential. For a circular device the angular momentum of a state increases proportionally to its distance r from the origin.

about, consider the states available in the different Landau levels in the vicinity of the edge of the quantum Hall device, as shown in fig.8. [56] It is clear that low–energy states only exist at the edge (the bulk quantum Hall state is well-known to be characterized by a finite excitation gap), and close to the Fermi energy (i.e. the edge) the states have a linear dispersion relation. This can be made particularly clear if one assumes a disk–shaped sample: the states have a wavefunction

$$\approx z^k \approx (e^{i\theta})^k \ ,$$

with k increasing linearly with radial position. The angular momentum quantum number k thus plays a role very similar to linear momentum in the linear geometry we have assumed up to now. One thus can linearize the the dispersion in fig.8 and obtains essentially the spinless model discussed in section IV A, the only difference being that here only right–going particles exist. This difference is the origin of the term *chiral* Luttinger liquid (in fact, the left going branch is to be found on the opposite edge of the device). Because there are no left–going particles (or at least they can be thought of as being at a macroscopic distance), there also is no right–left interaction, and consequently one expects the noninteracting value $K = 1$. Moreover all the left–going components of the fields have to be projected out, for example one has to replace $\phi \to \phi_+ = (\phi - \theta)/2$.

However, straightforward adoption of this scheme leads to trouble: from the preceding subsection we know that $K = 1$ leads to a conductance (which in the present case is the Hall conductance) of $G = e^2/h$, *different* from the well–known

$$G = \nu \frac{e^2}{h} \tag{5.7}$$

valid for a fractional quantum Hall state ($\nu = 1/m$ is the filling factor). To repair this problem one makes the *hypothesis* that instead of eqs.(4.18) and (4.23) one has

$$\rho(x) = -\frac{\sqrt{\nu}}{\pi}\frac{\partial\phi_+}{\partial x} \quad , \qquad j(x) = u\sqrt{\nu}\Pi_+(x) \quad , \tag{5.8}$$

where the subscripts indicate projection on right–going states. With these definitions following the calculations of the previous subsection one now straightforwardly reproduces the correct result, eq.(5.7). The appearance of the factors $\sqrt{\nu}$ in eq.(5.8) indicates that the objects occupying the states in fig.8 are not free electrons but rather strongly affected by the physics of the bulk of the samples. A more detailed derivation, starting from a Chern–Simons field theory for the bulk physics, has also been given by Wen. [55]

Beyond reproducing the correct value of the Hall conductance, the above hypothesis leads to a number of interesting conclusions. Consider first the creation operator for a real electron (charge e) on the edge. Following the arguments of section IV A, because of eq.(5.8), the bosonized version of the electron operator now must create a jump of ϕ of height $\pi/\sqrt{\nu}$, rather than of height π. This leads to

$$\psi_+(x) \approx e^{-i\phi_+(x)/\sqrt{\nu}} \quad , \tag{5.9}$$

x being the coordinate along the perimeter of the sample. Now, these operators obey the relation

$$\psi_+(x')\psi_+(x) = e^{\pm i\pi/\nu}\psi_+(x)\psi_+(x') \quad . \tag{5.10}$$

But the real electron is still a fermion, i.e. ψ_+ must obey anticommutation relations. Thus $m = 1/\nu$ *has to be an odd integer*. One thus reproduces one of the fundamental facts of the fractional quantum Hall effect. From eq.(5.9) one also finds a decay of the single–electron Green function as

$$G(x,t) \propto \frac{1}{(x-ut)^{1/\nu}} \quad . \tag{5.11}$$

Another fundamental property of the quantum Hall state appears when one considers the fractionally charged elementary excitation of charge $e\nu$ at the edge. A charge–$e\nu$ object is created by

$$\psi_{+\nu}(x) \approx e^{-i\sqrt{\nu}\phi_+(x)} \quad , \tag{5.12}$$

leading to a slow decay of the corresponding Green function, with exponent ν, instead of $1/\nu$ in eq.(5.11). One now has the relation

$$\psi_{+\nu}(x')\psi_{+\nu}(x) = e^{\pm i\pi\nu}\psi_{+\nu}(x)\psi_{+\nu}(x') \quad . \tag{5.13}$$

i.e. the fractionally charged objects also obey fractional statistics.

A single hypothesis, the insertion of the factors $\sqrt{\nu}$ in eq.(5.8), thus reproduces two of the fundamental facts about the fractional quantum Hall effect! In addition one obtains results for the asymptotics of Green functions.

C. Single barrier

The infinite conductivity in the ideally pure systems considered up to here is a natural but hardly realistic result: (almost) any realistic system will contain some form of inhomogeneity. This in general leads to a finite conductivity, and in one dimension one can anticipate even more dramatic effects: in a noninteracting system any form of disorder leads to localization of the single–particle eigenstates. [57,58] How this phenomenon occurs in interacting systems will be discussed in this and the following section.

Following Kane and Fisher, [59] consider first the case of a single inhomogeneity in an otherwise perfect one–dimensional system. The extra term in the Hamiltonian introduced by a localized potential $v(x)$ is (for spinless fermions)

$$H_{barrier} = \int \mathrm{d}x \psi^\dagger(x) \psi(x) \ . \tag{5.14}$$

Decomposing the product of fermion operators into right– and left–going parts, one has

$$\psi^\dagger \psi = \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- + \psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+ \ . \tag{5.15}$$

In the bosonic representation, the first two terms are proportional to $\partial_x \phi$ (see eq.(4.18)), and therefore the corresponding contribution in eq.(5.14) can in fact be eliminated by a simple unitary transformation of ϕ. These terms represent scattering with momentum transfer $q \ll 2k_F$, i.e. they do not transfer particles between k_F and $-k_F$ and therefore do not affect the conductance in any noticeable way. On the other hand, the last two terms in eq.(5.15) represent scattering with $|q| \approx 2k_F$, i.e. from the $+$ to the $-$ branch and vice versa. These terms certainly are expected to affect the conductance, because they change the direction of propagation of the particles. The bosonic representation of these terms is

$$H_{barrier} = \frac{V(2k_F)}{\pi\alpha} \cos 2\phi(0) \ , \tag{5.16}$$

where the potential $V(x)$ is assumed to be centered at $x = 0$. For this reason, only the value of the ϕ at $x = 0$ intervenes in eq.(5.16).

One now can integrate out all the degrees of freedom away from $x = 0$, to obtain an effective action implying only the time–dependence of $\phi(0)$. Then a renormalization group equation for $V \equiv V(2k_F)$ can be found as

$$\frac{\mathrm{d}V}{\mathrm{d}\ell} = (1 - K)V \ , \tag{5.17}$$

where $E = E_0 e^{-\ell}$, E_0 is the original cutoff, and E is the renormalized cutoff.

From eq.(5.17) it is clear that there are three regimes:

1. For $K > 1$ one has $V(\ell \to \infty) = 0$, i.e. as far as the low–energy physics is concerned, the system behaves like one without the barrier. In particular, the low–temperature conductance takes the "pure" value $G = e^2 K/h$, with corrections of order $T^{2(K-1)}$. We note that in this case superconducting fluctuations dominate, and the prefect transmission through the barrier can be taken as a manifestation of superconductivity in the one–dimensional system.

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2. For the noninteracting case $K = 1$, V is invariant, and one thus has partial transmission and a non–universal conductance depending on V.

3. For $K < 1$ $V(\ell)$ scales to infinity. Though the perturbative calculation does not provide any direct way to treat this case, it is physically clear that the transmission and therefore the conductance should vanish.

Note that the non–interacting case is marginal, separating the regions of perfect and zero transmission. These results are very similar to earlier ones by Luther and Peschel [60] who treat disorder in lowest–order perturbation theory.

