# 10. The electromagnetic field

We will now provide a quanto-mechanical description of the electro-magnetic field. Our main interest will be in analyzing phenomena linked to atomic physics and quantum optics, in which atoms interacts with radiation. Some processes can be analyzed with a classical description: for example we studied the precessing and the manipulation of a spin by classical static and rf magnetic fields. Absorption and emission of light by an atom can also be described as the interaction with a classical field. Some other phenomena, such as spontaneous emission, can only arise from a QM description of both the atom and the field. There are various examples in which the importance of a quantum treatment of electromagnetism becomes evident:

− Casimir force
− Spontaneous emission, Lamb Shift
− Laser linewidth, photon statistics
− Squeezed photon states, states with subpoissonian distribution,
− Quantum beats, two photon interference, etc.

## 10.1 Classical theory of the e.m. field

Before introducing the quantization of the field, we want to review some basic (and relevant) concepts about e.m. fields.
Maxwell equations for the electric and magnetic fields, E and B, are:

| | |
|---|---|
| Gauss's law | $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ |
| Gauss's law for magnetism | $\nabla \cdot \mathbf{B} = 0$ |
| Maxwell-Faraday equation (Faraday's law of induction) | $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ |
| Ampere's circuital law (with Maxwell's correction) | $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ |

We will be interested to their solution in empty space (and setting $c = 1/\sqrt{\mu_0 \varepsilon_0}$):

| | |
|---|---|
| Gauss's law | $\nabla \cdot \mathbf{E} = 0$ |
| Gauss's law for magnetism | $\nabla \cdot \mathbf{B} = 0$ |
| Maxwell-Faraday equation | $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ |
| Ampere's circuital law | $\nabla \times \mathbf{B} = \frac{1}{c}\frac{\delta \mathbf{E}}{\partial t}$ |

Combining Maxwell equation in vacuum, we find the wave equations:

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \qquad \nabla^2 B - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = 0$$

---

**? Question:** Show how this equation is derived
We need to take the curl of Maxwell-Faraday equation and the time derivative of Ampere's law and use the vector identity $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$ and Gauss law. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ☐

A general solution for these equations can be written simply as $E = E(\omega t - \vec{k}\cdot\vec{x})$. By fixing the boundary conditions, we can find a solution in terms of an expansion in normal modes, where the time dependence and spatial dependence are separated. The solution of the wave equation can thus be facilitated by representing the electric field as a sum of normal mode functions:

$$\mathbf{E}(\vec{x},t) = \sum_m f_m(t)\vec{u}_m(\vec{x}).$$

The normal modes $u_m$ are the equivalent of eigenfunctions for the wave equation, so they do not evolve in time (i.e. they are function of position only). The $u_m$ are orthonormal functions, called normal modes. The boundary conditions define the normal modes $u_m$ for the field, satisfying:

$$\nabla^2 u_m = -k_m^2 u_m, \quad \nabla \cdot u_m = 0, \quad \vec{n} \times u_m = 0$$

(where $n$ is a unit vector normal to a surface). This last condition is imposed because the tangential component of the electric field E must vanish on a conducting surface. We can also choose the modes to satisfy the orthonomality condition (hence *normal* modes):

$$\int \vec{u}_m(x)\vec{u}_n(x)d^3x = \delta_{n,m}$$

Substituting the expression for the electric field in the wave equation, we find an equation for the coefficient $f_m(t)$:

$$\sum_m \frac{d^2 f_m}{dt^2} + c^2 k_m^2 f_m(t) = 0.$$

Since the mode functions are linearly independent, the coefficients of each mode must separately add up to zero in order to satisfy the wave equation, and we find :

$$\frac{d^2 f_m}{dt^2} + c^2 k_m^2 f_m(t) = 0.$$

As it can be seen from this equation, the dynamics of the normal modes, as described by their time-dependent coefficients, is the same as that of the h.o. with frequency $\omega_m = ck_m$. Hence the electric field is equivalent to an infinite number of (independent) harmonic oscillators. In order to find a quantum-mechanical description of the e.m. field we will need to turn this h.o. into quantum harmonic oscillators.
We want to express the magnetic field in terms of the same normal modes $\vec{u}_m$ which are our basis. We assume for $B$ the expansion:

$$\mathbf{B}(x,t) = \sum_n h_n(t)\left[\nabla \times u_n(x)\right],$$

From Maxwell-Faraday law:

$$\nabla \times E = \sum_n f_n(t)\nabla \times u_n = -\frac{1}{c}\partial_t B$$

we see that we need to impose $h_n$ such that $\frac{d h_n}{d t} = -cf_n$ so that we obtain

$$\sum_n -\frac{1}{c}\frac{d h_n}{d t}\nabla \times u_n = -\frac{1}{c}\partial_t B$$

which indeed corresponds to the desired expression for the magnetic field. We now want to find as well an equation for the coefficient $h_n$ alone. From the expressions of the $E$ and $B$-field in terms of normal modes, using Ampere's law,

$$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t} \quad \rightarrow \quad \sum_n h_n(t)\nabla \times (\nabla \times u_n) = \frac{1}{c}\sum_n \frac{d f_n}{d t}u_n$$

$$\rightarrow \quad -\sum_n h_n \nabla^2 u_n = \frac{1}{c}\sum_n \frac{d f_n}{d t}u_n$$

