JOURNAL OF MATHEMATICAL PHYSICS VOLUME 5, NUMBER 7 JULY 1964

Note on Wigner's Theorem on Symmetry Operations

V. BARGMANN
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received 28 January 1964)

Wigner's theorem states that a symmetry operation of a quantum system is induced by a unitary or an anti-unitary transformation. This note presents a detailed proof which closely follows Wigner's original exposition.

INTRODUCTION

THE states of a quantum system S are described by unit vectors f (i.e., vectors of norm 1)[1] in some Hilbert space $\mathcal{H}$. Assume that, conversely, to every unit vector f in $\mathcal{H}$ corresponds a state of S.[2] This correspondence is, of course, not one-to-one since f and $f\tau$ describe the same state if τ is a scalar factor of modulus 1. The states of S are therefore in a one-one correspondence with *unit rays* $\mathbf{f}$, a unit ray being defined as the set of all vectors of the form $f_0\tau$, where f_0 is a fixed unit vector in $\mathcal{H}$ and τ any scalar of modulus 1. Any significant statement in quantum theory is therefore a statement about unit rays.

Every vector f contained in the ray f ($f \in \mathbf{f}$) will be called a *representative* of $\mathbf{f}$. The transition probability from a state $\mathbf{f}$ to a state $\mathbf{g}$ equals $|(f, g)|^2$ where f, g are representatives of the rays $\mathbf{f}$, $\mathbf{g}$, respectively. This suggests the introduction of the inner product of two rays by the definition

$$\mathbf{f}\cdot\mathbf{g} = |(f, g)| \qquad (f \in \mathbf{f},\, g \in \mathbf{g}),$$

which is evidently independent of the choice of the representatives f, g.

A symmetry operation $\mathbf{T}$ of the system S maps the states in $\mathcal{H}$ either onto themselves or onto the states in some other Hilbert space $\mathcal{H}'$, with preservation of transition probabilities. (The second alternative corresponds to the mapping of one coherent subspace onto another. See footnote 2.) In terms of rays, $\mathbf{T}$ defines a mapping, $\mathbf{f}' = \mathbf{T}\mathbf{f}$, of unit rays onto unit rays such that $\mathbf{f}_1'\cdot\mathbf{f}_2' = \mathbf{f}_1\cdot\mathbf{f}_2$ if $\mathbf{f}_i' = \mathbf{T}\mathbf{f}_i$.

It has been shown by Wigner[3] that every such ray mapping $\mathbf{T}$ may be replaced by a vector mapping U of $\mathcal{H}$ onto $\mathcal{H}'$ which is either unitary or anti-unitary.[4] (For a precise formulation see Sec. 1.3 below.) For a long time this theorem has played a fundamental role in the analysis of symmetry properties of quantum systems.

The reason for returning to this question is the following. In Wigner's book the theorem is not proved in full detail. The construction of the mapping U, however, is clearly indicated, so that it is not difficult to close the gaps in the proof. In recent years several papers have appeared in which a proof of Wigner's theorem is presented.[5,6] To this writer most of these proofs seem unsatisfactory in one significant aspect: They obscure the quite elementary nature of Wigner's theorem.[7]

In addition, some authors state—or imply—the view that it is desirable, if not necessary, to depart from Wigner's construction in order to arrive at a simple or rigorous demonstration of his theorem. This writer, on the contrary, has always felt that Wigner's construction provides an excellent basis for an elementary and straightforward proof.

The present note is expository and contains no new results. It gives a complete proof of Wigner's theorem, by a method which closely adheres to his original construction. The only change of any consequence is the following. While Wigner relates U to an orthonormal set defined once for all, the proof below uses orthonormal sets adjusted to the vectors under consideration. As a result, it suffices to employ sets of at most two or three vectors.

Remarks on the notation. Re λ and Im λ denote, respectively, the real and the imaginary part of the complex number λ, and λ^* its complex conjugate.

[1] Here f corresponds of course to a wavefunction ψ. In this note vectors will be denoted by italics and scalars by lower-case Greek letters. The product of a vector f by the scalar λ will be written $f\lambda$.

[2] If superselection rules hold for S, $\mathcal{H}$ will be considered a coherent subspace of the Hilbert space of all states. See the discussion in Wightman [A. S. Wightman, Nuovo Cimento Suppl. 14 (1959), p. 81].

[3] E. P. Wigner, *Gruppentheorie* (Frederick Vieweg und Sohn, Braunschweig, Germany, 1931), pp. 251–254; *Group Theory* (Academic Press Inc., New York, 1959), pp. 233–236.

[4] Although Wigner did not explicitly formulate his theorem in terms of rays, it is essentially equivalent to the one stated here and certainly follows from the theorem proved below.

[5] For a bibliography and a critical analysis of the proofs, see Uhlhorn, Ref. 6. The recent paper by Lomont and Mendelson [J. S. Lomont and P. Mendelson, Ann. Math. **78**, 548 (1963).] should be added to his list.

[6] U. Uhlhorn, Arkiv Fysik **23**, 307 (1963).

[7] This criticism does not apply at all to the very interesting papers by Emch and Piron [G. Emch and C. Piron, J. Math. Phys. **4**, 469 (1963)] and by Uhlhorn[6], who start from more general premises and, consequently, obtain more comprehensive results.

1. STATEMENT OF THE THEOREM

1.1. *Preliminary remarks on rays.* Let $\mathcal{H}$ be a complex Hilbert space—which may be finite dimensional—with vectors $f, g, \cdots$. The inner product (f, g) of two vectors f, g has Hermitian symmetry, i.e., $(g, f) = (f, g)^*$, and for any complex scalar λ

$$(f, g\lambda) = (f, g)\lambda. \qquad (1)$$

$||f|| = (f, f)^{\frac{1}{2}}$ is the norm of f.

A ray $\mathbf{f}$ in $\mathcal{H}$ is the set of all vectors $f_0\tau$, where f_0 is a fixed vector in $\mathcal{H}$ and τ any scalar of modulus 1.[8] Every vector $f \in \mathbf{f}$ is an element or a "representative" of $\mathbf{f}$. Two vectors f', f'' are *equivalent* if they belong to the same ray, which is the case if and only if $f'' = f'\omega$ ($|\omega| = 1$). It is clear that a ray $\mathbf{f}$ is uniquely determined by any one of its representatives f, and we write

$$\mathbf{f} = \{f\}. \qquad (1a)$$

$\mathbf{0}$ is the ray consisting of the vector 0.