The case of $K < 1$ can be further analyzed considering the case of two finite Luttinger liquids coupled by a weak tunneling barrier, as would be appropriate for a strong local potential. The barrier Hamiltonian then is

$$H_{barrier} = t[\psi_1^\dagger(0)\psi_2(0) + \psi_2^\dagger(0)\psi_1(0)] \approx \frac{t}{\pi\alpha}\cos 2\theta(0) \quad . \tag{5.18}$$

Here $\psi_{1,2}$ are the field operators to the left and to the right of the barrier. The operators have to satisfy the *fixed boundary condition* $\psi_i(x = 0) = 0$, different from the periodic boundary conditions we have used so far. Noting that the ψ_i can be decomposed into left– and right–going parts as $\psi_i = \psi_{i+} + \psi_{i-}$, and using eq.(4.19 this can be achieved by imposing the fixed boundary condition $\phi_i(x = 0) = \pi/2$ on the boson field. [61,62]

One can now proceed in complete analogy to the weak–V case to obtain the renormalization group equation

$$\frac{\mathrm{d}t}{\mathrm{d}\ell} = (1 - 1/K)t \quad . \tag{5.19}$$

Again, there are three different regimes: (i) for $K > 1$ now $t(\ell \to \infty) \to \infty$, i.e. the tunneling amplitude becomes very big. This can be interpreted as indicating perfect transmission, e.g. $G = e^2 K/h$; (ii) the case $K = 1$ remains marginal, leading to a t–dependent conductance; (iii) for $K < 1$ t scales to zero, there thus is no transmission, and $G = 0$.

The results obtained in the two limiting cases of small V (weak scattering) and of small t (weak tunneling) are clearly compatible: e.g. for $K < 1$ V becomes large, i.e. at sufficiently low energies one expects essentially a tunneling type behavior, and then from eq.(5.19) the tunneling amplitude actually does scale to zero, giving zero conductance in the low–energy (or low–temperature) limit. For $K \geq 1$ a similar compatibility of the two limiting cases is found. The global behavior can be represented by the "phase diagram" in fig.9.

For electrons with spin but spin–independent interactions, results are very similar: the separation between zero and perfect transmission is at $K_\rho = 1$, with $K_\rho = 1$ again the marginal case. In the transmitting region the conductance is $G = 2K_\rho e^2/h$.

These considerations can be generalized to the case of two barriers. [59,63] In particular, assuming that there are two identical, weakly scattering barrier at $\pm d$, the effective scattering potential becomes $V_{eff}(q) = 2v(q)\cos(qd/2)$. Though in general this is non–zero when $V(q)$ is non–zero, for particular values of k_F, so that $\cos(k_F d) = 0$ this potential vanishes, giving rise to perfect transmission even for $K < 1$. This *resonant scattering* condition corresponds to an average particle number between the two barriers of the form $\nu + 1/2$, with integer ν, i.e. the "island" between the two barriers is in a degenerate state. If interactions between
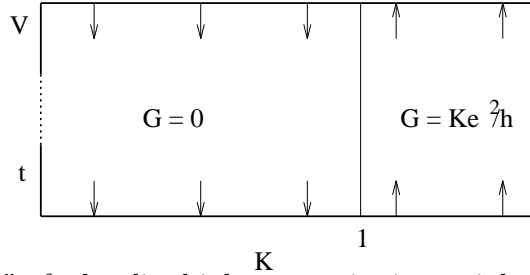
FIG. 9. "Phase diagram" of a localized inhomogeneity in a spinless Luttinger liquid, characterized by an exponent parameter K, according to [59]. The scaling trajectories calculated for weak V or t are indicated by arrows. It is clearly plausible to assume direct scaling from weak to strong coupling in the whole range of K.

the electrons in the island are included, one can recover the physics of the Coulomb blockade. [59,63]

For the chiral Luttinger liquid discussed in the preceding section backscattering events a priori seem to be excluded because all the particles are moving in the same direction. In that sense the chiral Luttinger liquid can be considered as "perfect". However, if the quantum hall device has a constriction that brings the two edges close to each other, scattering from one edge to the other becomes possible and is the equivalent of backscattering. Then similar considerations as made for the single–impurity case are possible [64], and in particular the crossover function describing the conductance through a resonance as a function of temperature has been obtained. [65]

D. Random potentials and localization

The discussion of the previous section was concerned with the effect of a single impurity, weak or strong. Clearly, in that case the effects of coherent scattering from many impurities which typically give rise to Anderson localization are absent. We now turn to this more complicated case which had been studied in fact well before the single impurity work. [66–69]

In the absence of electron–electron interactions, localization effects can be discussed in the framework of a scaling theory. [58] Under the assumption that at some short length scale one has elastic scattering of electrons off impurities, this theory leads to the following β–function for the variation of the conductance with linear dimension L:

$$\beta(G) = \frac{d\ln(G)}{d\ln(L)} = d - 2 - \frac{a}{G} + \ldots \;, \tag{5.20}$$

where a is a constant and d the spatial dimensionality. In particular in one dimension this leads to a conductance decaying exponentially with the length of the system, exhibiting clearly the localized character of the all single–electron states (a fact first shown by Mott [57] and studied in great detail since [70,71]).

In an interacting one–dimensional system (as described by the Luttinger liquid picture of the previous section) now a number of questions arise: what is the influence of disorder on the phase diagram obtained previously? What are the transport properties? Can one have true superconductivity in one dimension, i.e. infinite conductivity in a disordered system?

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To answer these question we discuss below the generalizations necessary to include disorder in our previous picture.

We start by the standard term in the Hamiltonian describing the coupling of a random potential to the electron density

$$H_{imp} = \sum_{R_i} \int dx \; V(x - R_i)\hat{\rho}(x) \; , \tag{5.21}$$

where the R_i are the random positions of impurity atoms, each acting with a potential V on the electrons. In one dimension one can distinguish two types of processes: (i) forward scattering, where the scattered particle remains in the vicinity of its Fermi point. As in the single–impurity case, this leads to a term proportional to $\partial\phi_\rho$ and can be absorbed by a simple redefinition of the ϕ_ρ field. The physical effects are minor, and in particular the dc conductivity remains infinite. (ii) backward scattering where an electron is scattered from k_F to $-k_F$ or vice versa. For small impurity density this can be represented by a complex field ξ with Gaussian distribution of width $D_\xi = n_i V(q = 2k_F)^2$:

$$H_b = \sum_\sigma \int dx \; [\xi(x)\psi_{R\sigma}^\dagger(x)\psi_{L\sigma}(x) + h.c.] \tag{5.22}$$

This term has has dramatic effects and in particular leads to Anderson localization in the noninteracting case. [72]

From a perturbative expansion in the disorder one now obtains a set of coupled renormalization group equations [69]:

$$\begin{aligned}
\frac{dK_\rho}{d\ell} &= -\frac{u_\rho}{2u_\sigma}K_\rho^2 \mathcal{D} \\
\frac{dK_\sigma}{d\ell} &= -\frac{1}{2}(\mathcal{D} + y^2)K_\sigma^2 \\
\frac{dy}{d\ell} &= 2(1 - K_\sigma)y - \mathcal{D} \\
\frac{d\mathcal{D}}{d\ell} &= (3 - K_\rho - K_\sigma - y)\mathcal{D}
\end{aligned} \tag{5.23}$$

where $\mathcal{D} = 2D_\xi\alpha/(\pi u_\sigma^2)\,(u_\sigma/u_\rho)^{K_\rho}$ is the dimensionless disorder, $y = g_{1\perp}/(\pi u_\sigma)$ is the dimensionless backscattering amplitude, and the K_ν are defined in eq.(4.32). These equations are valid for *arbitrary* K_ν (the usual strength of bosonization), but to lowest order in $\mathcal{D}$ and y.

As a first application of eqs.(5.23) one can determine the effect of the random potential on the "phase diagram", as represented in fig.7. In fact, there are three different regimes:

1. for $K_\rho > 2$ and $g_{1\perp}$ sufficiently positive the fixed point is $\mathcal{D}^*, y^* = 0, K_\rho^* \geq 2$. Because the effective random potential vanishes this is a delocalized region, characterized as in the pure case by the absence of a gap in the spin excitations and dominant TS fluctuations.

2. For $K_\rho > 3$ and $g_{1\perp}$ small or negative one has $\mathcal{D}^* = 0$, $y \to -\infty$, $K_\rho^* \geq 3$. Again, this is a delocalized region, but now because $y \to -\infty$ there is a spin gap and one has predominant SS fluctuations.

