Métodos numéricos

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Ecuación de Burgers (viscosa)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \qquad u(x,t) \in [0, 2\pi] \quad , \quad u(x) = u(x+2\pi)$$

Galerkin espectral:

$$u_N(x,t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t)\, e^{ikx}$$

$$(R_N, \phi_k) = \int_0^{2\pi} \left(\frac{\partial u_N}{\partial t} + u_N\frac{\partial u_N}{\partial x} - \nu\frac{\partial^2 u_N}{\partial x^2} \right) e^{-ikx}\, dx = 0 \quad ; \quad k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$$

$$\frac{\partial u_N}{\partial t} = \sum_{m=-N/2}^{N/2-1} \frac{d\hat{u}_m}{dt}\, e^{imx} \qquad \frac{\partial u_N}{\partial x} = \sum_{m=-N/2}^{N/2-1} im\, \hat{u}_m\, e^{imx} \qquad \frac{\partial^2 u_N}{\partial x^2} = \sum_{m=-N/2}^{N/2-1} -m^2 \hat{u}_m\, e^{imx}$$

$$u_N \frac{\partial u_N}{\partial x} = \left(\sum_{m=-N/2}^{N/2-1} \hat{u}_m \, e^{imx} \right) \left(\sum_{n=-N/2}^{N/2-1} in \, \hat{u}_n \, e^{inx} \right) = \sum_{m,n=-N/2}^{N/2-1} in \, \hat{u}_m \hat{u}_n \, e^{i(m+n)x}$$

De $(R_N, \phi_k) = 0$ y usando $\int_0^{2\pi} e^{i(m-k)x} dx = 2\pi\delta_{mk}$

$$\rightarrow \quad \frac{d\hat{u}_k}{dt} + \nu k^2 \hat{u}_k + \sum_{\substack{m,n=-N/2 \\ m+n=k}}^{N/2-1} in \, \hat{u}_m \hat{u}_n = 0 \quad ; \quad k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$$

→ acopla todos los modos

N^2 operaciones

Con método pseudoespectral (PS)

→ discretizo $x \to x_j$

Elijo: $x_j = \frac{2\pi}{N} j \; ; \; j = 0, \ldots, N-1$ $\quad \Delta x = \frac{2\pi}{N}$ $\quad x_0 = 0 \;,\; x_N = 2\pi \;,\; u(x_N) = u(x_0)$

$$u_N(x_j, t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t)\, e^{ikx_j} = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t)\, e^{2\pi i\, kj/N} \quad ; \quad j = 0, \ldots, N-1$$

$$\hat{u}_k \iff u_N(x_j, t)$$

DFT

$$\hat{u}_k(t) = \frac{1}{N} \sum_{j=0}^{N-1} u_N(x_j, t)\, e^{-ikx_j} \quad ; \quad k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$$

se cumple la relación de ortogonalidad: $\frac{1}{N} \sum_{j=0}^{N-1} e^{ipx_j} = \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi i\, pj/N} = \delta_{p,qN} \quad , \quad q = 0, \pm 1, \pm 2, \ldots.$

Idem antes (Galerkin) vale

$$\frac{\partial u_N}{\partial t} = \sum_{m=-N/2}^{N/2-1} \frac{d\hat{u}_m}{dt}\, e^{imx} \qquad \frac{\partial u_N}{\partial x} = \sum_{m=-N/2}^{N/2-1} im\, \hat{u}_m\, e^{imx} \qquad \frac{\partial^2 u_N}{\partial x^2} = \sum_{m=-N/2}^{N/2-1} -m^2 \hat{u}_m\, e^{imx}$$

y se pueden evaluar en $x = x_j$ y reemplazar con la expresión de $\hat{u}_m$

en términos de los $u_N(x_j, t)$ y resolver → colocación

Vamos a hacer algo “híbrido” → pseudoespectral

Empiezo con $\hat{u}_k(0)$ transformando la c.i. : $u_N(x_j, 0) \rightarrow \hat{u}_k(0)$

Evalúo $\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$ multiplicando por $ik\ ,\ -k^2$

Qué hago con $u\frac{\partial u}{\partial x}$? → Me voy al espacio real “x” y hago las N multiplicaciones en cada nodo

→ Luego vuelvo al espacio Fourier “k” y obtengo $\widehat{\left[u\frac{\partial u}{\partial x}\right]}_k$ → evoluciono en Fourier $\hat{u}_k(t)$