The inner product of two rays $\mathbf{f}, \mathbf{g}$ is defined by

$$\mathbf{f}\cdot\mathbf{g} = |(f, g)| \qquad (f \in \mathbf{f}, g \in \mathbf{g}) \qquad (1b)$$

and the *norm* of the ray $\mathbf{f}$ by

$$|\mathbf{f}| = (\mathbf{f}\cdot\mathbf{f})^{\frac{1}{2}} = ||f|| \qquad (f \in \mathbf{f}). \qquad (1c)$$

A *unit ray* is a ray of norm 1.

For nonnegative real scalars ρ we define

$$\mathbf{f}\rho = \{f\rho\} \qquad (f \in \mathbf{f}), \qquad (2)$$

i.e., if $f_0 \in \mathbf{f}$, the elements g of $\mathbf{f}\rho$ are given by $g = f_0\tau$ ($|\tau| = \rho$). Clearly,

$$(\mathbf{f}\rho)\sigma = \mathbf{f}(\rho\sigma), \qquad |\mathbf{f}\rho| = |\mathbf{f}|\,\rho, \qquad (2a)$$

$$\mathbf{f}\rho\cdot\mathbf{g}\sigma = (\mathbf{f}\cdot\mathbf{g})\rho\sigma. \qquad (2b)$$

Every ray $\mathbf{a}$ may be expressed in the form

$$\mathbf{a} = \mathbf{e}\rho \qquad (|\mathbf{e}| = 1, \rho \geq 0). \qquad (2c)$$

In all cases $\rho = |\mathbf{a}|$. If $\mathbf{a} = \mathbf{0}$, then $\rho = 0$, and the unit ray $\mathbf{e}$ may be chosen arbitrarily. If $\mathbf{a} \neq \mathbf{0}$, $\mathbf{e}$ is uniquely determined as $\mathbf{a}\rho^{-1}$.

1.2. It is reasonable to impose the following conditions on a symmetry operation $\mathbf{T}$.

(a) $\mathbf{T}$ is defined for every unit ray $\mathbf{e}$ in $\mathcal{H}$, and $\mathbf{e}' = \mathbf{T}\mathbf{e}$ is a unit ray in $\mathcal{H}'$.

(b) $\mathbf{T}\mathbf{e}_1\cdot\mathbf{T}\mathbf{e}_2 = \mathbf{e}_1\cdot\mathbf{e}_2$ (preservation of transition probabilities).

(c) If $\mathbf{T}\mathbf{e}_1 = \mathbf{T}\mathbf{e}_2$, then $\mathbf{e}_1 = \mathbf{e}_2$ (the mapping is one-to-one).

[8] Many authors define a ray differently, by including *all* multiples $f_0\lambda$ of a fixed vector f_0 (irrespective of $|\lambda|$) in one ray—provided $f_0 \neq 0$ and $\lambda \neq 0$—as is suggested by projective geometry. It seems to the writer that in the present context the definition of the text is more convenient.

(d) Every unit ray $\mathbf{e}'$ in $\mathcal{H}'$ is the image of some $\mathbf{e}$ in $\mathcal{H}$ (the mapping is *onto* the unit rays in $\mathcal{H}'$).

It is easily seen that (c) is superfluous, because it is an immediate consequence of (b). By Schwarz's inequality, two unit rays $\mathbf{f}, \mathbf{g}$ coincide if and only if $\mathbf{f}\cdot\mathbf{g} = 1$. Hence if $\mathbf{T}\mathbf{e}_1 = \mathbf{T}\mathbf{e}_2$, then $\mathbf{e}_1 = \mathbf{e}_2$ by (b).

In order to make the structure of the theorem more transparent we also drop the condition (d) and reinstate it in a corollary.

1.3. Thus our aim is the proof of the following

Main Theorem. Let $\mathbf{e}' = \mathbf{T}\mathbf{e}$ *be a mapping of the unit rays* $\mathbf{e}$ *of a Hilbert space* $\mathcal{H}$ *into the unit rays* $\mathbf{e}'$ *of a Hilbert space* $\mathcal{H}'$ *which preserves inner products, i.e., such that*

$$\mathbf{T}\mathbf{e}_1\cdot\mathbf{T}\mathbf{e}_2 = \mathbf{e}_1\cdot\mathbf{e}_2. \qquad (3)$$

Then there exists a mapping $a' = Ua$ *of all vectors* a *in* $\mathcal{H}$ *into the vectors* a' *in* $\mathcal{H}'$ *such that*

$$Ua \in \mathbf{T}\mathbf{a} \quad \text{if} \quad a \in \mathbf{a} \qquad (4)$$

(*if* $\mathbf{T}\mathbf{a}$ *is defined*) *and, in addition*

$$\left.\begin{aligned} &\text{(a)}\quad U(a + b) = Ua + Ub,\\ &\text{(b)}\quad U(a\lambda) = (Ua)\chi(\lambda),\\ &\text{(c)}\quad (Ua, Ub) = \chi((a, b)),\end{aligned}\right\} \qquad (5)$$

where either $\chi(\lambda) = \lambda$ *for all* λ *or* $\chi(\lambda) = \lambda^*$ *for all* λ.

A vector mapping U which satisfies (4) is called *compatible* with $\mathbf{T}$.

U is isometric since $||Ua|| = ||a||$ by (5). It is a *linear* or an *antilinear* isometry according as $\chi(\lambda) = \lambda$ or $\chi(\lambda) = \lambda^*$.

Corollary. If the $\mathbf{T}$ *of the theorem is a mapping onto all unit rays in* $\mathcal{H}'$*—in which case we call it a "ray correspondence"—*U *is a mapping onto* $\mathcal{H}'$*. It is unitary if* $\chi(\lambda) = \lambda$ *and anti-unitary if* $\chi(\lambda) = \lambda^*$.[9]

The corollary is an immediate consequence of the theorem.