(where we used the fact that $\nabla \cdot u = 0$) we find

$$\frac{df_n(t)}{dt} = ck_n^2 h_n(t).$$

since we have $\nabla^2 u_n = -k_n^2 u_n$. Finally we have:

$$\frac{d^2}{dt^2}h_n(t) + c^2 k_n^2 h_n(t) = 0$$

The Hamiltonian of the system represent the total energy[33]: $\mathcal{H} = \frac{1}{2}\frac{1}{4\pi}\int(E^2 + B^2)d^3x$.
We can show that $\mathcal{H} = \sum_n \frac{1}{8\pi}(f_n^2 + k_n^2 h_n^2)$:

$$\mathcal{H} = \frac{1}{8\pi}\int(E^2 + B^2)d^3x = \frac{1}{8\pi}\sum_{n,m}\left(f_n f_m \int u_n(x)u_m(x)d^3x + h_n h_m \int (\nabla \times u_n)\cdot(\nabla \times u_m)d^3x\right)$$

$$= \sum_n \frac{1}{8\pi}(f_n^2 + k_n^2 h_n^2)$$

where we used $\int (\nabla \times u_n)\cdot(\nabla \times u_m)d^3x = k_n^2 \delta_{n,m}$. We can then use the equation $\frac{df_n(t)}{dt} = ck_n^2 h_n(t)$ to eliminate $h_n$. Then $f_n$ can be associated with an equivalent position operator and $h_n$ (being a derivative of the position) with the momentum operator.
Notice that the Hamiltonian for a set of harmonic oscillators, each having unit mass, is

$$\mathcal{H}_{h.o.} = \sum_n \frac{1}{2}(p_n^2 + \omega_n^2 q_n^2)$$

with $q_n$, $p_n = \frac{dq_n}{dt}$ the position and momentum of each oscillator.

## 10.2 Quantization of the e.m. field

Given the Hamiltonian we found above, we can associate the energy $\frac{1}{2}(p_n^2 + \omega^2 q_n^2)$ to each mode. We thus make the identification of $f_n$ with an equivalent position:

$$Q_n = \frac{f_n}{2\omega_n\sqrt{\pi}}$$

and then proceed to quantize this effective position, associating an operator to the position $Q_n$:

$$\hat{Q}_n = \sqrt{\frac{\hbar}{2\omega_n}}(a_n^\dagger + a_n)$$

We can also associate an operator to the normal mode coefficients $f_n$:

$$\hat{f}_n = \sqrt{2\pi\omega_n\hbar}(a_n^\dagger + a_n)$$

---

[33] The factor $4\pi$ is present because I am using cgs units, in SI units the energy density is $\frac{\epsilon_0}{2}(E^2 + c^2 B^2)$.

Notice that $f_n(t)$ is a function of time, so also the operators $a_n(t)$ are (Heisenberg picture). The electric field is the sum over this normal modes (notice that now the position is just a parameter, no longer an operator):

$$\mathbf{E}(x,t) = \sum_n \sqrt{2\hbar\pi\omega_n}[a_n^\dagger(t) + a_n(t)]\mathbf{u}_n(x)$$

Of course now the electric field is an **operator field**, that is, it is a QM operator that is defined at each space-time point (x,t).
Notice that an equivalent formulation of the electric field in a finite volume $V$ is given by defining in a slightly different way the $u_n(x)$ normal modes and writing:

$$\mathbf{E}(x,t) = \sum_n \sqrt{\frac{2\hbar\pi\omega_n}{V}}[a_n^\dagger(t) + a_n(t)]\mathbf{u}_n(x).$$

We already have calculated the evolution of the operator $a$ and $a^\dagger$. Each of the operator $a_n$ evolves in the same way: $a_n(t) = a_n(0)e^{-i\omega_n t}$. This derives from the Heisenberg equation of motion $\frac{da_n}{dt} = \frac{i}{\hbar}[\mathcal{H}, a_n(t)] = -i\omega_n a_n(t)$.
The magnetic field can also be expressed in terms of the operators $a_n$:

$$\mathbf{B}(x,t) = \sum_n ic_n \sqrt{\frac{2\pi\hbar}{\omega_n}}[a_n^\dagger - a_n]\nabla \times \mathbf{u}_n(x)$$

The strategy has been to use the known forms of the operators for a harmonic oscillator to deduce appropriate operators for the e.m. field. Notice that we could have used the equation $\frac{dh_n(t)}{dt} = -cf_n(t)$ to eliminate $f_n$ and write everything in terms of $h_n$. This would have corresponded to identifying $h_n$ with position and $f_n$ with momentum. Since the Hamiltonian is totally symmetric in terms of momentum and position, the results are unchanged and we can choose either formulations. In the case we chose, comparing the way in which the raising and lowering operators enter in the $E$ and $B$ expressions with the way they enter the expressions for position and momentum, we may say that, roughly speaking, the electric field is analogous to the position and the magnetic field is analogous to the momentum of an oscillator.

### 10.2.1 Zero-Point Energy and the Casimir Force

The Hamiltonian operator for the e.m. field has the form of a harmonic oscillator for each mode of the field[34]. As we saw in a previous lecture, the lowest energy of a h.o. is $\frac{1}{2}\hbar\omega$. Since there are infinitely many modes of arbitrarily high frequency in any finite volume, it follows that there should be an infinite zero-point energy in any volume of space. Needless to say, this conclusion is unsatisfactory. In order to gain some appreciation for the magnitude of the zero-point energy, we can calculate the zero-point energy in a rectangular cavity due to those field modes whose frequency is less than some cutoff $\omega_c$. The mode functions $u_n(x)$, solutions of the mode equation for a cavity of dimensions $L_x \times L_y \times L_z$, have the vector components

$$u_{n,\alpha} = A_\alpha \cos(k_{n,x}r_x)\sin(k_{n,y}r_y)\sin(k_{n,z}r_z)$$

for $\{\alpha, \beta, \gamma\} = \{x, y, z\}$ and permutations thereof. The mode $u_{n,\alpha}(x)$ are labeled by the wave-vector $\vec{k}_n$ with components:

$$k_{n,\alpha} = \frac{n_\alpha \pi}{L_\alpha}, \quad n_\alpha \in \mathcal{N}$$

and the frequency of the mode is $\omega_n = \sqrt{k_{n,x}^2 + k_{n,y}^2 + k_{n,z}^2}$. At least two of the integers must be nonzero, otherwise the mode function would vanish identically.
The amplitudes of the three components $A_\alpha$ are related by the divergence condition $\nabla \cdot \vec{u}_n(x) = 0$, which requires that $\vec{A} \cdot \vec{k} = 0$, from which it is clear that there are two linearly independent polarizations (directions of A) for each

---

[34] See: Leslie E. Ballentine, "Quantum Mechanics A Modern Development", World Scientific Publishing (1998). We follow his presentation in this section.

k, and hence there are two independent modes for each set of positive integers $(n_x, n_y, n_z)$. If one of the integers is zero, two of the components of $u(x)$ will vanish, so there is only one mode in this exceptional case.