A diferencia del método de Galerkin, donde todo lo hacemos en el espacio “k” en el método pseudoespectral se hacen las derivadas en el espacio “k” y para hacer productos de funciones (que serían convoluciones en el espacio “k” y demandarían N^2 operaciones con Galerkin) se va y viene del espacio “k” al espacio “x” → usamos la DFT → también serían N^2 operaciones

Gauss 1805 — IBM — Princeton

→ en los 60’s se recupera la FFT (Fast Fourier Transform) → algoritmo Cooley-Tukey (paper 1965)

→ *divide and conquer*

$$N \log N \quad \text{operaciones}$$

Arrancamos con $\hat{u}_k$ calculamos derivada en “k” → $ik\hat{u}_k$ (N operaciones)

→ hacemos FFT^{-1} para obtener $u_N(x_j)\,,\ \dfrac{\partial u_N}{\partial x}(x_j)$ (2N log(N) operaciones)

→ multiplicamos $u_N(x_j)\dfrac{\partial u_N}{\partial x}(x_j)$ en cada punto x_j (N operaciones)

→ volvemos al espacio “k” $\left[\widehat{u\dfrac{\partial u}{\partial x}}\right]_k$ (N log N operaciones) → evolucionamos $\hat{u}_k(t+\Delta t)$

$$u_N(x_j) = \sum_{m=-N/2}^{N/2-1} \hat{u}_m \, e^{imx_j} \qquad \frac{\partial u_N}{\partial x}(x_j) = \sum_{n=-N/2}^{N/2-1} in\hat{u}_n \, e^{inx}$$

$$P(x_j) = u_N(x_j)\frac{\partial u_N}{\partial x}(x_j) = \left(\sum_{m=-N/2}^{N/2-1} \hat{u}_m \, e^{imx_j}\right)\left(\sum_{n=-N/2}^{N/2-1} in\hat{u}_n \, e^{inx_j}\right) = \sum_{m,n=-N/2}^{N/2-1} in\hat{u}_n\hat{u}_m e^{i(m+n)x_j}$$

$$\hat{P}_k = \frac{1}{N}\sum_{j=0}^{N-1} P(x_j)e^{-ikx_j} = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{m,n=-N/2}^{N/2-1} in\hat{u}_n\hat{u}_m e^{i(m+n)x_j}e^{-ikx_j}$$

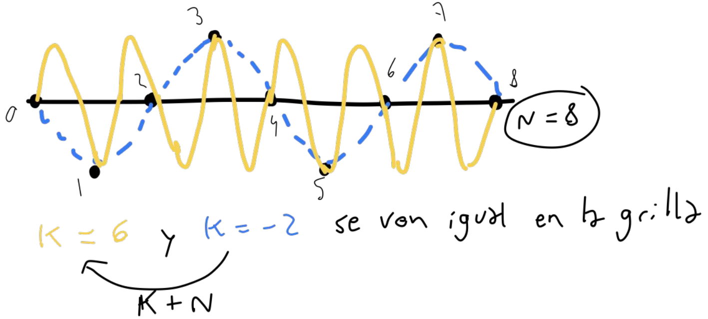
$$\hat{P}_k = \frac{1}{N}\sum_{m,n=-N/2}^{N/2-1} in\hat{u}_n\hat{u}_m \sum_{j=0}^{N-1} e^{i(m+n-k)x_j} \qquad \frac{1}{N}\sum_{j=0}^{N-1} e^{ipx_j} = \delta_{p,qN} \quad , \quad q = 0, \pm 1, \pm 2, \ldots .$$

$$p = m + n - k = qN = 0, \pm N$$

$$\hat{P}_k = \sum_{m+n=k} in\hat{u}_n\hat{u}_m + \sum_{m+n=k\pm N} in\hat{u}_n\hat{u}_m$$

valor de $\left[\widehat{u\frac{\partial u}{\partial x}}\right]_k$ si hago Galerkin

aliasing porque mi grilla no distingue modos k+N o k-N del modo k



técnica para *remover el aliasing*: regla de los 2/3

sup. uso $M \geq \frac{3N}{2}$ y hago que los modos con $|k| \geq \frac{N}{2}$ sean 0 (los filtro)