1.4. *The one-dimensional case* is of course trivial. $\mathcal{H}$ contains only one unit ray $\mathbf{e}$, and $\mathbf{T}$ is completely determined by $\mathbf{T}\mathbf{e} = \mathbf{e}'$. Let $e \in \mathbf{e}$, and $e' \in \mathbf{e}'$. The two vector mappings $U_1(e\alpha) = e'\alpha$ and $U_2(e\alpha) = e'\alpha^*$ are compatible with $\mathbf{T}$. The first is linear, the second antilinear.

Hereafter we assume $\mathcal{H}$ to be at least *two dimensional.*

1.5. It is worth mentioning that, if $\dim \mathcal{H} \geq 2$,[10] the linear or antilinear character of U may be

[9] By definition, a unitary (anti-unitary) mapping is a linear (antilinear) isometry which has an inverse.

[10] dim $\mathcal{H}$ denotes the dimension of $\mathcal{H}$.

expressed in terms of **T**. It describes, therefore, an *intrinsic* property of the mapping **T** and is independent of the choice of U.

Consider three rays $\mathbf{a}_i$, and let $a_i \in \mathbf{a}_i$. The expression

$$\Delta(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = (a_1, a_2)(a_2, a_3)(a_3, a_1)$$

is *independent* of the choice of the representatives a_i and is therefore a function of the rays $\mathbf{a}_i$. In fact, if a_i are replaced by $a'_i = a_i\tau_i$ ($|\tau_i| = 1$) the factors τ_i cancel in Δ. It follows now from (5c) that

$$\Delta(\mathbf{Te}_1, \mathbf{Te}_2, \mathbf{Te}_3) = \chi(\Delta(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)).$$

As it should be, this criterion is vacuous if $\dim \mathcal{H} = 1$ because then $\mathbf{e}_i = \mathbf{e}$, and $\Delta = 1$. But if $\dim \mathcal{H} \geq 2$, Δ is not always real and may serve to distinguish linear from antilinear mappings. [Let e and f be two orthogonal unit vectors in $\mathcal{H}$, and set $e_1 = e$, $e_2 = 2^{-\frac{1}{2}}(e - f)$, $e_3 = 3^{-\frac{1}{2}}(e + f(1 - i))$. Then $||e_i|| = 1$, $\Delta = i/6$.]

1.6. A vector mapping U which transforms equivalent vectors into equivalent vectors *induces* (i.e., is compatible with) a uniquely defined *ray mapping* **T** by the equation

$$\mathbf{T}\{a\} = \{Ua\}$$

[see (1a)]. It is clear that every (linear or antilinear) isometry—in view of (5b) and (5c)—induces a ray mapping **T** which preserves inner products. Wigner's theorem asserts that no other ray mappings of this kind exist.

2. EXTENSION OF THE MAPPING T

Before constructing U we extend, following Wigner, **T** from a mapping of unit rays to a mapping of *all* rays **a** in $\mathcal{H}$ into the rays $\mathbf{a}'$ in $\mathcal{H}'$ by defining

$$\mathbf{T}(\mathbf{e}\rho) = (\mathbf{Te})\rho \qquad (\rho \geq 0,\ |\mathbf{e}| = 1). \tag{6}$$

Note that (6) defines **Ta** unambiguously for every ray $\mathbf{a} = \mathbf{e}\rho$. If $\mathbf{a} = 0$, then $\rho = 0$, hence $\mathbf{T}0 = 0$. If $\mathbf{a} \neq 0$, both ρ and **e** are uniquely determined [see (2c)].

For the extended mapping we have

$$\left.\begin{array}{ll}\text{(a)}\quad \mathbf{T}(\mathbf{a}\sigma) = (\mathbf{Ta})\sigma \qquad (\sigma \geq 0), & \\ \text{(b)}\quad \mathbf{Ta}_1\cdot\mathbf{Ta}_2 = \mathbf{a}_1\cdot\mathbf{a}_2, \qquad \text{(c)}\quad |\mathbf{Ta}| = |\mathbf{a}|. & \end{array}\right\} \tag{7}$$

[(a) If $\mathbf{a} = \mathbf{e}\rho$, both sides of (7a) equal $(\mathbf{Te})\rho\sigma$. (b) Set $\mathbf{a}_i = \mathbf{e}_i\rho_i$. The assertion follows from (2b) and (3), and if $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{a}$, we obtain (c).]

In the sequel we deal throughout with the *extended* mapping **T**—which is assumed to be given—and construct U so as to be compatible with it. Equation (4) is the *only* condition imposed on U. [(5) results from the construction.]

For later use we note the following. If, in the course of the construction, U has been defined [in accordance with (4)] for all multiples $a\lambda$ of a vector $a \neq 0$, then by (7)

$$||Ua|| = ||a||, \qquad U(a\lambda) = (Ua)\chi_a(\lambda), \tag{8}$$

$$\chi_a(1) = 1, \qquad |\chi_a(\lambda)| = |\lambda|, \tag{8a}$$

where $\chi_a(\lambda)$ is a uniquely defined function of a and λ. If U has been defined for a and b,

$$|(Ua, Ub)| = |(a, b)|. \tag{8b}$$

In Sec. 3 the mapping U is constructed for a subclass of all vectors. The partially defined U is analyzed in Sec. 4, and in Sec. 5 the construction of U—and the proof of the main theorem—is completed.

3. PARTIAL CONSTRUCTION OF U

3.1. *Preliminary remarks.* Let $\mathbf{f}_\rho$ ($\rho = 1, \cdots, m$) be m *orthonormal* rays (m finite!) so that $\mathbf{f}_\rho\cdot\mathbf{f}_\sigma = \delta_{\rho\sigma}$, and set $\mathbf{f}'_\rho = \mathbf{Tf}_\rho$. If $f_\rho \in \mathbf{f}_\rho$ and $f'_\rho \in \mathbf{f}'_\rho$, then $(f_\rho, f_\sigma) = (f'_\rho, f'_\sigma) = \delta_{\rho\sigma}$. Let $a = \sum_\rho f_\rho\alpha_\rho$, and $\mathbf{a} = \{a\}$. For any $a' \in \mathbf{Ta}$,

$$a' = \sum_\rho f'_\rho\alpha'_\rho \qquad |\alpha'_\rho| = |\alpha_\rho|. \tag{9}$$

Proof: Note that $||a'|| = ||a||$, $|(f'_\rho, a')| = |(f_\rho, a)|$, and $(f_\rho, a) = \alpha_\rho$. Therefore

$$||a' - \sum_\rho f'_\rho(f'_\rho, a')||^2 = ||a'||^2 - \sum_\rho |(f'_\rho, a')|^2$$
$$= ||a||^2 - \sum_\rho |(f_\rho, a)|^2 = ||a - \sum_\rho f_\rho(f_\rho, a)||^2 = 0,$$

hence $a' = \Sigma_\rho f'_\rho(f'_\rho, a')$, and the assertion follows, with $\alpha'_\rho = (f'_\rho, a')$.