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3. In all other cases one has $\mathcal{D} \to \infty, y \to -\infty$. This corresponds to a localized regime. For small K_ρ the bosonized Hamiltonian in this regime is that of a charge density wave in a weak random potential with small quantum fluctuations parameterized by K_ρ. This region can therefore be identified as a weakly pinned CDW, also called a "charge density glass" (CDG). The transition from the CDG to the SS region then can be seen as depinning of the CDW by quantum fluctuations.

One should notice that the CDG is a nonmagnetic spin singlet, representing approximately a situation where localized single–particle states are doubly occupied. Though this is acceptable for attractive or possibly weakly repulsive interactions, for strong short–range repulsion single occupancy of localized states seems to be more likely. One then has a spin in each localized state, giving probably rise to a localized antiferromagnet with random exchange (RAF). A detailed theory of the relative stability of the two states is currently missing and would certainly at least require higher-order perturbative treatment. The boundaries of the different regimes can be determined in many cases from eqs.(5.23), and the resulting phase diagram is shown in fig.10.
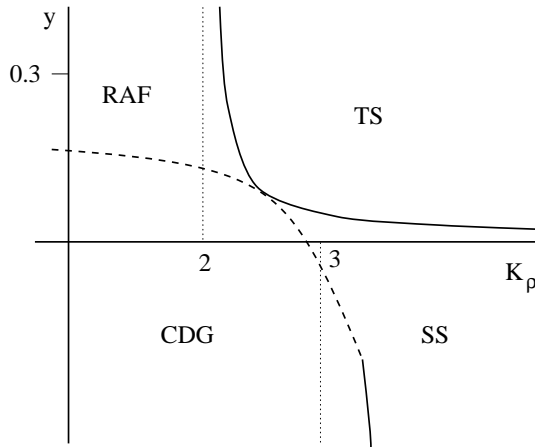

FIG. 10. Phase diagram of a Luttinger liquid in the presence of a weak random potential ($\mathcal{D} = 0.05$). The full lines represent results obtained directly from the scaling equations (5.23), the dashed lines are qualitative interpolations. The dotted lines are the phase boundaries in the limit $\mathcal{D} \to 0$.

The localization length for small disorder can be obtained from standard scaling arguments: suppose that a system with some fixed disorder $\mathcal{D}_0$ has a localization length ξ_0. Then in the general case one has $\xi(\mathcal{D}) = \xi_0 e^{\ell(\mathcal{D}_0, \mathcal{D})}$, where $\ell(\mathcal{D}_0, \mathcal{D})$ is the time it takes for the "bare" disorder $\mathcal{D}$ to scale up to $\mathcal{D}_0$. From this reasoning one finds, for the case without a spin gap ($g_1 > 0$) and weak disorder

$$\xi(\mathcal{D}) \propto (1/\mathcal{D})^{1/(2-K_\rho)} \quad . \tag{5.24}$$

Note that for $K_\rho > 1$, i.e. superconducting fluctuations predominating in the pure case ξ is greater than the mean free path $\lambda \propto 1/\mathcal{D}$, i.e. there is a kind of diffusive regime, contrary to the noninteracting case. On the other hand, for $K_\rho < 1$ one has $\xi < \lambda$. In the vicinity of the TS–RAF boundary one has

$$\xi(\mathcal{D}) \propto \exp\left(\frac{K_\rho - 2}{\mathcal{D} - y(K_\rho - 2)} \right) \quad . \tag{5.25}$$

The analogous results for the case with a spin gap ($g_1 < 0$) are

$$\xi(\mathcal{D}) \propto (1/\mathcal{D})^{1/(3-K_\rho)} \tag{5.26}$$

$$\xi(\mathcal{D}) \propto \exp\left(\frac{2\pi}{\sqrt{9\mathcal{D} - (K_\rho - 3)^2}}\right) \tag{5.27}$$

There are two points to be noted about this result: (i) for $K_\rho = 0$ one has $\xi \propto \mathcal{D}^{-1/3}$, which is the same result as that found for the pinning length of a classical CDW. [73] (ii) the results (5.24), (5.25) and (5.26), eq.(5.27) are qualitatively different, both in the vicinity of the phase boundaries and in the localized states. The transitions are thus in different universality classes, and this strongly supports the idea that the localized phases reached through the transition are themselves different (RAF or CDG).

The temperature dependence of the dc conductivity can be obtained noting that at finite temperature there are no coherent effects on length scales larger than v_F/T. One therefore stops renormalization at $e^{\ell^*} = v_F/(\alpha T)$. As long as $\mathcal{D}$ remains weak one can still use the Born approximation to obtain

$$\sigma(\mathcal{D}) = \sigma_0 \frac{e^\ell \mathcal{D}}{\mathcal{D}(\ell)} \quad, \tag{5.28}$$

where $\sigma_0 = e^2 v_F^2/2\pi\hbar D_\xi$ is the lowest order conductivity. In the delocalized phases one then finds a conductivity diverging as $\sigma(T) \sim T^{-1-\gamma}$ where $\gamma = K_\rho^* - 2$ in the TS case and $\gamma = K_\rho^* - 3$ in the SS case. On the phase boundaries one has universally $\sigma \sim 1/T$. In the localized region $\mathcal{D}$ diverges at low temperatures, and a perturbative calculation thus becomes meaningless. However, the conducting–localized crossover can still be studied at not too low temperatures. [69] In particular, the high–temperature conductivity is found to vary as $\sigma \sim T^{1-K_\rho}$. This is the perturbative result first found by Luther and Peschel [60] and also reproduced by the single–impurity calculations. [59] The high–temperature behavior thus can be understood in terms of scattering off the individual impurities. On the other hand, at lower temperatures one necessarily comes into the region where $\mathcal{D}$ increases sharply. This has its origin in coherent scattering from many impurities and ultimately gives rise to localization.

One can finally notice the effects of different types of interactions on localization. Roughly speaking, for forward scattering repulsion ($g_2 > 0$) enhances localization whereas attraction weakens it. In particular, strong attraction leads to vanishing effective random potential. The delocalized state then can be considered to be a true superconductor in the sense that there is infinite conductivity even in an impure system. The effect of backward scattering interactions is opposite to that of forward interactions.

VI. THE BETHE ANSATZ: A PEDESTRIAN INTRODUCTION

The methods and results described in the preceding two sections were to a large extent based on a continuum description of the physics of one–dimensional electrons, and often relied on perturbation theory. Complementary insight can be gained by considering models defined on a lattice, and in particular some exactly solvable models. The physically most

important cases are the Heisenberg spin chain (the subject of Bethe's original work) and its generalization for itinerant fermions, the Hubbard model. In the following, I will discuss the exact solution of the Heisenberg model in some detail, in the hope to clarify the essential features. Subsequently, the necessary generalizations for the Hubbard model will be indicated.

The exact solutions give exact energies of the ground state and all the excited states in terms of the solution of a system of coupled nonlinear equations. On the other hand, the corresponding wavefunctions have a form so complicated that the explicit calculation of matrix elements, correlation functions and other physical quantities has remained impossible so far. In the subsequent sections I shall describe how the knowledge of the energy spectrum obtained from exact solutions can be combined with the results of the preceding two chapters to obtain a rather detailed picture of the low–energy properties, in particular of correlation functions.