→ aliasing es $\sum_{m+n=k\pm M} in\hat{u}_n \hat{u}_m$ y solo es $\neq 0$ si $m+n-k = \pm M$

sabemos que $$-\frac{3M}{2} + 1 \leq m+n-k \leq \frac{3M}{2} - 2$$

y no se cumple nunca que $m+n-k = \pm M$ para modos $\neq 0$

→ en la práctica lo que se hace es filtrar los modos $|k| \geq \frac{N}{2} = \frac{M}{3}$

es decir, si trabajo con M modos, filtro los modos $|k| \geq \frac{M}{3}$ en cada evaluación de $\left[\widehat{u\frac{\partial u}{\partial x}}\right]_k$

Balance de energía

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \qquad 0 \leq x \leq L$$

$$E = \frac{1}{2L}\int_0^L u^2\, dx$$

$$u\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = \nu u\frac{\partial^2 u}{\partial x^2} \qquad \frac{1}{2}\frac{\partial u^2}{\partial t} + \frac{1}{3}\frac{\partial u^3}{\partial x} = \nu\, u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \nu\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) - \nu\left(\frac{\partial u}{\partial x}\right)^2$$

Integrando, $$\frac{dE}{dt} + \frac{u^3}{3L}\bigg|_0^L = \frac{\nu}{L}\left[u\frac{\partial u}{\partial x}\right]\bigg|_0^L - \frac{\nu}{L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$\frac{dE}{dt} = -2\nu\Omega \quad , \quad \Omega = \frac{1}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2 dx$$ enstrofía

Si $\nu = 0 \;\Rightarrow\; E = cte$

Sup. $L = 2\pi$ Parseval $\rightarrow$ $$E = \frac{1}{4\pi}\int_0^{2\pi} u^2\, dx = \frac{1}{2}\sum_k |\hat{u}_k|^2 = \frac{1}{2N}\sum_j u_N(x_j,t)^2$$

$$x_j = (j-1)\frac{2\pi}{N} \quad , \quad dx = \frac{2\pi}{N}$$

$$\frac{d\hat{u}_k}{dt} + \left[\widehat{u\frac{\partial u}{\partial x}}\right]_k = -\nu k^2 \hat{u}_k \quad , \quad |\hat{u}_k|^2 = \hat{u}_k\hat{u}_k^* = \hat{u}_k\hat{u}_{-k}$$

Si $u(x) \in \mathbb{R} \rightarrow \hat{u}_{-k} = \hat{u}_k^*$

Ec. para $\hat{u}_{-k}$ $$\frac{d\hat{u}_{-k}}{dt} + \left[\widehat{u\frac{\partial u}{\partial x}}\right]_{-k} = -\nu k^2 \hat{u}_{-k}$$

$$\hat{u}_{-k}\frac{d\hat{u}_k}{dt} + \hat{u}_{-k}\left[\widehat{u\frac{\partial u}{\partial x}}\right]_k + \hat{u}_k\frac{d\hat{u}_{-k}}{dt} + \hat{u}_k\left[\widehat{u\frac{\partial u}{\partial x}}\right]_{-k} = -\nu k^2\hat{u}_{-k}\hat{u}_k - \nu k^2 \hat{u}_k\hat{u}_{-k}$$

$$\frac{d(\hat{u}_k\hat{u}_{-k})}{dt} + \hat{u}_{-k}\left[\widehat{u\frac{\partial u}{\partial x}}\right]_k + \hat{u}_k\left[\widehat{u\frac{\partial u}{\partial x}}\right]_{-k} = -2\nu k^2\hat{u}_k\hat{u}_{-k}$$

Sumamos sobre todo k y usamos $\hat{u}_k \hat{u}_{-k} = |\hat{u}_k|^2 \quad , \quad E = \frac{1}{2}\sum_k |\hat{u}_k|^2$

$$2\frac{dE}{dt} + \sum_k \left\{ \hat{u}_{-k}\left[\widehat{u\frac{\partial u}{\partial x}}\right]_k + \hat{u}_k\left[\widehat{u\frac{\partial u}{\partial x}}\right]_{-k} \right\} = -2\nu\sum_k k^2|\hat{u}_k|^2 = -4\nu\,\Omega$$

$$\sum_k \hat{u}_{-k}\left[\widehat{u\frac{\partial u}{\partial x}}\right]_k = \sum_k \hat{u}_{-k}\sum_{m+n=k} im\,\hat{u}_m\hat{u}_n = \sum_{m+n-k=0} im\,\hat{u}_m\hat{u}_n\hat{u}_{-k} = \sum_{m+n+k'=0} im\,\hat{u}_m\hat{u}_n\hat{u}_{k'}$$

$$= \frac{1}{3}\sum_{m+n+k'=0} im\,\hat{u}_m\hat{u}_n\hat{u}_{k'} + ik'\,\hat{u}_{k'}\hat{u}_n\hat{u}_m + in\,\hat{u}_n\hat{u}_{k'}\hat{u}_m = \frac{1}{3}\sum_{m+n+k'=0} i(m+n+k')\hat{u}_m\hat{u}_n\hat{u}_{k'} = 0$$