3.2. Fix a unit ray **e** in $\mathcal{H}$, and let $\mathbf{e}' = \mathbf{Te}$, so that $|\mathbf{e}'| = 1$. Select $e \in \mathbf{e}$, and $e' \in \mathbf{e}'$. We define

A. $$Ue = e'$$

in accordance with (4).[11]

Denote by $\mathcal{P}$ the set of vectors in $\mathcal{H}$ orthogonal to e, by $\mathcal{P}'$ the set of vectors in $\mathcal{H}'$ orthogonal to e'.

Every vector a in $\mathcal{H}$ has a unique decomposition

$$a = e\alpha + z \qquad (z \in \mathcal{P}); \tag{10}$$

viz., $\alpha = (e, a)$, $z = a - e(e, a)$. In this section we construct U for those a for which $\alpha = 0$ or 1.

Let $a = e + z$ ($z \in \mathcal{P}$, $z \neq 0$), and set $f = z/||z||$.

[11] The selection of e' constitutes the only arbitrary choice in the construction of U. All other definitions are uniquely determined by e'.

Let **a**, **f**, **z** be the corresponding rays, so that

$$\mathbf{z} = \mathbf{f}\,\|z\|, \qquad |\mathbf{f}| = 1.$$

If $a' \in \mathbf{Ta}$, and $f' \in \mathbf{f}' = \mathbf{Tf}$, then, by (9), $a' = e'\alpha_0' + f'\alpha_1'$, $|\alpha_0'| = 1$, $|\alpha_1'| = \|z\|$.[12] Hence **Ta** contains a *uniquely determined* vector a'' $(= a'\alpha_0'^{-1})$ of the form $e' + f'\beta'$ $(|\beta'| = \|z\|)$. Setting $f'\beta' = Vz$ we define[13]

B. $\quad U(e + z) = e' + Vz \qquad (z \in \mathcal{P},\ Vz \in \mathcal{P}').$

Clearly, $Vz = f'\beta' \in (\mathbf{Tf})\|z\| = \mathbf{Tz}$, and we are allowed to set

C. $\quad Uz = Vz(= U(e + z) - Ue) \qquad (z \in \mathcal{P}).$

If $z = 0$, we set $Vz = 0$, so that B reduces to A.

In the next section we analyze the mapping V of $\mathcal{P}$ into $\mathcal{P}'$ in greater detail.

4. ANALYSIS OF THE MAPPING V

4.1. *The real part of* (Vw, Vx). Let w, x be in $\mathcal{P}$. By C, B, and (8b),

$$|(Vw, Vx)|^2 = |(w, x)|^2, \tag{11}$$

and $|(e' + Vw, e' + Vx)|^2 = |(e + w, e + x)|^2$, or $|1 + (Vw, Vx)|^2 = |1 + (w, x)|^2$. Since for every complex number ζ, $|1 + \zeta|^2 = 1 + |\zeta|^2 + 2\,\mathrm{Re}\,\zeta$, it follows from (11) that

$$\mathrm{Re}\,(Vw, Vx) = \mathrm{Re}\,(w, x), \tag{12}$$

$$(Vw, Vx) = (w, x) \quad \text{if} \quad (w, x) \quad \text{is real.} \tag{12a}$$

4.2. It will now be shown—in the remainder of this section—that, for any two nonvanishing y, z in $\mathcal{P}$, (a) $V(y + z) = Vy + Vz$, (b) $\chi_y(\lambda) = \chi_z(\lambda)$ [see (8)], (c) $(Vy, Vz) = \chi_y((y, z))$.

4.3 Set $f_1 = y/\|y\|$. If $\dim \mathcal{H} = 2$ all vectors in $\mathcal{P}$ are multiples of f_1, hence

$$y = f_1\rho, \qquad z = f_1\sigma. \tag{13a}$$

If $\dim \mathcal{H} \geq 3$ choose a second unit vector f_2 in $\mathcal{P}$ *orthogonal* to f_1 (whether or not y and z are independent) such that

$$y = f_1\rho, \qquad z = f_1\sigma + f_2\tau. \tag{13b}$$

In both cases let $\mathcal{L}$ be the set of linear combinations of the m orthonormal vectors f_ρ ($m = 1$ or $m = 2$).

4.4. *The functions* $\chi_\rho(\alpha)$. Set $\mathbf{f}_\rho = \{f_\rho\}$. Then $f_\rho' = Vf_\rho \in \mathbf{Tf}_\rho$, and the vectors f_ρ' are *orthonormal*. By (8),

$$V(f_\rho\alpha) = f_\rho'\chi_\rho(\alpha), \qquad |\chi_\rho(\alpha)| = |\alpha|. \tag{14}$$

[12] e and f are orthonormal, so are e' and f'.

[13] The definitions A and B are the crucial steps in Wigner's construction.