In this case the electric field can be written as:

$$E(x, t) = \sum_{\alpha=1,2} (\mathcal{E}_\alpha + \mathcal{E}_\alpha^\dagger) = \sum_{\alpha=1,2} \hat{e}_\alpha \sum_n \sqrt{\frac{\hbar\omega_n}{2\epsilon_0 V}}[a_n^\dagger e^{i(\vec{k}_n \cdot \vec{r} - \omega t)} + a_n e^{i(\vec{k}_n \cdot \vec{r} - \omega t)}]$$

Here $V = L_x L_y L_z$ is the volume of the cavity. Notice that the electric field associated with a single photon of frequency $\omega_n$ is

$$\mathcal{E}_n = \sqrt{\frac{2\pi\hbar\omega_n}{V}}$$

This energy is a figure of merit for any phenomena relying on atomic interactions with a vacuum field, for example, cavity quantum electrodynamics. In fact, $\mathcal{E}_n$ may be estimated by equating the quantum mechanical energy of a photon $\hbar\omega_n$ with its classical energy $\frac{1}{2}\int dV(E^2 + B^2)$.

Going back to the calculation of the energy density, if the dimensions of the cavity are large, the allowed values of $k$ approximate a continuum, and the density of modes in the positive octant of $k$ space is $\rho(k) = 2V/\pi^3$ (the factor 2 comes from the two possible polarizations). The zero-point energy density for all modes of frequency less that $\omega_c$ is then given by

$$E_0 = \frac{2}{V}\sum_{k=1}^{k_c} \frac{1}{2}\hbar\omega_k \approx \frac{2}{V}\frac{1}{8}\int d^3k \rho(k)\frac{1}{2}\hbar\omega(k)$$

where we sum over all positive $k$ (in the first sum) and multiply by the number of possible polarizations (2). The sum is then approximated by an integral over the positive octant (hence the $1/8$ factor). Using $\omega(k) = kc$ (and $d^3k = 4\pi k^2 dk$), we obtain

$$E_0 = \frac{2}{V}\frac{2V}{\pi^3}\frac{4\pi}{8}\int_{k=0}^{k_c} dk\frac{1}{2}\hbar k^3 c = \frac{c\hbar}{2\pi^2}\int_{k=0}^{k_c} dk k^3 = \frac{\hbar c k_c^4}{8\pi^2}$$

where we set the cutoff wave-vector $k_c = \omega_c/c$. The factor $k_c^4$ indicates that this energy density is dominated by the high-frequency, short-wavelength modes. Taking a minimum wavelength of $\lambda_c = 2\pi/k = 0.4 \times 10^{-6}$m, so as to include the visible light spectrum, yields a zero-point energy density of 23 J/m$^3$. This may be compared with energy density produced by a 100 W light bulb at a distance of 1 m, which is $2.7 \times 10^{-8}$ J/m$^3$. Of course it is impossible to extract any of the zero-point energy, since it is the minimum possible energy of the field, and so our inability to perceive that large energy density is not incompatible with its existence. Indeed, since most experiments detect only energy differences, and not absolute energies, it is often suggested that the troublesome zero-point energy of the field should simply be omitted. One might even think that this energy is only a constant background to every experimental situation, and that, as such, it has no observable consequences. On the contrary, the vacuum energy has direct measurable consequences, among which the Casimir effect is the most prominent one.

In 1948 H. B. G. Casimir showed that two electrically neutral, perfectly conducting plates, placed parallel in vacuum, modify the vacuum energy density with respect to the unperturbed vacuum. The energy density varies with the separation between the mirrors and thus constitutes a force between them, which scales with the inverse of the forth power of the mirrors separation. The Casimir force is a small but well measurable quantity. It is a remarkable macroscopic manifestation of a quantum effect and it gives the main contribution to the forces between macroscopic bodies for distances beyond 100nm.

We consider a large cavity of dimensions $V = L^3$ bounded by conducting walls (see figure). A conducting plate is inserted at a distance $R$ from one of the yz faces ($R \ll L$). The new boundary condition at $x = R$ alters the energy (or frequency) of each field mode. Following Casimir, we shall calculate the energy shift as a function of $R$. Let $W_X$ denote the electromagnetic energy within a cavity whose length in the $x$ direction is $X$. The change in the energy due to the insertion of the plate at $x = R$ will be

$$\Delta W = (W_R + W_{L-R}) - W_L$$

Each of these three terms is infinite, but the difference will turn out to be finite. Each mode has a zero-point energy of $\frac{1}{2}\hbar kc$. But while we can take the continuous
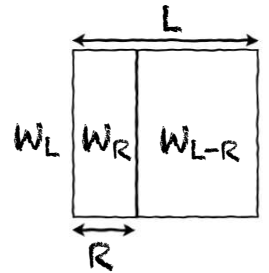


Fig. 14: Geometry of Casimir Effect

97

approximation in calculating $W_L$ and $W_{L-R}$, for $W_R$ we have to keep the discrete
sum in the $x$ direction (if $R$ is small enough). With some calculations (see Ballentine) we find that the change in energy is

$$\Delta W = -\hbar c \frac{\pi^2}{720} \frac{L^2}{R^3}$$

When varying the position $R$, an attractive force (minus sign) is created between the conducting plates, equal to

$$F = -\frac{\partial \Delta W}{\partial R} = -\hbar c \frac{\pi^2}{240} \frac{L^2}{R^4}$$

The force per unit area (pressure) is then $P = -\frac{\pi^2}{240} \frac{\hbar c}{R^4}$. This is the so-called Casimir force. This force is very difficult to measure. The surfaces must be flat and clean, and free from any electrostatic charge. However, there has been measurements of the Casimir effect, since the experiment by by Sparnaay (1958).