It should be noticed that Bethe's method has been successfully applied to a number of other interesting cases like the Kondo model [74,75] and also is by no means restricted to lattice models. [76–79] For more general and detailed discussions, the reader is referred to the literature. [80–82]

A. The Heisenberg spin chain

1. Bethe's solution

We will write the Hamiltonian of the Heisenberg spin chain of L sites in the form

$$
\begin{aligned}
H &= \sum_{i=1}^{L}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z) \\
&= \sum_{i=1}^{L}\left[\frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + \Delta S_i^z S_{i+1}^z\right] \quad .
\end{aligned}
\tag{6.1}
$$

Here $\vec{S}_i = (S_i^x, S_i^y, S_i^z)$ is a spin–1/2 operator acting on site i, δ is an anisotropy parameter that allows one to treat the antiferromagnetic ($\Delta = 1$), the ferromagnetic ($\Delta = -1$), and general anisotropic cases[2], and periodic boundary conditions imply $\vec{S}_{L+1} = \vec{S}_1$. We notice that the Hamiltonian conserves the $z-$ component of the total spin (for $|\Delta| = 1$ total spin is also conserved), and consequently one can write any eigenfunction of a state with N down and $L - N$ up spins as

$$
|\psi\rangle = \sum_{x_1 < x_2 < \ldots < x_N} f(x_1, x_2, \ldots, x_N)|x_1, x_2, \ldots, x_N\rangle \quad ,
\tag{6.2}
$$

---

[2]In the Hamiltonian, eq.(6.1), an overall sign opposite to that of some of the original literature [83,84] is used

where the x_i are the positions of the down spins. Applying the Hamiltonian to this state, one obtains a "Schrödinger equation" for the coefficients f. In particular, if none of the x_i are adjacent, one has

$$\varepsilon f(\{x_i\}) = \frac{1}{2}\sum_{i=1}^{N}\delta_i f(\{x_i\}) \ , \tag{6.3}$$

where $\varepsilon = E + \Delta(N - L/4)$, E is the eigenenergy, and the finite–difference operator δ_i is defined by

$$\delta_i f(\ldots, x_i, \ldots) = f(\ldots, x_i - 1, \ldots) + f(\ldots, x_i + 1, \ldots) \ . \tag{6.4}$$

If two down spins are adjacent, e.g. $x_{k+1} = x_k + 1$, displacement of the spin is partly suppressed, and one finds instead of eq.(6.3)

$$\begin{aligned}
\varepsilon f(\ldots, x_k, x_k + 1, \ldots) = {} & \frac{1}{2}\sum_{i\neq k,k+1}^{N}\delta_i f(\ldots, x_k, x_k + 1, \ldots) \\
& + \frac{1}{2}[f(\ldots, x_k - 1, x_k + 1, \ldots) + f(\ldots, x_k, x_k + 2, \ldots)] \\
& + \Delta f(\ldots, x_k, x_k + 1, \ldots)
\end{aligned} \tag{6.5}$$

It should be quite clear how this generalizes to the case of more than two x_i's adjacent: whenever an down spin has an down neighbor, the term representing displacement to that site disappears, and one finds an "interaction term" (Δ) instead.

Bethe's solution starts by the observation that any linear combination of plane waves of the form

$$f(x_1, \ldots, x_N) = \sum_{P} A_P \exp\left[i\sum_{j=1}^{N}k_{Pj}x_j\right] \tag{6.6}$$

is a solution of the "free" eq.(6.3) for arbitrary coefficients A_P. Here the summation is over all permutations P of the N different wavenumbers k_j, i.e. in eq.(6.6) the momenta are permuted amongst all the particles (down spins).

The essential point in Bethe's solution is now to determine the coefficients A_P so that eqs.(6.3) and (6.5) become identical, i.e. one requires

$$\begin{aligned}
2\Delta f(\ldots, x_k, x_k + 1, \ldots) = {} \\
f(\ldots, x_k + 1, x_k + 1, \ldots) + f(\ldots, x_k, x_k, \ldots) \ .
\end{aligned} \tag{6.7}$$

Inserting this into eq.(6.5) one sees that one recovers eq.(6.3) which is automatically solved by the plane wave form (6.6). One should notice in eq.(6.7) that coefficients with two equal arguments like $f(\ldots, x_k, x_k, \ldots)$ have no direct physical meaning but are of course well–defined mathematically from eq.(6.6).

To see how this works let us first consider the two–particle case. Then there are only two permutations to sum over in eq.(6.6), and eq.(6.7) becomes (after canceling common factors)

$$2\Delta(A_0 e^{ik_2} + A_1 e^{ik_1}) = (A_0 + A_1)(1 + e^{i(k_1+k_2)}) \ . \tag{6.8}$$

This then gives

$$\frac{A_1}{A_0} = -\frac{1 + e^{i(k_1+k_2)} - 2\Delta e^{ik_2}}{1 + e^{i(k_1+k_2)} - 2\Delta e^{ik_1}} = -\exp[-i\Theta(k_2,k_1)] \quad, \tag{6.9}$$

where the second equation defines the *phase shift* Θ. Note that this means that the plane–wave form of the solution, eq.(6.6), remains valid even when two down spins are at nearest–neighbor sites where they interact. For longer range interactions this would not in general be the case.

We now consider eq.(6.7) for general N. It is then convenient to associate with each permutation P another permutation P' which differs from P only by the exchange of two adjacent elements: $P'k = Pk+1, P'k+1 = Pk$. One then has

$$f(\ldots, x_k + m, x_k + n, \ldots) = \sum_P{}' \exp\left[i \sum_{j \neq k,k+1} k_{Pj} x_j\right] e^{i(k_{Pk}+k_{P'k})x_k}$$
$$\times \left(A_P e^{i(mk_{Pk}+nk_{P'k})} + A_{P'} e^{i(mk_{P'k}+nk_{Pk})}\right) \quad, \tag{6.10}$$

where in eq.(6.7) the cases $m, n = 0, 1$ are relevant, and the summation is over half of the permutations (for example those with $Pk < P'k$), the other permutations being included explicitly by the term proportional to $A_{P'}$. Given that eq.(6.7) is supposed to be valid for any set $\{x_i\}$, the coefficients of the plane wave factors in the left and right hand sides have to be equal, e.g.

$$2\Delta(A_P e^{ik_{P'k}} + A_{P'} e^{ik_{Pk}}) = (A_P + A_{P'})(1 + e^{i(k_{Pk}+k_{P'k})}) \quad. \tag{6.11}$$

This is an obvious generalization of eq.(6.9) and leads to

$$\frac{A_P}{A_P'} = -\frac{1 + e^{i(k_{Pk}+k_{P'k})} - 2\Delta e^{ik_{Pk}}}{1 + e^{i(k_{Pk}+k_{P'k})} - 2\Delta e^{ik_{P'k}}} = -\exp(-i\Theta(k_{Pk}, k_{P'k})) \quad. \tag{6.12}$$

This now fixes all the coefficients in eq.(6.6) unambiguously, up to an overall normalization, because every permutation can be build up from a sequence of elementary permutation of two adjacent elements. To each such sequence corresponds a product of factors like eq.(6.12), one for each elementary permutation, and moreover this product is independent of the sequence of individual permutations, e.g.

$$(123) \to (213) \to (231) \to (321)$$
$$\text{and} \quad (123) \to (132) \to (312) \to (321) \tag{6.13}$$

lead to the same factor. One should also notice that eq.(6.12) was derived without any assumption on the other x_i's, and consequently situations with three or more adjacent down spins are equally covered and the generalizations of eq.(6.5) to these cases are therefore also fulfilled.

It remains to determine the allowed k–values. This follows from the periodic boundary condition

$$f(1, x_2, \ldots, x_N) = f(x_2, \ldots, x_N, L+1) \quad. \tag{6.14}$$

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Inserting into eq.(6.6) and noting that eq.(6.14) has to be satisfied for all $(x_2, \ldots, x_N)$, one obtains the condition

$$A_P/A_{P'}\mathrm{e}^{ik_{P'1}L} = 1 \;, \tag{6.15}$$

which must be satisfied for every permutation P. Here P' is a permutation obtained from P by a "right shift" of all elements:

$$P' = (PN, P1, P2, \ldots, PN-1) \;. \tag{6.16}$$

The ratio of coefficients in eq.(6.15) now can be calculated by repeatedly permuting PN in eq.(6.16) to the right and using eq.(6.12). The resulting equation is

$$(-1)^{N-1}\exp\left(i\sum_{j=1}^{N}\Theta(k_j, k_l)\right)\mathrm{e}^{ik_l L} = 1 \;, \tag{6.17}$$

with Θ defined in eq.(6.12).