Applying (12) to $f_\rho\alpha$ and $f_\rho\beta$ we obtain

$$\mathrm{Re}\,(\chi_\rho(\alpha)^*\chi_\rho(\beta)) = \mathrm{Re}\,(\alpha^*\beta). \tag{15}$$

Set $\alpha = 1$. Since $\chi_\rho(1) = 1$ we conclude from (12) and (12a) that

$$\mathrm{Re}\,\chi_\rho(\beta) = \mathrm{Re}\,\beta, \tag{15a}$$

$$\chi_\rho(\beta) = \beta \quad \text{for} \quad \textit{real}\ \beta. \tag{15b}$$

4.5.[14] Let $x = \sum_\rho f_\rho\alpha_\rho$. By (9), $Vx = \sum_\rho f_\rho'\alpha_\rho'$, $|\alpha_\rho'| = |\alpha_\rho|$. We prove first that $\alpha_\rho' = \chi_\rho(\alpha_\rho)$. This is trivial if $\alpha_\rho = 0$. If $\alpha_\rho \neq 0$, set $\gamma_\rho = \alpha_\rho^{*-1}$. Then $(f_\rho\gamma_\rho, f_\rho\alpha_\rho) = (f_\rho\gamma_\rho, x) = 1$. Hence, by (12a), $\chi_\rho(\gamma_\rho)^*\chi_\rho(\alpha_\rho) = \chi_\rho(\gamma_\rho)^*\alpha_\rho' = 1$, i.e., $\alpha_\rho' = \chi_\rho(\alpha_\rho)$.

We show next that $\chi_2(\alpha) = \chi_1(\alpha)$ if $m = 2$. Let $w = \sum_\rho f_\rho$. Then $Vw = \sum_\rho f_\rho'$, and $V(w\alpha) = \sum_\rho f_\rho'\chi_\rho(\alpha) = (Vw)\chi_w(\alpha)$, by (8). Thus $\chi_1(\alpha) = \chi_2(\alpha) = \chi_w(\alpha)$. As a result,

$$V(\sum_\rho f_\rho\alpha_\rho) = \sum_\rho f_\rho'\chi_1(\alpha_\rho). \tag{16}$$

4.6. *Determination of* $\chi_1(\beta)$. (1) Set $\beta = i$. Then $|\chi_1(i)| = 1$, $\mathrm{Re}\,\chi_1(i) = 0$; thus $\chi_1(i) = \eta i$, $\eta = 1$ *or* $\eta = -1$. (2) For any complex ζ, $\mathrm{Im}\,\zeta = \mathrm{Re}\,(i^*\zeta)$. Hence, from (15), $\mathrm{Im}\,\chi_1(\beta) = \mathrm{Re}\,(i^*\chi_1(\beta)) = \eta\,\mathrm{Re}\,(\chi_1(i)^*\chi_1(\beta)) = \eta\,\mathrm{Re}\,(i^*\beta) = \eta\,\mathrm{Im}\,\beta$. Combining this with (15a),

$$\chi_1(\beta) = \beta \ \text{if}\ \eta = 1, \quad \chi_1(\beta) = \beta^* \ \text{if}\ \eta = -1. \tag{17}$$

Note the obvious relations:

(a) $\chi_1(\alpha + \beta) = \chi_1(\alpha) + \chi_1(\beta)$,

(b) $\chi_1(\alpha\beta) = \chi_1(\alpha)\chi_1(\beta)$,

(c) $\chi_1(\alpha)^* = \chi_1(\alpha^*)$.

4.7. *The structure of* V. Let $w = \sum_\rho f_\rho\alpha_\rho$ and $x = \sum_\rho f_\rho\beta_\rho$ be two vectors in $\mathcal{L}$. From (16) and from the properties of χ_1 just stated we draw the following conclusions. By (a), $V(w + x) = Vw + Vx$, by (b), $V(x\lambda) = (Vx)\chi_1(\lambda)$. Since both y and z belong to $\mathcal{L}$ this proves the assertions (a) and (b) of Sec. 4.2, with $\chi_y(\lambda) = \chi_z(\lambda) = \chi_1(\lambda)$. To establish (c) in (4.2) we note that $(y, z) = \rho^*\sigma$ [by (13)], and $(Vy, Vz) = \chi_1(\rho)^*\chi_1(\sigma) = \chi_1(\rho^*)\chi_1(\sigma) = \chi_1(\rho^*\sigma)$, Q.E.D.

By (b) in Sec. 4.2, $\chi_z(\lambda)$ is the *same* function, say, $\chi(\lambda)$, for every nonvanishing vector z in $\mathcal{P}$. To sum up, the mapping V has the following properties:

$$\begin{aligned} &\text{(a)} \quad V(y + z) = Vy + Vz, \\ &\text{(b)} \quad V(z\lambda) = (Vz)\chi(\lambda), \\ &\text{(c)} \quad (Vy, Vz) = \chi((y, z)), \end{aligned} \tag{18}$$

[14] If $m = 1$, Sec. 4.5 may be omitted since Eq. (16) reduces then to Eq. (14).

where χ is one of the functions in (17). [The equations (18) have been explicitly proved for nonvanishing y and z. But they hold trivially if $y = 0$ or $z = 0$.]

5. THE CONSTRUCTION OF U COMPLETED

It remains to define U for vectors $a = e\alpha + z$ ($z \in \mathcal{P}$) for which $\alpha \neq 0$, 1 [see (10)]. Set $b = e + z\alpha^{-1}$, so that $a = b\alpha$, and $\mathbf{Ta} = (\mathbf{Tb})\ |\alpha|$ if **a**, **b** are the corresponding rays. Ub ($\in \mathbf{Tb}$) is defined by B in Sec. 3. Hence $(Ub)\chi(\alpha) \in \mathbf{Ta}$, and we may therefore define $Ua = (e' + V(z\alpha^{-1}))\chi(\alpha)$, or, by (18),

$$\text{D.} \qquad U(e\alpha + z) = e'\chi(\alpha) + Vz \qquad (z \in \mathcal{P}).$$

If $\alpha = 1$ or 0, D coincides with A, B, or C of Sec. 3. Thus it defines the mapping U for all vectors a in $\mathcal{H}$.

By virtue of (18) it is an immediate consequence of D that U satisfies *all conditions* (5) of the theorem. [For an example, let $a_j = e\alpha_j + z_j$ ($j = 1, 2$). Then $(a_1, a_2) = \alpha_1^*\alpha_2 + (z_1, z_2)$, and $(Ua_1, Ua_2) = \chi(\alpha_1)^*\chi(\alpha_2) + Vz_1, Vz_2) = \chi(\alpha_1^*\alpha_2) + \chi((z_1, z_2)) = \chi(\alpha_1^*\alpha_2 + (z_1, z_2)) = \chi((a_1, a_2))$.]

This concludes the proof of the main theorem.