The availability of experimental set-ups that allow accurate measurements of surface forces between macroscopic objects at submicron separations has recently stimulated a renewed interest in the Casimir effect and in its possible applications to micro- and nanotechnology. The Casimir force is highly versatile and changing materials and shape of the boundaries modifies its strength and even its sign. Modifying strength and even sign of the Casimir force has great potential in providing a means for indirect force transmission in nanoscale machines, which is at present not achievable without damaging the components. A contactless method would represent a breakthrough in the future development of nanomachines. More generally, a deeper knowledge of the Casimir force and Casimir torque could provide new insights and design alternatives in the fabrications of micro- and nanoelectromechanical-systems (MEMS and NEMS). Another strong motivation comes from the need to make advantage of the unique properties of Carbon Nanotubes in nanotechnology.

Measuring the Casimir force is also important from a fundamental standpoint as it probes the most fundamental physical system, that is, the quantum vacuum. Furthermore, it is a powerful experimental method for providing constraints on the parameters of a Yukawa-type modification to the gravitational interaction or on forces predicted by supergravity and string theory.

## 10.3 Quantization of the e.m. field in the Coulomb gauge

The quantization procedure and resulting interactions detailed above may appear quite general, but in fact we made an important assumption at the very beginning which will limit their applicability: we considered only the situation with no sources, so we implicitly treated only transverse fields where $\nabla \cdot \vec{E} = 0$. Longitudinal fields result from charge distributions $\rho$ and they do not satisfy a wave equation. By considering only transverse fields, however, we have further avoided the issue of gauge. Since a transverse electric field $E_T$ satisfies the wave equation, we were able to directly quantize it without intermediate recourse to the vector potential $A$ and thus we never encountered a choice of gauge. In fact, the procedure can be viewed as corresponding to an implicit choice of gauge $\phi = 0$, $\nabla \cdot \vec{A} = 0$ that corresponds to a Lorentz gauge.

A more general approach may use the canonical Hamiltonian for a particle of mass $m$ and charge $q$ in an electromagnetic field. In this approach, the particle momentum $p$ is replaced by the canonical momentum $p - qA/c$, so the Hamiltonian contains terms like $\mathcal{H} \sim (p - qA/c)^2/2m$. In this case, it is still possible to write a wave equation for the potentials. Then the potential are quantized and for an appropriate choice of gauge we find again the same results. Specifically, for an appropriate choice of gauge, the $p \cdot A$ terms imply the dipole interaction $E \cdot d$ that we will use in the following.

Within the Coulomb gauge, the vector potential obeys the wave equation

$$\frac{\partial^2 A}{\partial t^2} - c^2 \nabla^2 A = 0$$

Taking furthermore periodic boundary conditions in a box of volume $V = L^3$ the quantized electromagnetic field in the Heisenberg picture is:

$$\vec{A}(t,x) = \sum_{\alpha=1,2} \sum_{k} \sqrt{\frac{2\pi\hbar}{V\omega_k}} \left[ a_{k\alpha} e^{-i(\omega_k t - \vec{k} \cdot \vec{x})} + a_{k\alpha}^\dagger e^{i(\omega_k t - \vec{k} \cdot \vec{x})} \right] \hat{e}_\alpha(k)$$

The field can then be written in terms of the potential as $E = -\frac{\partial A}{\partial t}$ and we find the similar result as before:

$$E(t,x) = \sum_{\alpha=1,2} \sum_{k} \sqrt{\frac{2\pi\hbar\omega_k}{V}} \left[ a_{k\alpha} e^{-i(\omega_k t - \vec{k} \cdot \vec{x})} - a_{k\alpha}^\dagger e^{i(\omega_k t - \vec{k} \cdot \vec{x})} \right] \hat{e}_\alpha(k)$$

and

$$B(t,x) = \sum_{\alpha=1,2} \sum_{k} \sqrt{\frac{2\pi\hbar\omega_k}{V}} \left[ a_{k\alpha} e^{-i(\omega_k t - \vec{k} \cdot \vec{x})} - a_{k\alpha}^\dagger e^{i(\omega_k t - \vec{k} \cdot \vec{x})} \right] (k \times \hat{e}_\alpha(k))$$

## 10.4 States of the e.m. field

Because of the analogies of the e.m. with a set of harmonic oscillators, we can apply the knowledge of the h.o. states to describe the states of the e.m. field. Specifically, we will investigate number states and coherent states.

### 10.4.1 Photon number eigenstates

We can define number states for each mode of the e.m. field. The Hamiltonian for a single mode is given by $\mathcal{H}_m = \hbar\omega_m(a_m^\dagger a_m + \frac{1}{2})$ with eigenvectors $|n_m\rangle$. The state representing many modes is then given by

$$|n_1, n_2, \ldots\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \cdots = |\vec{n}\rangle$$

Therefore the $m^{th}$ mode of this state is described as containing $n_m$ photons. These elementary excitations of the e.m. field behave in many respects like particles, carrying energy and momentum. However, the analogy is incomplete, and it is not possible to replace the e.m. field by a gas of photons.