Idem $\sum_k \hat{u}_k\left[\widehat{u\frac{\partial u}{\partial x}}\right]_{-k} = 0 \quad \Rightarrow \quad \frac{dE}{dt} = -2\nu\Omega$ Si $\nu = 0 \quad \Rightarrow \quad E = cte$

Métodos espectrales conservan las cantidades cuadráticas

Estabilidad Burgers RK2

Si no, linealizo

$$u = c + \epsilon \quad , \quad u\frac{\partial u}{\partial x} \simeq c\frac{\partial \epsilon}{\partial x}$$

$$\frac{\partial \epsilon}{\partial t} + c\frac{\partial \epsilon}{\partial x} = 0$$

Veamos el caso de advección lineal → $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$

Galerkin → $\dfrac{d\hat{u}_k}{dt} + ik\, c\, \hat{u}_k = 0$

→ sol. exacta $\hat{u}_k(t) = \hat{u}_k(0)e^{-ikct}$

$w = k\, c$ <u>no hay dispersión ni difusión</u>

RK2 →

$$\hat{u}_k^{n+1/2} = \hat{u}_k^n - \frac{\Delta t}{2} ikc\, \hat{u}_k^n$$

$$\hat{u}_k^{n+1} = \hat{u}_k^n - \Delta t\, ikc\, \hat{u}_k^{n+1/2}$$

$$\lambda^{n+1/2} = \lambda^n - \frac{\Delta t}{2} ikc\, \lambda^n = \left(1 - \frac{\Delta t}{2} ikc\right)\lambda^n$$

$$\lambda^{n+1} = \lambda^n - \Delta t\, ikc\, \lambda^{n+1/2} = \lambda^n - \Delta t ikc\left(1 - \frac{\Delta t}{2} ikc\right)\lambda^n$$

$$\lambda = 1 - \frac{\Delta t^2 k^2 c^2}{2} - i\Delta t\, kc$$

$$|\lambda|^2 = 1 + \frac{\Delta t^4 k^4 c^4}{4}$$

→ inestable, pero crece lento...

Criterio de estabilidad amplio → alcanza con que $|\lambda| \leq 1 + \alpha\Delta t$ con α chico

A $T = n\Delta t$ fijo , $\epsilon^{n+1} = \lambda\epsilon^{n} \sim (1+\alpha\Delta t)\epsilon^{n} \sim (1+\alpha\Delta t)^{n+1}\epsilon^{0}$

$\Delta t = T/n \Rightarrow \epsilon^{n} \sim \left(1 + \frac{\alpha T}{n}\right)^{n}\epsilon^{0} \sim e^{\alpha T}\epsilon^{0}$ → error crece cómo $e^{\alpha T}$ (fijo para T fijo)

$\alpha =$ tasa de crecimiento del error

error en la energía crece como $e^{2\alpha T}$

En Burgers RK2, $|\lambda|^{2} = 1 + \frac{\Delta t^{4} k^{4} c^{4}}{4}$ $|\lambda| \sim 1 + \frac{\Delta t^{4} k^{4} c^{4}}{8} \leq 1 + \alpha\Delta t$

Peor caso → k=N/2 → $\alpha = \frac{\Delta t^{3} N^{4} c^{4}}{16 \times 8}$

notar que la tasa de crecimiento aumenta con N (y con c)

→ si quiero error fijo, al aumentar N tengo que achicar $\Delta t \sim \frac{1}{N^{4/3} c^{4/3}}$ → es cómo un CFL

Por otro lado, en el caso no lineal c → u y puede cambiar con el tiempo

Método PS para Navier-Stokes incompresible 2D

→ planteamos con vorticidad-función corriente

$$\mathbf{v} = v_x(x,y,t)\,\hat{\mathbf{x}} + v_y(x,y,t)\,\hat{\mathbf{y}}$$

$$v_x = \frac{\partial \psi}{\partial y} \quad , \quad v_y = -\frac{\partial \psi}{\partial x}$$

$$\mathbf{w} = \nabla \times \mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$w = w_z = -\nabla^2 \psi$$

$$\frac{\partial w}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial w}{\partial y} = \nu \nabla^2 w$$