6. UNIQUENESS OF U

It is of course important to know to what extent U is determined by a ray mapping **T** which preserves inner products. Without using the main theorem in this section the following can be asserted. (**T** stands for the *extended* mapping)

(a) Let U be a vector mapping compatible with **T**. If a_1, a_2 are (linearly) independent, so are Ua_1, Ua_2. *Proof:* Two vectors a_j are independent if and only if $G(a_1, a_2) = (a_1, a_1)(a_2, a_2) - |(a_1, a_2)|^2 > 0$. Since, by (8b), G is not changed by the mapping U, the assertion follows.

(b) If U_2 and U_1 are compatible with the same **T**, then $U_20 = U_10 = 0$, and for every $a \neq 0$

$$U_2a = (U_1a)\tau(a), \qquad |\tau(a)| = 1.$$

If $\tau(a) = \theta$ (independent of a), we write $U_2 = U_1\theta$.

A mapping U is *additive* if $U(a + b) = Ua + Ub$.

Theorem 2. If two additive vector mappings U_2 and U_1 are compatible with the same **T**, *and* dim $\mathcal{H} \geq 2$, *then $U_2 = U_1\theta$.*[15] ($|\theta| = 1$.)

Proof: We proceed in two steps. (1) *If a, b are independent, $\tau(a) = \tau(b)$.* Set $c = a + b$. Then $U_2c = U_2a + U_2b$, $U_1c = U_1a + U_1b$, and $U_2c = (U_1c)\tau(c)$. Therefore

$$(U_1a)\tau(a) + (U_1b)\tau(b) = (U_1a + U_1b)\tau(c).$$

Since U_1a and U_1b are independent, $\tau(c) = \tau(a) = \tau(b)$.

(2) Fix a vector $a_0 \neq 0$ in $\mathcal{H}$, and set $\tau(a_0) = \theta$. *For every vector $a \neq 0$, $\tau(a) = \theta$.* If a and a_0 are independent, this follows from (1). If $a = a_0\mu$ ($\mu \neq 0$), choose b independent of a_0 (and hence of a). Then, by (1), $\tau(b) = \tau(a)$, $\tau(b) = \theta$, Q.E.D.

Let $U_2 = U_1\theta$. *If $U_1(a\lambda) = (U_1a)\chi(\lambda)$, then also $U_2(a\lambda) = (U_2a)\chi(\lambda)$.* In fact,

$$U_2(a\lambda) = (U_1a)\chi(\lambda)\theta = (U_1a)\theta\chi(\lambda) = (U_2a)\chi(\lambda) \qquad (19)$$

in accordance with our result in Sec. 1.5.

APPENDIX. WIGNER'S THEOREM IN QUATERNION QUANTUM THEORY

In recent years there has been some interest in a modification of the quantum theoretical formalism which consists in replacing the complex Hilbert space of quantum states by a quaternion Hilbert space.[16] We wish to indicate the changes in the theorem and in its proof that must be made. The above exposition is so arranged that these changes are concentrated in a few places.

1. *Preliminary remarks on quaternions.*[17] Let Q be the set of all quaternions. We write a quaternion λ in the form $\lambda = \sum_{\nu=0}^{3} \lambda_\nu i_\nu$. Here, λ_ν are real numbers, $i_0 = 1$ (i.e., $i_0\lambda = \lambda i_0 = \lambda$ for every $\lambda \in Q$), while i_r ($r = 1, 2, 3$) are the imaginary units, with the multiplication rules

$$i_r^2 = -1, \qquad i_ri_s = -i_si_r = i_t \qquad \text{(A1)}$$

where r, s, t is an *even* permutation of 1, 2, 3.

The conjugate λ^* of λ is defined by

$$\lambda^* = \lambda_0 i_0 - \sum_{r=1}^{3} \lambda_r i_r$$

so that $(\lambda^*)^* = \lambda$. A quaternion is *real* if $\lambda^* = \lambda$, i.e., $\lambda = \lambda_0 \cdot 1 = \lambda_0$. (The real quaternions, and only they, *commute* with all of Q.) In general we denote the real part of λ by

$$\text{Re}\ \lambda = \tfrac{1}{2}(\lambda + \lambda^*) = \lambda_0$$

Note that $\text{Re}\ \lambda^* = \text{Re}\ \lambda$. Let $\kappa = \sum_\nu \kappa_\nu i_\nu$. Since $(i_\mu i_\nu)^* = i_\nu^* i_\mu^*$ (for all μ, ν), $(\kappa\lambda)^* = \lambda^*\kappa^*$.

For all μ, ν

$$\text{Re}\ (i_\mu^* i_\nu) = \text{Re}\ (i_\mu i_\nu^*) = \delta_{\mu\nu}. \qquad \text{(A2)}$$

It follows that

$$\text{Re}\ (\kappa^*\lambda) = \text{Re}\ (\lambda^*\kappa) = \sum_\nu \kappa_\nu\lambda_\nu, \qquad \text{(A2a)}$$

$$\text{Re}\ (\kappa^*\lambda) = \text{Re}\ (\kappa\lambda^*). \qquad \text{(A2b)}$$

[15] In terms of the construction in Sec. 3.2 this merely means that A is replaced by $U_2e = e'\theta$ (see footnote 11). It follows from Sec. 1.4 that dim $\mathcal{H} \geq 2$ is a necessary assumption.

[16] D. Finkelstein, J. M. Jauch, S. Schiminovich, and D. Speiser, J. Math. Phys. 3, 207 (1961).

[17] The reader is assumed to be familiar with quaternions. These introductory remarks fix the notation and review several relations to be used later on.

In particular, $\lambda\lambda^*$ and $\lambda^*\lambda$ being real,

$$\lambda\lambda^* = \lambda^*\lambda = \sum_\nu \lambda_\nu^2 = |\lambda|^2,$$

where $|\lambda|$ is the modulus of λ. We have $|\kappa\lambda| = |\kappa|\,|\lambda|$ since $|\kappa\lambda|^2 = \kappa(\lambda\lambda^*)\kappa^* = (\kappa\kappa^*)(\lambda\lambda^*) = |\kappa|^2\,|\lambda|^2$. If $\lambda \neq 0$, $\lambda^{-1} = |\lambda|^{-2}\lambda^*$, $\lambda\lambda^{-1} = \lambda^{-1}\lambda = 1$.