In a state with definite photon numbers, the electric and magnetic fields are indefinite and fluctuating. The probability distributions for the electric and magnetic fields in such a state are analogous to the distributions for the position and momentum of an oscillator in an energy eigenstate. Thus we have for the expectation value of the electric field operator:

$$\langle E(x,t)\rangle = \langle \vec{n}| \sum_m \sqrt{2\hbar\pi\omega_m}[a_m^\dagger(t) + a_m(t)]u_m(x) |\vec{n}\rangle = 0$$

However, the second moment is non-zero:

$$\left\langle |E(x,t)|^2 \right\rangle = 2\pi\hbar \sum_{p,m} \sqrt{\omega_p\omega_m} \left\langle [a_p^\dagger(t) + a_p(t)][a_m^\dagger(t) + a_m(t)] \right\rangle \vec{u}_p(x) \cdot \vec{u}_m(x)$$

$$= 2\pi\hbar \sum_m \omega_m |u_m|^2 \left\langle [a_m^\dagger(t) + a_m(t)]^2 \right\rangle = 2\pi\hbar \sum_m \omega_m |u_m(x)|^2 (2n_m + 1)$$

The sum over all modes is infinite. This divergence problem can often be circumvented (but not solved) by recognizing that a particular experiment will effectively couple to the EM field only over some finite bandwidth, thus we can set cut-offs on the number of modes considered.

Notice that we can as well calculate $\Delta B$ for the magnetic field, to find the same expression.

### 10.4.2 Coherent states

A coherent state of the e.m. field is obtained by specifying a coherent state for each of the mode oscillators of the field. Thus the coherent state vector will have the form

$$|\vec{\alpha}\rangle = |\alpha_1\alpha_2\ldots\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots$$

It is parameterized by a denumberably infinite sequence of complex numbers. We now want to calculate the evolution of the electric field for a coherent state. In the Heisenberg picture it is:

$$E(x,t) = \sum_m \sqrt{2\hbar\pi\omega_m}[a_m^\dagger e^{i\omega_m t} + a_m e^{-i\omega_m t}]u_m(x)$$

then, taking the expectation value we find:

$$\langle E(x,t)\rangle = \sum_m \sqrt{2\hbar\pi\omega_m}[\alpha_m^* e^{i\omega_m t} + \alpha_m e^{-i\omega_m t}]u_m(x)$$

This is exactly the same form as a normal mode expansion of a classical solution of Maxwell's equations, with the parameter $\alpha_m$ representing the amplitude of a classical field mode. In spite of this similarity, a coherent state of the quantized EM field is not equivalent to a classical field, although it does give the closest possible quantum operator, in terms of its expectation value. Even if the average field is equivalent to the classical field, there are still the characteristic quantum fluctuations. A coherent state provides a good description of the e.m. field produced by a laser. Most ordinary light sources emit states of the e.m. field that are very close to a coherent state (lasers), or to a statistical mixture of coherent states (classical sources).

### A. Fluctuations

We calculate the fluctuations for a single mode $\Delta E_m$. From $\left\langle |E_m|^2 \right\rangle = \langle \alpha_m| E_m \cdot E_m |\alpha_m\rangle$ we obtain $\Delta E_m^2 = 2\pi\hbar\omega_m |\vec{u}_m(x)|^2$. Indeed, from $\left\langle (a_m + a_m^\dagger)^2 \right\rangle$ we obtain:

$$\langle \alpha_m| (a_m^\dagger)^2 + a_m^2 + a_m^\dagger a_m + a_m a_m^\dagger |\alpha_m\rangle \, 2\pi\hbar\omega|u_m(x)|^2 = \left[1 + ((\alpha_m^*)^2 + \alpha_m^2 + 2\alpha_m^*\alpha_m)\right] 2\pi\hbar\omega|u_m(x)|^2$$

while we have $\left\langle a_m + a_m^\dagger \right\rangle = (\alpha_m + \alpha_m^*)\sqrt{2\pi\hbar\omega}u_m(x)$, so that we obtain

$$\left\langle (a_m + a_m^\dagger)^2 \right\rangle - \left\langle a_m + a_m^\dagger \right\rangle^2 = 2\pi\hbar\omega_m |\vec{u}_m(x)|^2$$

This is independent of $\alpha_m$, and is equal to the mean square fluctuation in the ground state. The Heisenberg inequality is therefore saturated when the field is in a coherent state,

## B. Photon statistics

The photon number distribution for each mode in a coherent state is obtained as for the h.o. The probability of finding a total of $n$ photons in the field mode is governed by the Poisson distribution.

The probability of finding $n$ photons in the mode $m$ is $P_\alpha(n_m) = |\langle n_m | \alpha \rangle|^2$. Using the expansion of the coherent state in terms of the number states that we found for the h.o., we obtain $P_\alpha(n_m) = \frac{|\alpha_m|^{2n}}{n!} e^{-|\alpha_m|^2}$. This is a Poisson distribution, with parameter $|\alpha_m|^2$. Thus we have $\langle n_m \rangle = |\alpha_m|^2$, so that we can rewrite the pdf as $P(n) = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle}$. From the known properties of the Poisson distribution, we also find $\Delta n^2 = \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle$.

In particular we have the well-know shot-noise scaling $\frac{\Delta n}{\langle n \rangle} = 1/\sqrt{\langle n \rangle}$ (i.e. the fluctuations go to zero when there are many photons.)

### 10.4.3 Measurement Statistics

We saw in the previous section the photon number distribution for a coherent state. This corresponds to the experimental situation in which we want to measure the number of photons in a field (such as laser light) which is well approximated by a classical field and thus can be represented by a coherent state.

This is not the only type of measurement of the e.m. field that we might want to do. Two other common measurement modalities are homodyne and heterodyne detection[35].