2. The Heisenberg antiferromagnet

We now turn to the particularly interesting and physically relevant case of the Heisenberg antiferromagnet ($\Delta = 1$). In this case, the fundamental eq.(6.17) can be transformed into a somewhat simpler form making the transformation of variables

$$\lambda = -\frac{1}{2}\cot(k/2) \Longleftrightarrow \mathrm{e}^{ik} = \frac{2\lambda - i}{2\lambda + i} \;. \tag{6.18}$$

Equation (6.17) then becomes

$$\left(\frac{2\lambda_l - i}{2\lambda_l + i}\right)^L = -\prod_{j=1}^{N}\frac{\lambda_l - \lambda_j - i}{\lambda_l - \lambda_j + i} \;, \tag{6.19}$$

and similarly one finds the total momentum

$$P = \sum_{i=j}^{N}k_j = \sum_{j=1}^{N}(2\arctan(2\lambda_j) - \pi) \tag{6.20}$$

and the energy

$$E = L/4 + \sum_{j=1}^{N}(\cos k_j - 1) = L/4 - \sum_{j=1}^{N}\frac{2}{1 + 4\lambda_j^2} \;. \tag{6.21}$$

The z–component of the total spin of the states obtained from eq.(6.19) is obviously $L/2 - N$. In fact it can be shown that this is also the total spin of these states, and that these states, together with those obtained by repeated application of the lowering operator of the total spin, form the complete set of eigenstates.

If all the λ's are real, eq.(6.19) can be simplified by taking the logarithm. One then obtains

$$2L\arctan(2\lambda_l) = 2\pi J_l + 2\sum_{j=1}^{N}\arctan(\lambda_l - \lambda_j) \;, \tag{6.22}$$

where the J_l are integers if $L - N$ is odd and half–odd integers (of the form $1/2$, $3/2$, ...) is $L - N$ is even and are restricted to

$$|J_l| < (L - N + 1)/2 \quad . \tag{6.23}$$

In the ground state one has $N = L/2$, and then all the J_l are uniquely determined. In the thermodynamic limit, the allowed λ's are very close to each other, and their distribution then can be determined by a linear integral equation. The ground state energy then is $E_0 = L(1/4 - \ln 2)$.

Excited states can be obtained by taking out one of the down spins. One then has a state of total spin 1. One now has to choose $N = L/2 - 1$, subject to the constraint (6.23), i.e. one has to take two (half–)integers out of the sequence $-L/4...L/4$. This is thus a two–parameter family of states, which, in the thermodynamic limit has energy

$$E(p_1, p_2) = \varepsilon(p_1) + \varepsilon(p_2) \quad , \quad \text{with} \quad \varepsilon(p) = \frac{\pi}{2} \sin p \tag{6.24}$$

and momentum $P = p_1 + p_2$. At fixed total momentum P there is thus a $continuum$ of allowed states, as shown in fig.11. This energy–momentum relation suggests that the excitation is actually the combination of two elementary excitations which don't interact,
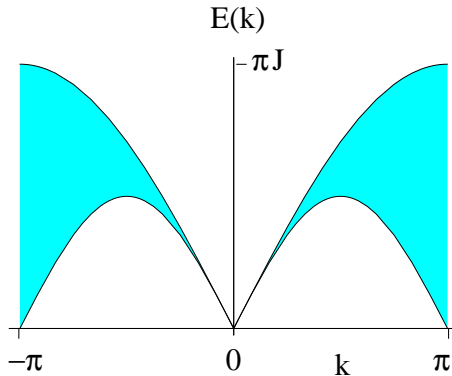

FIG. 11. The continuum of low–lying triplet and singlet states. The lower and upper limit of the continuum are $(\pi/2)|\sin k|$ and $\pi|\sin(k/2)|$, respectively.

so that there energies and momenta just add up. This interpretation finds further support in the fact that excitations with total spin 0 exist that have the same energetics as the spin 1 excitations (these correspond to the appearance of one pair of complex conjugate λ's in eq.(6.19)). These singlet and triplet excitations thus can in fact be seen as excitations of pairs of spin $1/2$ objects, called "spinons", which do not interact but form states of total spin 0 or 1. Spinons can be visualized as kinks in an antiferromagnetic order parameter, as shown in fig.12. In an isotropic Heisenberg chain, these individual spinons get delocalized into plane wave states due to the exchange interaction. For an even number of sites the total spin is always integer, so that the spinons always have to be excited pairwise. However, once created, they behave essentially as noninteracting objects.

In the Heisenberg model, the spinons are only noninteracting in the thermodynamic limit $L \to \infty$. However, in a modified model with $1/r^2$ interaction, this separation is true for all

↑ ↓ ↑ [↓ ↓] ↑ ↓ ↑ ↓ ↑ ↓ ↑ ↓ ↑ [↓ ↓] ↑ ↓
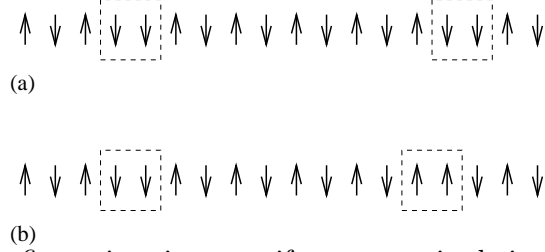
(a)

↑ ↓ ↑ [↓ ↓] ↑ ↓ ↑ ↓ ↑ ↓ ↑ ↓ [↑ ↑] ↓ ↑ ↓

(b)

FIG. 12. Two–spinon configurations in an antiferromagnetic chain with $m_z = 0$ (a) and $m_z = 1$ (b).

states even in a finite system [85,86]. One can further show that individual spinons behave as semions, i.e. their statistical properties are in many respects intermediate between fermions and bosons [87].

3. The Jordan–Wigner transformation and spinless fermions

The spin model discussed above can be transformed into a model of spinless fermions, noting that S_i^+ and S_i^- anticommute. The *Jordan–Wigner transformation* [88] then relates spin to fermion operators $(a_i, a_i^\dagger)$ via

$$S_i^+ = a_i^\dagger \exp\left[i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j \right] \quad , S_i^z = a_i^\dagger a_i - \frac{1}{2} \quad . \tag{6.25}$$

Presence or absence of a fermion now represent an up or down spin, and the exponential factor insures that spin operators on different sites commute, whereas fermionic operators of course anticommute. The transformation can now be used to rewrite the spin Hamiltonian (6.1) in terms of fermions as

$$H = \sum_{i=1}^{L} \left[\frac{1}{2}(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) + \Delta (a_i^\dagger a_i - \frac{1}{2})(a_{i+1}^\dagger a_{i+1} - \frac{1}{2}) \right] \quad . \tag{6.26}$$

The spin model thus is transformed into a fermionic model, with the "spin–flip" terms giving rise to motion of the fermions, whereas the S^z–S^z interaction gives rise to a fermion–fermion interaction between adjacent sites.

The fermionic analogy now allows one to study correlation functions of the spin chain using the bosonization method developed in section IV A. In particular, the spin–spin correlations for the isotropic antiferromagnet decays as [89]

$$\langle \vec{S}_i \cdot \vec{S}_j \rangle \approx (-1)^{|i-j|} \frac{\ln^{1/2} |i-j|}{|i-j|} \quad , \tag{6.27}$$

where the logarithmic corrections term comes from contributions of a marginally irrelevant umklapp interaction term [38,37,39]. The alternation in eq.(6.27) indicates the expected tendency towards antiferromagnetic order, but correlations do decay (with a rather slow power law), and there thus is no long–range order, as to be expected in one dimension. The analysis can be extended to give the full dynamic structure factor [90], and recent experiments on KCuF$_3$ confirm these results. [91]

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For $\Delta < 1$, a similar law holds, but with a Δ–dependent exponent and no logarithmic term. On the other hand, for $\Delta > 1$ the spins are preferentially aligned along the z–direction, and one then has a long–range ordered ground state of the Ising type. There thus is a phase transition exactly at the isotropic point $\Delta = 1$. In the fermionic language, this corresponds to a metal–insulator transition [92].

B. The Hubbard model

The Hubbard model is the prototypical model used for the description of correlated fermions in a large variety of circumstances, ranging from high–T_c superconductors to heavy fermion compounds and organic conductors. In spite of its apparent simplicity, there is still no general solution, or even a consensus on its fundamental properties. Notable exceptions are the cases of one and infinite dimensions [93,94]. In particular, in one dimension an exact solution is available. [95]

The Hamiltonian in one dimension has the well–known form

$$H = -t \sum_{i,s}(a^\dagger_{i,s}a_{i+1,s} + a^\dagger_{i+1,s}a_{i,s}) + U \sum_i n_{i,\uparrow}n_{i,\downarrow} \quad , \tag{6.28}$$

where $n_{i,s}$ is the number operator for fermions of spin s, and the sum is over the L sites of a one–dimensional chain with periodic boundary conditions.