Se puede reescribir como $$\frac{\partial w}{\partial t} = \frac{\partial}{\partial y}\left(w\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(w\frac{\partial \psi}{\partial y}\right) + \nu \nabla^2 w$$

$$w = w(x,y,t) \to w_N(x,y,t) = \sum_{\mathbf{k}} \hat{w}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \qquad \mathbf{k} = (k_x, k_y) \quad , \quad \mathbf{r} = (x,y)$$

$$w_N(x,y,t) = \sum_{k_x,k_y=-N/2}^{N/2-1} \hat{w}_{\mathbf{k}} e^{ik_x x} e^{ik_y y}$$

discretización →

$$x_j = \frac{2\pi}{N} j \quad , \quad j = 0, \ldots, N-1$$

$$y_l = \frac{2\pi}{N} l \quad , \quad l = 0, \ldots, N-1$$

$$w_N(x,y,t) = \sum_{k_x,k_y=-N/2}^{N/2-1} \hat{w}_{\mathbf{k}} e^{ik_x x_j} e^{ik_y y_l}$$

FFT 2D

$$\hat{w}_k(t) = \frac{1}{N^2} \sum_{j,l=0}^{N-1} w_N(x_j, y_l, t) e^{-ik_x x_j} e^{-ik_y y_l}$$

$$\frac{\partial w_N}{\partial x} \to ik_x \hat{w}_{\mathbf{k}} \qquad \frac{\partial w_N}{\partial y} \to ik_y \hat{w}_{\mathbf{k}}$$

$$\nabla^2 w \to -k_x^2 \hat{w}_{\mathbf{k}} - k_y^2 \hat{w}_{\mathbf{k}} = -k^2 \hat{w}_{\mathbf{k}}$$

$$w = -\nabla^2 \psi \to \hat{w}_{\mathbf{k}} = k^2 \hat{\psi}_{\mathbf{k}} \to \hat{\psi}_{\mathbf{k}} = \frac{\hat{w}_{\mathbf{k}}}{k^2}$$

(elijo $\hat{\psi}_{\mathbf{k}=\mathbf{0}} = 0$)

Método PS:

1) Dada $w(t=0) \rightarrow$ obtengo (FFT) $\hat{w}_{\mathbf{k}}(0)$

2) Calculo $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \nabla^2$ en $\mathbf{k}$ $\rightarrow ik_x\hat{w}_{\mathbf{k}},\ ik_y\hat{w}_{\mathbf{k}}\ , -k^2\hat{w}_{\mathbf{k}}$

$$\text{y}\ \hat{\psi}_{\mathbf{k}} = \frac{1}{k^2}\hat{w}_{\mathbf{k}} \rightarrow ik_x\hat{\psi}_{\mathbf{k}},\ ik_y\hat{\psi}_{\mathbf{k}}$$

3) Hago FFT^{-1} $\hat{w}_{\mathbf{k}} \rightarrow w$

$ik_x\hat{\psi}_{\mathbf{k}} \rightarrow \frac{\partial\psi}{\partial x}$ → hago $w\frac{\partial\psi}{\partial x}(x_j, y_l)$ y $w\frac{\partial\psi}{\partial y}(x_j, y_l)$

$ik_y\hat{\psi}_{\mathbf{k}} \rightarrow \frac{\partial\psi}{\partial y}$ → hago FFT $\left[\widehat{w\frac{\partial\psi}{\partial x}}\right]_k$ y $\left[\widehat{w\frac{\partial\psi}{\partial y}}\right]_k$ $\rightarrow ik_y\left[\widehat{w\frac{\partial\psi}{\partial x}}\right]_k$ y $ik_x\left[\widehat{w\frac{\partial\psi}{\partial y}}\right]_k$

$\left[\frac{\partial}{\partial y}\left(w\frac{\partial\psi}{\partial x}\right)\right]_k$ $\left[\frac{\partial}{\partial x}\left(w\frac{\partial\psi}{\partial y}\right)\right]_k$

4) Integro $\rightarrow \hat{w}_{\mathbf{k}}(t+\Delta t)$

dealiasing: anulo modos con $|k_x| > N/3$, $|k_y| > N/3$