Homodyne detection corresponds to the measurement of one quadrature amplitude. In practice, one mixes the e.m. field with a local oscillator at with a fixed frequency $\omega$ (same as the field frequency) before collecting the signal with
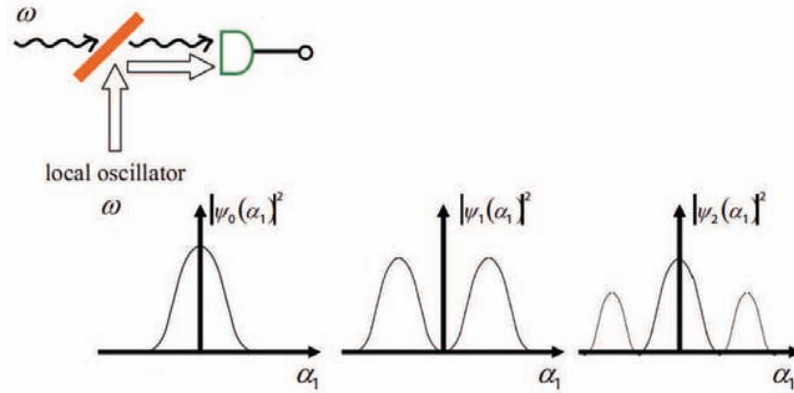
Fig. 15: Homodyne detection scheme and measurement statistics of the first three photon number eigenstates.

a photon counting detector.

Thus the measurement corresponds to the observable $O_{ho} = |\alpha_1\rangle \langle \alpha_1|$ (or $O_{ho} = |\alpha_2\rangle \langle \alpha_2|$, depending on the phase of the local oscillator), where $|\alpha_{1,2}\rangle$ are the eigenstates of the quadrature operators $a_1 = \frac{1}{2}(a + a^\dagger)$ and $a_2 = \frac{i}{2}(a^\dagger - a)$. The measurement statistics for a number state $|m\rangle$ is thus:

$$P_m(\alpha_1) = |\langle \alpha_1 | m \rangle|^2 = \langle \alpha_1 | \frac{(a^\dagger)^n}{n!} | n \rangle = \sqrt{\frac{\omega}{\pi \hbar}} H_m^2(\alpha_1/2) e^{-\alpha_1^2/2}, \qquad \langle O_{ho} \rangle = \langle m | O_{he} | m \rangle = 0, \qquad \langle \Delta O_{ho} \rangle = \frac{1}{2}$$

where $H_n$ is the $n^{th}$ Hermite polynomial and $\langle \Delta O \rangle = \sqrt{\langle O^2 \rangle - \langle O \rangle^2}$. We note that these results correspond to what we had found for the $x$ operator in the case of the quantum harmonic oscillator.

Heterodyne detection corresponds to the simultaneous measurement of the two quadratures of a field. Operationally, one mixes the e.m. field with a local oscillator of frequency $\omega$, modulated at the Intermediate Frequency $\omega_{IF}$; the

---

[35] We follow here the presentation in Prof. Yamamoto's Lectures
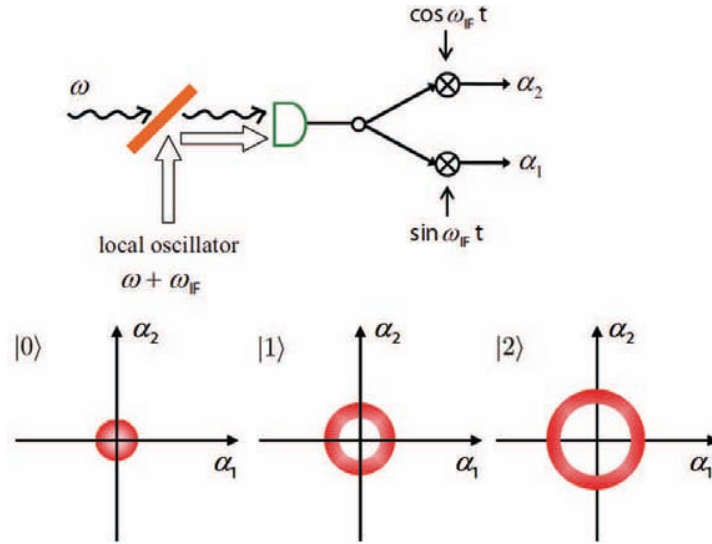
Fig. 16: Heterodyne detection scheme and measurement statistics of the first three photon number eigenstates.

signal, after collection, is demodulated by mixing it with $\sin(\omega_{IF})$ and $\cos(\omega_{IF}t)$. Thus the measurement corresponds to the observable $O_{he} = |\alpha\rangle\langle\alpha|$. The measurement statistics for a number state $|m\rangle$ is thus:

$$P_m(\alpha) = |\langle\alpha|m\rangle|^2 = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}, \qquad \langle O_{he}\rangle = \langle m|\, O_{he}\, |m\rangle = |\alpha|^2, \qquad \langle\Delta O_{he}\rangle = |\alpha|^2$$

(note that of course this is equivalent to the case were we measured a number state for a coherent state). The measurement statistics for a coherent state $|\beta\rangle$, would be

$$P_\beta(\alpha) = |\langle\alpha|\beta\rangle|^2 = e^{-|\beta-\alpha|^2}$$
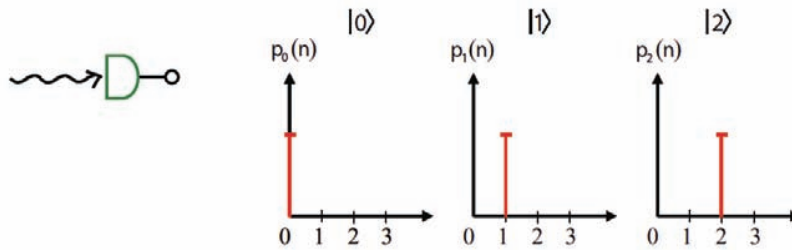
Fig. 17: Photon counting detection scheme and measurement statistics of the first three photon number eigenstates.