The model has *two* global SU(2) symmetries [96–98]: the first is the well–known spin rotation invariance and the second is particular to the Hubbard model and relates sectors of different particle numbers. The total symmetry thus is $SU(2) \times SU(2) \simeq SO(4)$. One should notice that more complicated interactions, e.g. involving further neighbors, will conserve the spin rotation invariance but in general not the "charge" SU(2) invariance. Rather, this second symmetry will become the standard global $U(1)$ invariance associated with particle number conservation.

1. The exact solution

The exact solution of the one–dimensional Hubbard model has been found by Lieb and Wu [95]. To obtain the wavefunctions of the Hubbard model, one writes a general N–particle state as

$$|F\rangle = \sum_{x_1}\cdots\sum_{x_N}\sum_{s_1}\cdots\sum_{s_N} F_{s_1,\ldots,s_N}(x_1,\ldots,x_N)\prod_{i=1}^{N} a^\dagger_{x_i,s_i}|vac\rangle \quad . \tag{6.29}$$

The simplest case is the two–particle problem. Then in the two parts of configuration space $x_2 > x_1$ (region I) and $x_1 > x_2$ (region II) the wavefunction is just a product of two plane waves, and the only nontrivial effects occur for $x_1 = x_2$. The full wavefunction F then is

$$F_{s_1,s_2} = e^{i(k_1 x_1 + k_2 x_2)}[\theta(x_2 - x_1)\xi^I_{s_1,s_2} + \theta(x_1 - x_2)\xi^{II}_{s_1,s_2}] \quad , \tag{6.30}$$

where the factors ξ depend only on the spin quantum numbers of the two particles, and θ is the usual step function, with $\theta(0) = 1/2$. In order to satisfy the Schrödinger equation, these coefficients have to obey the equation

$$\xi^{II}_{s_1,s_2} = \sum_{t_1,t_2} S_{s_1 s_2,t_t t_2}\xi^I_{t_1 t_2} \quad . \tag{6.31}$$

The *S–matrix* has the form

$$S_{\alpha\beta,\gamma\delta} = \frac{t(\sin k_1 - \sin k_2) + iUP_{\alpha\beta,\gamma\delta}}{t(\sin k_1 - \sin k_2) + iU} \quad , \tag{6.32}$$

where $P_{\alpha\beta,\gamma\delta}$ is the operator permuting two spins. This operator acts on the spin part of the two–particle Hilbert space.

For N particles, similar to the Heisenberg case, one has matching conditions whenever two particles cross. The major complication in the Hubbard model comes from the fact that these matching conditions obviously involve the spin of the two particles and therefore involve the S–matrix, eq.(6.32), rather than just complex phase factors (compare with eq.(6.9)). In order to fulfill compatibility conditions for different paths in configuration space like those of eq.(6.13) one has has to have

$$S^{23}S^{13}S^{12} = S^{12}S^{13}S^{23} \quad . \tag{6.33}$$

These are the famous *Yang–Baxter equations* [77] which have to be satisfied for a system to be solvable by Bethe Ansatz. One can verify that these equations are indeed satisfied by the S–matrix of the Hubbard model, eq. (6.32). The subsequent analysis is based on a "generalized Bethe ansatz" [76,77]. A detailed description of the methods used can be found in the specialized literature [80,81], and a very detailed derivation is given in Gaudin's book [82].

Imposing periodic boundary conditions, the allowed values of k_j are obtained from the solution of the coupled set of nonlinear equations

$$e^{ik_jL} = \prod_{\alpha=1}^{M} e\left(\frac{4(\sin k_j - \lambda_\alpha)}{U}\right) \tag{6.34}$$

$$\prod_{j=1}^{N} e\left(\frac{4(\lambda_\alpha - \sin k_j)}{U}\right) = -\prod_{\beta=1}^{M} e\left(\frac{2(\lambda_\alpha - \lambda_\beta)}{U}\right) \quad , \tag{6.35}$$

Here N is the total number of electrons, M is the number of down–spin electrons ($M \leq N/2$), and $e(x) = (x + i)/(x - i)$. The λ_α are parameters characterizing the spin dynamics. We note that in general, both the k_j's and the λ's are allowed to be complex. The energy and momentum of a state are

$$E = -2t \sum_{j=1}^{N} \cos k_j \quad , \qquad P = \sum_{j=1}^{N} k_j \quad . \tag{6.36}$$

The determination of all the solutions of eqs.(6.34, 6.35) is not easy. It has recently been shown that under certain assumptions these equations do indeed give all the "lowest weight" (with respect to $SU(2) \times SU(2)$) eigenstates of the Hubbard model. The complete set of eigenstates then is obtained acting repeatedly with raising operators on the Bethe ansatz states. [99]

In the ground state all the k's and λ's are real. Numerical results for the ground state energy as a function of particle density and U have been given by Shiba [100]. The excited states are discussed in some detail in the literature [100–103,46]. I here just briefly summarize the different types:

1. "$4k_F$" singlet states. The excitation energy goes to zero at $q = 0$ and $q = 4k_F$. These excitations form a two–parameter continuum and only exist away from half–filling.

2. "$2k_F$" singlet and triplet states. These are degenerate two–parameter families of solutions, similar to the Heisenberg chain. The energy goes to zero for $q = 0$ and $q = 2k_F$.

3. There are also states with complex k. The energy of these states is proportional to U for large $U : t$, i.e. some sites are doubly occupied ("upper Hubbard band").

In addition, there are of course states with added particles or holes.

For $U \to \infty$ the ratio $\sin k_j / U$ in eqs.(6.34,6.35) clearly vanishes, however there is no restriction on the λ's. Introducing the scaled variables $\Lambda_\alpha = 2\lambda_\alpha / U$, the "spin equation" (6.35) becomes

$$\left(\frac{2\Lambda_\alpha + i}{2\Lambda_\alpha - i} \right)^N = -\prod_{\beta=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta + i}{\Lambda_\alpha - \Lambda_\beta - i} \ , \tag{6.37}$$

which are the well–known Bethe ansatz equations for the spin–1/2 Heisenberg chain, eq.(6.19), i.e. the spin wavefunction of N particles is just that of an N–site (not L–site!) Heisenberg chain, even when there is less than one particle per site.

<center>2. Luttinger liquid parameters</center>

The above results provide a complete picture of the energy spectrum of the one–dimensional Hubbard model (and also of its thermodynamics [104]). On the other hand, the wavefunctions obtained are prohibitively complicated, and no practical scheme for the evaluation of expectation values or correlation functions has been found. Of course, in a weakly interacting system the coefficients K_ρ and u_ν can be determined perturbatively. For example, for the Hubbard model one finds

$$K_\rho = 1 - U/(\pi v_F) + ... \ , \tag{6.38}$$

where $v_F = 2t \sin(\pi n/2)$ is the Fermi velocity for n particles per site. For larger U higher operators appear in the continuum Hamiltonian (4.30), e.g. higher derivatives of the fields or cosines of multiples of $\sqrt{8}\phi_\sigma$. These operators are irrelevant, i.e. they renormalize to zero and do not qualitatively change the long-distance properties, but they do lead to nontrivial corrections to the coefficients u_ν, K_ρ. In principle these corrections can be treated order by order in perturbation theory. However, this approach is obviously unpractical for large U, and moreover it is quite possible that perturbation theory is not convergent. To obtain the physical properties for arbitrary U a different approach [105] can be used which will now be explained.