Q may be considered a four-dimensional *real vector space*. (A2a) defines then an inner product in Q, and by (A2) the units i_ν form an *orthonormal basis*. Hence

$$\lambda_\nu = \operatorname{Re}\,(i_\nu^*\lambda). \tag{A3}$$

More generally, let j_ν ($\nu = 0, 1, 2, 3$) be four orthonormal quaternions, so that $\operatorname{Re}\,(j_\mu^* j_\nu) = \delta_{\mu\nu}$. Then every κ may be written as

$$\kappa = \sum_\nu \operatorname{Re}\,(j_\nu^*\kappa) j_\nu. \tag{A3a}$$

For a later application we add a few words about the automorphism

$$\lambda' = \sigma_\gamma(\lambda) = \gamma\lambda\gamma^{-1} = \gamma\lambda\gamma^* \tag{A4}$$

where γ is a fixed quaternion of modulus 1. Clearly

(a) $\sigma_\gamma(\kappa + \lambda) = \sigma_\gamma(\kappa) + \sigma_\gamma(\lambda)$,

(b) $\sigma_\gamma(\kappa\lambda) = \sigma_\gamma(\kappa)\sigma_\gamma(\lambda)$,

(c) $\sigma_\gamma(\lambda^*) = \sigma_\gamma(\lambda)^*$.

In addition,

(d) $\operatorname{Re}\,\sigma_\gamma(\lambda) = \operatorname{Re}\,\lambda$,

(e) $\operatorname{Re}\,(\sigma_\gamma(\kappa)^*\sigma_\gamma(\lambda)) = \operatorname{Re}\,(\kappa^*\lambda)$.

[(d) follows from $\operatorname{Re}\,((\gamma\lambda)\gamma^*) = \operatorname{Re}\,(\gamma^*\gamma\lambda) = \operatorname{Re}\,\lambda$, and this in turn implies (e) because $\sigma_\gamma(\kappa)^*\sigma_\gamma(\lambda) = \sigma_\gamma(\kappa^*\lambda)$.] By (e), σ_γ is an orthogonal transformation on Q. Conjugation is also an orthogonal mapping, by (A2a). Combining it with σ_γ one obtains a second type of orthogonal transformation,

$$\lambda' = \sigma_\gamma(\lambda^*). \tag{A4a}$$

2. *Wigner's theorem*. The states of the system S are again put in a one-one correspondence with the unit rays in the Hilbert space $\mathcal{H}$. The discussion in Sec. 1.1 remains unchanged except that the scalars λ and the values of the inner products (f, g)—in $\mathcal{H}$ and $\mathcal{H}'$—are now quaternions. More generally, the whole content of Secs. 1–6 remains applicable with the exception of those instances where (a) specific properties of complex numbers are used or (b) the commutative law of multiplication is applied.

The only instance of type (a) is the determination of χ_1 in Sec. 4.6 and, consequently, the characterization of $\chi(\lambda)$ in the statement of the main theorem. The only instance of Type (b) is Eq. (19) in Sec. 6.[18]

These two points are now re-examined.

3. *The two-dimensional case*. In the quaternion case Wigner's theorem *no longer holds* if dim $\mathcal{H} = 2$.[19]

Every vector z in $\mathcal{P}$ has now the form $f_1\alpha$, and $V(f_1\alpha) = f_1'\chi_1(\alpha)$ [see (14)] where χ_1 satisfies the three equations (15), (15a), and (15b). From (A4) and (A4a) we obtain two types of solutions, viz.,

$$(1)\quad \chi_1(\alpha) = \sigma_\gamma(\alpha), \qquad (2)\quad \chi_1(\alpha) = \sigma_\gamma(\alpha^*). \tag{A5}$$

(It is not difficult to show that no other solutions exist.)

Now the proof of the main theorem in Sec. 5 is based, in part, on Eq. (18b), whose derivation in turn depends on the relation (b) at the end of Sec. 4.6, namely, $\chi_1(\alpha\beta) = \chi_1(\alpha)\chi_1(\beta)$. While the first solution in (A5) satisfies this relation we find for the second $\chi_1(\alpha\beta) = \sigma_\gamma((\alpha\beta)^*) = \sigma_\gamma(\beta^*\alpha^*) = \sigma_\gamma(\beta^*)\sigma_\gamma(\alpha^*) = \chi_1(\beta)\chi_1(\alpha)$. The order of the factors is reversed: χ_1 is an *antiautomorphism*.

Choose, for simplicity, $\gamma = 1$, so that $V(f_1\beta) = f_1'\beta^*$. As the arguments of Sec. 5 show we may set $U(e\alpha + f_1\beta) = (e' + f_1'(\beta\alpha^{-1})^*)\alpha$ if $\alpha \neq 0$. For convenience we define a new mapping U_0 compatible with **T** by

$$U_0(e\alpha + f_1\beta) = (e' + f_1'(\beta\alpha^{-1})^*)\alpha \quad (\alpha \neq 0),$$

$$U_0(f_1\beta) = f_1'\beta \quad (\alpha = 0).$$

(U_0 differs from U only if $\alpha = 0$.)

To disprove Wigner's theorem in this case it remains to show (1) that U_0 actually induces a ray mapping **T** with the required properties [i.e., that no conditions have been overlooked that might rule out the second solution in (A5)]; (2) that no additive vector mapping is compatible with **T**.

(1) By straightforward computation one verifies that, for every vector $a = e\alpha + f_1\beta$, $U_0(a\lambda) = (U_0a)\lambda$, and that $|(U_0a_1, U_0a_2)| = |(a_1, a_2)|$. Thus U_0 induces indeed a ray mapping **T** which preserves inner products.

(2) will be proved by contradiction. Let W be an additive vector mapping compatible with **T**. Then

$$W(e\alpha + f_1\beta) = (U_0(e\alpha + f_1\beta))\Phi(\alpha, \beta) \quad |\Phi(\alpha, \beta)| = 1$$

if $(\alpha, \beta) \neq (0, 0)$. In particular, $W(e\alpha) = e'\alpha\Phi(\alpha, 0)$, $W(f_1\beta) = f_1'\beta\Phi(0, \beta)$. Setting $\eta(\alpha) = \alpha\Phi(\alpha, 0)$ and

[18] It should be added that the remarks in Sec. 1.5 do not apply to the quaternion case. The proof that Δ does not depend on the choice of the representatives a_i uses commutativity of multiplication.