For comparison, photon counting is of course the measurement of the observable $O_n = |n\rangle\langle n|$, with statistics for a number state $|m\rangle$

$$P_m(n) = |\langle n|m\rangle|^2 = \delta_{m,n}, \qquad \langle O_n\rangle = \langle m|\, O_n\, |m\rangle = \delta_{m,n}, \qquad \langle\Delta O_n\rangle = 0$$

## 10.5 Atomic interactions with the quantized field

Let us consider the interaction of isolated neutral atoms with optical fields. Such atoms alone have no net charge and no permanent electric dipole moment. In an electric field $\vec{E}$ associated, e.g., with an electromagnetic wave, the atoms

102

do develop an electric dipole moment $\vec{d}$ which can then interact with the electric field with an interaction energy $V$ given by

$$V = \vec{d} \cdot \vec{E}$$

We have already treated a similar case in a semiclassical way, although we were interested in the interaction with a magnetic field. The semi-classical treatment of this interaction, is quite similar: we treat the atom quantum mechanically and therefore consider $\vec{d}$ as an operator, but treat the electromagnetic field classically and so consider $E$ as a vector. We can write the dipole moment operator as

$$\vec{d} = e\vec{r} = \sum_{k,h} |k\rangle \langle k| \vec{d} |h\rangle \langle h|$$

where $\{|k\rangle\}$ forms a complete basis. Transitions are only possible between states with different $h$ and $k$:

$$\vec{d}_{h,k} = \langle h| \vec{d} |k\rangle \neq 0 \quad \text{iif} \quad k \neq h$$

and we will consider for simplicity a two-level atom:

$$\vec{d} = |0\rangle\langle 1|\vec{d}_{01} + |1\rangle\langle 0|\vec{d}_{10}$$

Let's first consider a single mode **classical** electromagnetic field, given by $E = \vec{\mathcal{E}}e^{-i\omega t} + \vec{\mathcal{E}}^* e^{i\omega t}$. The full *semi-classical* (sc) Hamiltonian is then:

$$\mathcal{H}_{sc} = \frac{1}{2}\hbar\omega_0(|1\rangle\langle 1| - |0\rangle\langle 0|) - (|0\rangle\langle 1|\vec{d}_{01} + |1\rangle\langle 0|\vec{d}_{10}) \cdot (\vec{\mathcal{E}}e^{-i\omega t} + \vec{\mathcal{E}}^* e^{i\omega t})$$

If we assume $\{\vec{d}_{10}, \ \mathcal{E}\} \in \mathbb{R}$, we can rewrite this as

$$\mathcal{H}_{sc} = \frac{1}{2}\hbar\omega_0\sigma_z - 2\sigma_x\vec{d}_{10} \cdot \vec{\mathcal{E}}\cos(\omega t)$$

Notice the correspondence with the spin Hamiltonian $\mathcal{H}_{spin} = \Omega\sigma_z + B\cos(\omega t)\sigma_x$ describing the interaction of a spin with a time-varying, classical magnetic field.

We can now go into the interaction frame defined by the Hamiltonian $\mathcal{H}_0 = \frac{1}{2}\omega_0\sigma_z$. Then we have:

$$\tilde{\mathcal{H}}_{sc} = -(|0\rangle\langle 1|\vec{d}_{01}e^{i\omega_0 t} + |1\rangle\langle 0|\vec{d}_{10}e^{-i\omega_0 t}) \cdot (\vec{\mathcal{E}}e^{-i\omega t} + \vec{\mathcal{E}}^* e^{i\omega t})$$

On resonance ($\omega_0 = \omega$) we retain only time-independent contributions to the Hamiltonian (RWA), then

$$\tilde{\mathcal{H}}_{sc} \approx -(|1\rangle\langle 0|d\mathcal{E} + |0\rangle\langle 1|d^*\mathcal{E}^*)$$

Assuming for example that $d\mathcal{E}$ is real, we obtain an Hamiltonian $-d\mathcal{E}\sigma_x$, in perfect analogy with the TLS already studied. (A more general choice of $d\mathcal{E}$ just gives an Hamiltonian at some angle in the xy plane).

Now let us consider a full quantum-mechanical treatment of this problem. The interaction between an atom and a quantized field appears much the same as the semiclassical interaction. Starting with the dipole Hamiltonian for a two-level atom, we replace E by the corresponding operator, obtaining the interaction Hamiltonian

$$V = -\vec{d} \cdot \vec{E} = -\sum_{\alpha}(\vec{\mathcal{E}}_{\alpha} + \vec{\mathcal{E}}_{\alpha}^{\dagger}) \cdot (\vec{d}|1\rangle\langle 0| + \vec{d}^*|0\rangle\langle 1|)$$

$$= -\sum_{\alpha}\sum_{m}\sqrt{\frac{2\pi\hbar\omega_m}{V}}[a_m^{\dagger}e^{-i\vec{k}_m \cdot \vec{r}} + a_m e^{i\vec{k}_m \cdot \vec{r}}](d_{\alpha}|1\rangle\langle 0| + d_{\alpha}^*|0\rangle\langle 1|)$$

(where $\alpha$ is the polarization and $m$ the mode). As in the semiclassical analysis, the Hamiltonian contains four terms, which now have a clearer physical picture:

$a_m^\dagger|0\rangle\langle1|$      Atom decays from $|1\rangle \rightarrow |0\rangle$ and emits a photon (in the $m^{th}$ mode).

$a_m|1\rangle\langle0|$      Atom is excited from $|0\rangle \rightarrow |1\rangle$ and absorbs a photon (from the $m^{th}$ mode).

$a_m^\dagger|1\rangle\langle0|$      Atom is excited from $|0\rangle \rightarrow |1\rangle$ and emits a photon (in the $m^{th}$ mode).

$a_m|0\rangle\langle1|$      Atom decays from $|1\rangle \rightarrow |0\rangle$ and absorbs a photon (from the $m^{th}$ mode).

For photons near resonance with the atomic transition, the first two processes conserve energy; the second two processes do not conserve energy, and intuition suggests that they may be neglected. In fact, there is a direct correspondence between the RWA and energy conservation: the second two processes are precisely those fast-rotating terms we disregarded previously.