I note two points: (i) in the small-U perturbative regime, interactions renormalize to the weak-coupling fixed point $g_1^* = 0, K_\sigma^* = 1$; (ii) the exact solution [95] does not show any singular behavior at nonzero U, i.e. large U and small U are the same phase of the model, so that the long-range behavior even of the large U case is determined by the fixed point $g_1^* = 0$. Thus, the low energy properties of the model are still determined by the three parameters $u_{\rho,\sigma}$ and K_ρ.
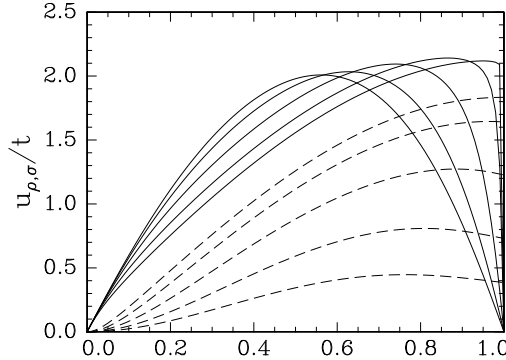
FIG. 13. *The charge and spin velocities u_ρ (full line) and u_σ (dashed line) for the Hubbard model, as a function of the band filling for different values of U/t: for u_σ $U/t = 1, 2, 4, 8, 16$ from top to bottom, for u_ρ $U/t = 16, 8, 4, 2, 1$ from top to bottom in the left part of the figure.*

The velocities $u_{\rho,\sigma}$ can be obtained from the long wavelength limit of the excitations discussed above. In particular, the $4k_F$ component of the charge correlation function only involves the ϕ_ρ field, and it is therefore natural to obtain u_ρ from the $q \to 0$ limit of the "$4k_F$" singlets. Similarly, u_σ is obtained from the "$2k_F$" excitations. In the thermodynamic limit the corresponding excitation energies are easily found from the numerical solution of a linear integral equation. [101] Results are shown in fig.13 for various values of U/t. Note that for $U = 0$ one has $u_\rho = u_\sigma = 2t\sin(\pi n/2)$, whereas for $U \to \infty$ $u_\rho = 2t\sin(\pi n)$, $u_\sigma = (2\pi t^2/U)(1 - \sin(2\pi n)/(2\pi n))$. In the noninteracting case $u_\sigma \propto n$ for small n, but for *any* positive U $u_\sigma \propto n^2$.

To obtain the parameter K_ρ from the exact solution note that the gradient of the phase field ϕ_ρ is proportional to the particle density, and in particular a constant slope of ϕ_ρ represents a change of total particle number. Consequently, the coefficient u_ρ/K_ρ in eq. (4.31) is proportional to the variation of the ground state energy E_0 with particle number:

$$\frac{1}{L}\frac{\partial^2 E_0(n)}{\partial n^2} = \frac{\pi}{2}\frac{u_\rho}{K_\rho} = \frac{1}{n^2\kappa} \quad . \tag{6.39}$$

Equation (6.39) now allows the direct determination of K_ρ: $E_0(n)$ can be obtained solving (numerically) Lieb and Wu's [95] integral equation, and u_ρ is already known. The results for K_ρ as a function of particle density are shown in fig.14 for different values of U/t. For small U one finds in all cases agreement with the perturbative expression, eq. (6.38), whereas for large U $K_\rho \to 1/2$. The limiting behavior for large U can be understood noting that for $U = \infty$ the charge dynamics of the system can be described by noninteracting *spinless* fermions (the hard-core constraint then is satisfied by the Pauli principle) with k_F replaced by $2k_F$. Consequently one finds a contribution proportional to $\cos(4k_Fx)x^{-2}$ in the density-density correlation function, which from eq. (4.42) implies $K_\rho = 1/2$. One then finds an asymptotic decay like $\cos(2k_Fx)x^{-3/2}\ln^{1/2}(x)$ for the spin-spin correlations, eq.(4.43), and an exponent $\alpha = 1/8$ in the momentum distribution function. The result $\alpha = 1/8$ has also been found by Anderson and Ren, [106] and by Parola and Sorella [107]. Ogata and Shiba's numerical results [108] are quite close to these exact values.

We note that in the whole parameter region, as long as the interaction is repulsive one always has $K_\rho < 1$, which means that magnetic fluctuations are enhanced over the noninteracting case. On the other hand, superconducting pairing is always suppressed.
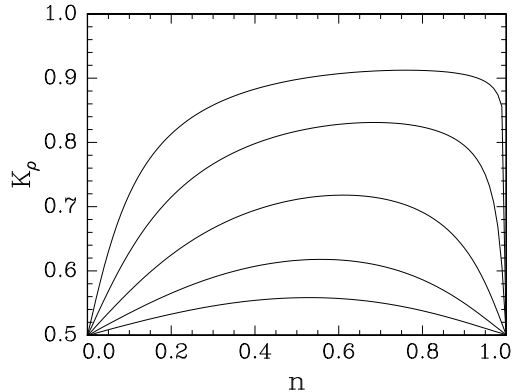
50

FIG. 14. *The correlation exponent K_ρ as a function of the bandfilling n for different values of U ($U/t = 1, 2, 4, 8, 16$ for the top to bottom curves). Note the rapid variation near $n = 1$ for small U.*

It should be emphasized here that the results of fig.14 are valid for $n \to 1$, but not for $n = 1$. In fact, in that latter case, there is a gap in the charge excitation spectrum, as expected from the umklapp term discussed below, and the correlations of ϕ_ρ become long ranged. Close to half–filling, the asymptotic behavior of the charge part of correlation functions like (4.42) is essentially determined by the motion of the holes. Writing the density of holes as $\rho = 1/(1 - n)$ one then expects a crossover of the form [109]

$$\langle n(x)n(0) \rangle \approx \cos(2k_F x)[1 + (\rho x)^2]^{K_\rho/2} x^{-1} \ln^{-3/2}(x) \tag{6.40}$$

for the $2k_F$ part of the density correlation function, and similarly for other correlation functions. Clearly, only for $x \gg 1/\rho$ are the asymptotic power laws valid, whereas at intermediate distances $1 \ll x \ll 1/\rho$ one has effectively $K_\rho = 0$. Clearly, the form (6.40) provides a smooth crossover as $n \to 1$.

Results equivalent to the present ones can be obtained using the conformal invariance of the Hubbard model [110,111]. These results have subsequently be generalized to the case with an applied magnetic field [112]. The method for calculating the exponent K_ρ is quite general and has been used for example for generalized Hubbard models [105,113] and the t–J model. [114,115]

3. Transport properties and the metal–insulator transition

The exact solution of Lieb and Wu can also be combined with the long–wavelength effective Hamiltonian (4.30) to obtain some information on the frequency–dependent conductivity $\sigma(\omega)$. In particular, from eq. (4.37) there is a delta function peak at zero frequency of weight $\sigma_0 = 2K_\rho u_\rho$, with the results plotted in fig.15. As expected, far from half–filling, the dc weight is nearly independent of interaction strength. peak. On the other hand, for $n \to 1$ σ_0 vanishes linearly, implying a linear variation of the ratio n/m^* with "doping", and for exactly half–filling the dc conductivity vanishes. There thus is a *metal–insulator transition* at $n = 1$, with the insulating state existing only at $n = 1$. One can also obtain the total frequency–integrated conductivity. [105] One then sees that as $n \to 1$ more and
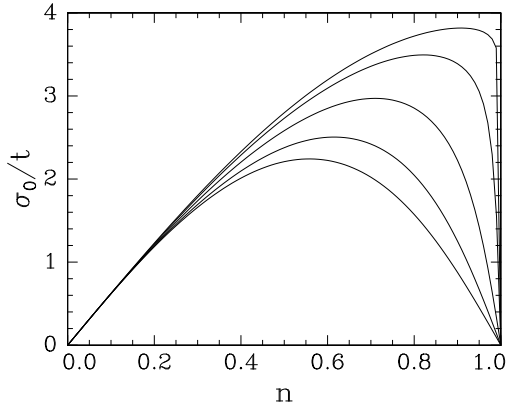
51

FIG. 15. *The weight of the dc peak in $\sigma(\omega)$ as a function of bandfilling for different values of U/t ($U/t = 1, 2, 4, 8, 16$ for the top to bottom curves).*

more weight is transferred from the zero–frequency peak to higher frequencies, essentially but not exclusively to frequencies above a "Mott–Hubbard gap".