[19] Uhlhorn's contrary assertion (Ref. 6, pp. 335, 336) is incorrect. He overlooked the second solution in (A5).

$\zeta(\beta) = \beta\Phi(0, \beta)$ we conclude from the additivity of W that

$$e'\eta(\alpha) + f_1'\zeta(\beta) = (e' + f_1'\alpha^{*-1}\beta^*)\alpha\Phi(\alpha, \beta).$$

Assume $\alpha \neq 0$, $\beta \neq 0$. Then η, ζ and $\Phi \neq 0$, and

$$\alpha^*\zeta(\beta) = \beta^*\eta(\alpha). \tag{A6}$$

Setting, in succession, $\alpha = \beta = 1$, $\beta = 1$, and $\alpha = 1$, one finds $\zeta(1) = \eta(1)$, $\eta(\alpha) = \alpha^*\eta(1)$, $\zeta(\beta) = \beta^*\eta(1)$. Multiplying (A6) with $\eta(1)^{-1}$ from the right, we finally obtain $\alpha^*\beta^* = \beta^*\alpha^*$ or $\beta\alpha = \alpha\beta$, which is absurd.

4. *Determination of* χ_1 *if dim* $\mathcal{H} \geq 3$. Here $m = 2$, and we first derive a further condition on χ_1 to supplement Eqs. (15), (15a), (15b). Let $w = f_1 + f_2\alpha$. Then $Vw = f_1' + f_2'\chi_1(\alpha)$, and $V(w\beta) = f_1'\chi_1(\beta) + f_2'\chi_1(\alpha\beta) = (Vw)\chi_w(\beta)$, so that $\chi_1(\beta) = \chi_w(\beta)$, $\chi_1(\alpha\beta) = \chi_1(\alpha)\chi_w(\beta)$. Thus

$$\chi_1(\alpha\beta) = \chi_1(\alpha)\chi_1(\beta). \tag{A7}$$

Set now $j_\nu = \chi_1(i_\nu)$. By (15b), $j_0 = 1$, and by (15) and (A2)

$$\text{Re}\,(j_\mu^* j_\nu) = \text{Re}\,(i_\mu^* i_\nu) = \delta_{\mu\nu}.$$

Let $\beta = \sum_{\nu=0}^{3} \beta_\nu i_\nu$. By (15), $\text{Re}\,(j_\nu^*\chi_1(\beta)) = \text{Re}\,(i_\nu^*\beta) = \beta_\nu$ [see (A3)], so that, by (A3a)

$$\chi_1(\beta) = \sum_\nu \beta_\nu j_\nu. \tag{A8}$$

By (15b), $\chi_1(-1) = -1$; hence by (A7), $\chi_1(-\beta) = -\chi_1(\beta)$. Applying (A7) to j_r $(r > 0)$ we find

$$j_r^2 = \chi_1(i_r^2) = -1,$$
$$j_r j_s = \chi_1(i_r i_s) = \chi_1(i_t) = j_t,$$
$$j_s j_r = \chi_1(-i_t) = -j_t,$$

if (r, s, t) is an even permutation of (1, 2, 3). Together with $j_0 = 1$, this shows that the j_ν satisfy the multiplication rules of the units i_ν.

It is well known—and easily proved—that then $j_\nu = \gamma i_\nu \gamma^{-1} = \sigma_\gamma(i_\nu)$ for some fixed γ of modulus 1. Inserting this in (A8) we finally obtain

$$\chi_1(\beta) = \sigma_\gamma(\beta). \tag{A9}$$

This solution satisfies all three relations listed at the end of Sec. 4.6, and the arguments in Sec. 4.7 and Sec. 5 apply without change. Thus the main theorem is valid, but $\chi(\lambda)$ is an automorphism $\sigma_\gamma(\lambda)$, and U a *semilinear* isometry.

5. *Theorem* 2 *of Sec.* 6 *holds.* The transition from U_1 to $U_2 = U_1\theta$, however, has now more radical consequences. Instead of Eq. (19) we have

$$U_2(a\lambda) = U_1(a)\chi(\lambda)\theta = (U_1a)\theta(\theta^{-1}\chi(\lambda)\theta).$$

Assuming $\chi(\lambda) = \sigma_\gamma(\lambda)$,

$$U_2(a\lambda) = (U_2a)\chi'(\lambda),\qquad \chi'(\lambda) = \theta^{-1}\sigma_\gamma(\lambda)\theta = \sigma_{\theta^{-1}\gamma}(\lambda). \tag{A10}$$

$\chi'(\lambda) = \chi(\lambda)$ *if and only if* $U_2 = \pm U_1$. In fact, θ must commute with all $\sigma_\gamma(\lambda)$ and hence must be *real*. Since $|\theta| = 1$, $\theta = \pm 1$.

In particular, if $\theta = \gamma$, then $\chi'(\lambda) = \lambda$, so that U_2 is *linear*.

To sum up: *In the quaternion case, if dim* $\mathcal{H} \geq 3$, *every ray mapping* **T** *which preserves inner products is induced by a linear mapping* U, *and* **T** *determines* U *up to a sign.*

6. *Remarks on Uhlhorn's theorem.* The following remarks apply to the complex as well as the quaternion case. Uhlhorn has obtained the very interesting result that Wigner's theorem holds under considerably weaker assumptions. In terms of the conditions listed in Sec. 1.2 it suffices to maintain (a) and (d) while (b) is replaced by the condition b': $\mathbf{Te}_1 \cdot \mathbf{Te}_2 = 0$ if and only if $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ (preservation only of the transition probability zero!) On the other hand it is necessary to assume dim $\mathcal{H} \geq 3$.

Since, however, the condition (d)—or possibly some weaker substitute—is actually needed for the proof of Uhlhorn's result it seems to the writer that the main theorem proved in the present note retains an independent mathematical interest.

In conclusion it may be mentioned that a minor modification of Wigner's construction also yields a simple proof of Uhlhorn's theorem.