Consider the total Hamiltonian :

$$\mathcal{H} = \mathcal{H}_0 + V = \frac{\hbar}{2}\omega_0\sigma_z + \sum_m \hbar\omega_m\left(a_m^\dagger a_m + \frac{1}{2}\right) + V$$

If we go to the interaction frame defined by the Hamiltonian $\mathcal{H}_0$, each mode acquires a time dependence $e^{\pm i\omega_m t}$ while the atom acquires a time dependence $e^{\pm i\omega_0 t}$:

$$\sum(\mathcal{E}_m a_m^\dagger e^{-i\vec{k}_m\cdot\vec{r}}e^{i\omega_m t} + \mathcal{E}_m^* a_m e^{i\vec{k}_m\cdot\vec{r}}e^{-i\omega_m t})\cdot(e^{+i\omega_0 t}d_\alpha|1\rangle\langle0| + d_\alpha^* e^{-i\omega_0 a t}|0\rangle\langle1|)$$

where $\mathcal{E}_m = \sqrt{\frac{2\pi\hbar\omega_m}{V}}$. Thus the time-dependent factors acquired are

$a_m^\dagger|0\rangle\langle1| \rightarrow a_m^\dagger|0\rangle\langle1|e^{+i(\omega_0-\omega_m)t}$      $a_m|1\rangle\langle0| \rightarrow a_m|1\rangle\langle0|e^{-i(\omega_0-\omega_m)t}$

$a_m^\dagger|1\rangle\langle0| \rightarrow a_m^\dagger|1\rangle\langle0|e^{i(\omega_0+\omega_m)t}$      $a_m|0\rangle\langle1| \rightarrow a_m|0\rangle\langle1|e^{-i(\omega_0+\omega_m)t}$

For frequencies $\omega_m$ near resonance $\omega_m \approx \omega_0$, we only retains the first two terms.

Then, defining the *single-photon Rabi frequency*, $g_{m,\alpha} = -\frac{d_\alpha}{\hbar}\sqrt{\frac{2\pi\hbar\omega_m}{V}}e^{i\vec{k}_m\cdot\vec{r}}$, the Hamiltonian in the interaction picture and in the RWA approximation is

$$\mathcal{H} = \sum_m \hbar\left(g_{m,\alpha}a_m|1\rangle\langle0| + g_{m,\alpha}^* a_m^\dagger|0\rangle\langle1|\right)$$

From now on we assume an e.m. with a single mode (or we assume that only one mode is on resonance). We can write a general state as $|\psi\rangle = \sum_n \alpha_n(t)|1n\rangle + \beta_n(t)|0n\rangle$, where $|n\rangle = |n_m\rangle$ is a state of the given mode $m$ we retain and here I will call the Rabi frequency for the mode of interest $g$. The evolution is given by:

$$i\hbar\sum_n \dot{\alpha}_n|1n\rangle + \dot{\beta}_n|0n\rangle = \hbar\sum_n g\left[\alpha_n\sigma_- a^\dagger|1n\rangle + \beta_n\sigma_+ a|0n\rangle\right]$$

$$= \hbar\sum_n g\left[\alpha_n\sqrt{n+1}|0,n+1\rangle + \beta_n\sqrt{n}|1,n-1\rangle\right]$$

We then project these equations on $\langle1n|$ and $\langle0n|$:

$$i\hbar\dot{\alpha}_n = \hbar g\beta_{n+1}(t)\sqrt{n+1}$$

$$i\hbar\dot{\beta}_n = \hbar g\alpha_{n-1}(t)\sqrt{n}$$

to obtain a set of equations:

$$\begin{cases} \dot{\alpha}_n = -ig\sqrt{n+1}\beta_{n+1} \\ \dot{\beta}_{n+1} = -ig\sqrt{n+1}\alpha_n \end{cases}$$

This is a closed system of differential equations and we can solve for $\alpha_n$, $\beta_{n+1}$.

We consider a more general case, where the field-atom are not exactly on resonance. We define $\Delta = \frac{1}{2}(\omega_0 - \omega)$, where $\omega = \omega_m$ for the mode considered. Then the Hamiltonian is:

$$\mathcal{H} = \hbar\left(g\,a|1\rangle\langle0| + g^* a^\dagger|0\rangle\langle1|\right) + \hbar\Delta\left(|1\rangle\langle1| - |0\rangle\langle0|\right)$$

We can assume that initially the atom is in the the excited state $|1\rangle$ (that is, $\beta_n(0) = 0$, $\forall n$). Then we have:

$$\alpha_n(t) = \alpha_n(0)e^{i\Delta t/2}\left[\cos\left(\frac{\Omega_n t}{2}\right) - \frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]$$

$$\beta_n(t) = -\alpha_n(0)e^{-i\Delta t/2}\left[\frac{2ig\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]$$

with $\Omega_n^2 = \Delta^2 + 4g^2(n+1)$. If initially there is no field (i.e. the e.m. field is in the vacuum state) and the atom is in the excited state, then $\alpha_0(0) = 1$, while $\alpha_n(0) = 0$ $\forall n \neq 0$. Then there are only two components that are different than zero:

$$\alpha_0(t) = e^{i\Delta t/2}\left[\cos\left(\frac{\Omega_0 t}{2}\right) - \frac{i\Delta}{\sqrt{\Delta^2 + 4g^2}}\sin\left(\frac{\Omega_0 t}{2}\right)\right]$$

$$\beta_0(t) = -e^{-i\Delta t/2}\frac{2ig}{\sqrt{\Delta^2 + 4g^2}}\sin\left(\frac{\Omega_0 t}{2}\right)$$

or on resonance ($\Delta = 0$)

$$\langle 1, n = 0|\psi(t)\rangle = \alpha_0(t) = \cos\left(\frac{gt}{2}\right)$$

$$\langle 0, n = 0|\psi(t)\rangle = \beta_0(t) = -i\sin\left(\frac{gt}{2}\right)$$

Thus, even in the absence of field, it is possible to make the transition from the ground to the excited state! In the semiclassical case (where the field is treated as classical) we would have no transition at all. These are called Rabi vacuum oscillations.

22.51 Quantum Theory of Radiation Interactions

Fall 2012