It is interesting to discuss the metal–insulator transition in more detail. In the bosonization description, the insulating state at half filling is due to an extra term

$$H_u = \frac{2g_3}{(2\pi\alpha)^2} \int dx \cos(\sqrt{8}\phi_\rho) \tag{6.41}$$

which is due to umklapp scattering. The charge part of the Hamiltonian can then be transformed into a model of massive fermions, with energy–momentum relation $\varepsilon_k = \pm(v^2k^2 + \Delta_c^2)^{1/2}$, [116] where Δ is the charge excitation gap created by umklapp scattering. At half–filling, this term leads to a gap Δ_c in the charge excitation spectrum and therefore to insulating behavior. At half-filling all negative energy states are filled, all positive energy states are empty, and one has an insulator. Doping with a concentration n^* of holes ($n < 1$), some of the negative energy states become empty and only states with $|k| > k_F^* \propto n^*$ are filled. Because of the vanishing interaction, a standard formula can be used [117] and gives a positive thermopower, i.e. *approaching the metal–insulator transition from $n < 1$, the thermopower is hole–like*, [118,119] whereas obviously far from the transition ($n \ll 1$) it is electron–like. The exactly opposite behavior can be found for $n > 1$.

The fact that u_ρ and σ_0 vanish linearly as $n \to 1$ seems to be consistent with a divergent effective mass at constant carrier density because $u_\rho \approx 1/m^*, \sigma_0 \approx n/m^*$. A constant carrier density is also consistent with the fact that $k_F = \pi n/2$ is independent of U. It is *not consistent* with the hole–like sign of the thermopower as $n \to 1$ from below, nor with the electron–like sign as $n \to 1$ from above: if the carriers are holes, the carrier density is the density of holes: $n^* = 1 - n$. Treating the holes as spinless fermions, as already mentioned before, one expects $\sigma_0 \to 0$ because $n^* \to 0$, and $\gamma \to \infty$ because the density of states of a one–dimensional band diverges at the band edges. This agrees with what was found explicitly in section VI B 2. What is not so easily understood in this picture is the fact that k_F (i.e. the location of the singularity of n_k) is given by its free–electron value $\pi n/2$, rather then being proportional to n^*. One should however notice that n_k is given by the single–particle Green's function, which contains both charge and spin degrees of freedom. The location of k_F then may possibly be explained by phase shifts due to holon–spinon

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interaction. This is in fact suggested by the structure of the wavefunction of the exact solution [108].

The magnetic properties do not agree with what one expects from an effective mass diverging as $n \to 1$: u_σ and therefor χ remain finite. Moreover, the NMR relaxation rate would have the behavior $1/T_1 = \alpha T + \beta\sqrt{T}$, where the first (Korringa) term comes from fluctuations with $q \approx 0$, whereas the second term comes from antiferromagnetic fluctuations with $q \approx 2k_F$. None of these properties is strongly influenced by the diverging effective mass observed e.g. in the specific heat. This fact is of course a manifestation of the separation between spin and charge degrees of freedom.

The discussion can be generalized to metal–insulator transitions near other rational band-fillings, i.e. one–third or one–quarter filled bands. [120] In the Hubbard model, however, an insulating phase occurs only at half–filling.

It is worthwhile to point out here that the behavior of the one–dimensional system is quite different from that of scenarios for the metal–insulator transition in Fermi liquids. For example, in the "nearly–localized" description, [121] effective mass effects dominate, giving rise to a simultaneous divergence of specific heat and spin susceptibility. On the other hand, in infinite dimension [93,94] a behavior similar to the one–dimensional case is observed: approaching the insulating state by varying the bandfilling the spin susceptibility remains finite but the specific heat coefficient diverges. [122]

4. Spin–charge separation

The Hubbard model also provides a rather straightforward interpretation of the spin–charge separation discussed above. Consider a piece of a Hubbard chain with a half–filled band. Then for strong U there will be no doubly–occupied sites, and because of the strong short–range antiferromagnetic order the typical local configuration will be

$$\cdots \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \cdots$$

Introducing a hole will lead to

$$\cdots \uparrow\downarrow\uparrow\downarrow\uparrow \, O \, \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \cdots$$

and after moving the hole one has (note that the kinetic term in the Hamiltonian does not flip spins)

$$\cdots \uparrow\downarrow \, O \, \uparrow\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \cdots$$

Now the hole is surrounded by one up and one down spin, whereas somewhere else there are two adjacent up spins. Finally, a few exchange spin processes lead to

$$\cdots \uparrow\downarrow \, O \, \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow \cdots$$

Note that the original configuration, a hole surrounded by *two* up spins has split into a hole surrounded by antiferromagnetically aligned spins ("holon") and a domain–wall like configuration, two adjacent up spins, which contain an excess spin 1/2 with respect to the initial antiferromagnet ("spinon"). The exact solution by Lieb and Wu contains two types of quantum numbers which can be associated with the dynamics of the spinons and holons,

respectively. We thus notice that spinons and holons [123,124] have a well-defined meaning in the present one–dimensional case.

The above pictures suggest that, as far as charge motion is concerned, the Hubbard model away from half–filling can be considered as a one–dimensional harmonic solid, the motion of the holes providing for an effective elastic coupling between adjacent electrons. This picture has been shown to lead to the correct long–distance correlation functions for spinless fermions [27,125]. For the case with spin, this suggests that one can consider the system as a harmonic solid with a spin at each site of the elastic lattice (lattice site = electron in this picture), and this gives indeed the correct spin correlations [46].

VII. CONCLUSION AND OUTLOOK

In these notes, the different behavior of interacting many–fermion systems in one and higher dimension have been contrasted. In dimension bigger than one, Landau's Fermi liquid theory is based on the postulated existence of quasiparticles which have properties closely related to those of a noninteracting system. On the other hand in one dimension, essentially because of the geometric constraints on the particle dynamics no well–defined quasiparticle excitations exist. Instead, the low–lying excitations are collective spin or charge density oscillations, giving rise to the interesting phenomenon of spin–charge separation, as well as to non–universal power law behavior of correlation functions. This *Luttinger liquid* physics is most appropriately described by the bosonization formalism developed in sec.IV. We may note here that bosonization can equally be applied in higher dimension, [126,127] however at least for "normal" interactions one finds Fermi liquid properties.

It is clearly of interest to investigate the possibility of a crossover between Luttinger and Fermi liquid. A most natural candidate for such behavior are systems of parallel chains coupled by a small interchain hooping integral $t_\perp$. Such a model is appropriate for the discussion or real *quasi*–one-dimensional conductors. This type of models has been studied for some time but there is still a number of open questions. Nevertheless, most work points to a crossover between Luttinger (at high temperature) and Fermi liquid (at low temperature) behavior at a temperature determined by $t_\perp$. [128,129,14,130,131] At lower temperatures three–dimensionally ordered phases can appear. A different scenario has been advocated by Anderson and collaborators [132,133] who argue that even well below $T \approx t_\perp$ particle movement in the direction transverse to the chains remains incoherent, i.e. there would still be no three–dimensional Fermi liquid. Another approach is to consider the spatial dimension as a continuously varying parameter. [134] One then finds that the Luttinger liquid disappear as soon as $d > 1$.

It is worthwhile to mention here that the question of Fermi liquid or non–Fermi liquid is of interest in a number of other contexts. For example, the low–temperature properties of a "normal" Kondo impurity can be described as a local Fermi liquid. [135] On the other hand, the so–called "overscreened" Kondo effect [136] is characterized by non–Fermi liquid behavior, in particular a divergent local spin susceptibility. [137] A review of this subject has been given by Emery and Kivelson. [138]

The other major area of current interest in possible non–Fermi liquid behavior is the physics of strongly correlated systems in two dimensions, and in particular of the high–T_c superconducting copper oxides. Here a number of experimental results in the normal

conducting state suggest non–Fermi liquid behavior. [139–141] The theoretical situation is much less clear. On the one hand, a number of radically non–Fermi liquid pictures have been proposed, e.g. a two–dimensional Luttinger liquid [140] or the so–called gauge theory approach. [142] On the other hand, there are also models with only relatively minor modifications of the Fermi liquid picture. A rather complete overview of the rather unclear current situation (1994) can be found in recent conference proceedings. [143] Finally, an interesting non–Fermi liquid state has been proposed for the half–filled Landau level in two dimensions. [144]

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